GENERALIZED SYNCHRONIZATION: MASTER-SLAVE RELATIONSHIP IN THREE COUPLED SYSTEMS

GİZEM DOĞAN

A thesis submitted to
the School of Graduate Studies of Kadir Has University
in partial fulfilment of the requirements for the degree of
Master of Science in
Faculty of Engineering and Natural Sciences

Istanbul, June, 2022

APPROVAL

This thesis titled **GENERALIZED SYNCHRONIZATION: MASTER-SLAVE RELATIONSHIP IN THREE COUPLED SYSTEMS** submitted by GİZEM DOĞAN, in partial fulfillment of the requirements for the degree of Master of Science in Faculty of Engineering and Natural Sciences is approved by

Asst. Prof. Dr. Deniz Eroğlu (Advisor) ....
Kadir Has University

Prof. Dr. Ayşe Bilge ....
Kadir Has University

Assoc. Prof. Dr. Şükrü Yalçınkaya ....
Istanbul University

I confirm that the signatures above belong to the aforementioned faculty members.

....
Prof. Dr. Mehmet Timur Aydemir
Director of School of Graduate Studies
Date of Approval: 20.06.2022

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In addition, I acknowledge that any claim of irregularity that may arise in relation to this work will result in a disciplinary action in accordance with university legislation.

GİZEM DOĞAN

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20.06.2022

To my lovely sister, for everything

ACKNOWLEDGEMENT

I want to express my gratitude to the following people for providing invaluable support. First of all, I would like to thank my supervisor Deniz Eroğlu for inspiring and productive discussions, excellent help, and continued motivation. I am grateful to all of my friends and NODDS Lab members, Narçiçeği Kıran, Nuray Söğünmez, Elif Yunt, İrem Topal, Çelik Özdeş, Arash Rezaeinazhad, Toprak Fırat, Mustafa Küçük, Ebru Özen, Murat Akın Çiçek, Ayberk Ayhan, Mehmet Kırtışoğlu, İlkin Tırnaklı. I would like to especially thank Sajjad Bakrani for his kind support and help during my study. I am grateful for the funding received from Scientific and Technological Research Council of Turkey (TUBITAK) under Grant No. 119F125 and Kadir Has University for the full scholarship and additional financial life support. I am also thankful to my amazing friends Yiğit Turan, Ceyda Gamze Tosun, and Elif Ertekin for their generous support. Finally, my warmest thanks to my wonderful parents for all their love and support.

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ABSTRACT

Synchronization is an important phenomenon for complex, biological, and physical systems such as the brain, i.e., Parkinson's disease, heart beating, hand-clapping, power grids, lasers, and many others. Intuitively, we can express synchronization as strong correlations between coupled systems. We can state two scenarios in this manner. One is synchronization between identical systems, which is called complete synchronization; the other is the synchronization between the non-identical systems, called generalized synchronization. In this thesis, initially, we considered the two coupled systems and calculated the critical coupling value for the generalized synchronization analytically. More precisely, the Lorenz system drives two Rössler systems. We investigated the critical coupling value for synchronization numerically. However, real-world examples are much more complex. The most straightforward case was the two coupled systems for the generalized synchronization, and next, we focus on three coupled systems. In particular, suppose that we have three coupled one Lorenz and two Rössler systems. In our example, the Lorenz system drives the first Rössler system, and first Rössler system drives the second Rössler system, and finally, the second Rössler system also drives the Lorenz system. We calculated the critical coupling of the whole system for generalized synchronization and analyzed the time series for each system.

Keywords: synchronization, coupled systems

ÖZET

Senkronizasyon, beyin gibi karmaşık, biyolojik ve fiziksel sistemler, yani Parkinson hastalığı, kalp atışı, alkışlama, elektrik şebekeleri, lazerler ve diğerleri için önemli bir olgudur. Sezgisel olarak, senkronizasyonu, bağlı sistemler arasındaki güçlü korelasyonlar olarak ifade edebiliriz. Bu şekilde iki senaryo belirtebiliriz. Biri, tam senkronizasyon olarak adlandırılan özdeş sistemler arasındaki senkronizasyondur; diğeri ise, genelleştirilmiş senkronizasyon olarak adlandırılan, özdeş olmayan sistemler arasındaki senkronizasyondur. Bu tezde ilk olarak iki bağlı sistemi ele aldık ve genelleştirilmiş senkronizasyon için kritik eşleşme değerini analitik olarak hesapladık. Daha açık bir şekilde, Lorenz sistemi iki Rössler sistemini çalıştırır. Bu senaryodaki senkronizasyonu gözlemleyeceğimiz kritik eşleşme değerini sayısal olarak araştırdık. Ancak, gerçek dünyadaki örnekler çok daha karmaşıktır. Bu bağlamda, en basit durum, genelleştirilmiş senkronizasyon için iki bağlı sistemdi ve daha sonra, üç bağlı sisteme odaklandık. Üç adet birleştirilmiş bir Lorenz ve iki adet Rössler sistemimiz olduğunu varsayalım. Bu örnekte Lorenz sistemi birinci Rössler sistemini ve ilk Rössler sistemi ikinci Rössler sistemini çalıştırır ve son olarak ikinci Rössler sistemi de Lorenz sistemini çalıştır. Genelleştirilmiş senkronizasyon için tüm sistemin kritik eşleşme değerini hesapladık ve her sistem için zaman serilerini analiz ettik.

Anahtar Sözcükler: senkronizasyon, birleştirilmiş sistemler

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LIST OF SYMBOLS

$\mathbb{R}^{n\times n}$	The set of n by n matrices
$C(\mathcal{I},\mathbb{R}^n)$	The space of continuous functions from the interval $\mathcal{I}$ to $\mathbb{R}^n$
$\phi(t,x)$	Flow of a system of ODEs
$\dot{x}$	Derivative of x with respect to the $t\in\mathbb{R}$ variable
$\|A\|$	The norm of the n by n matrix A
e^{At}	The exponential of the matrix $A\in\mathbb{R}^{n\times n}$
E^s, E^u, E^c	The stable, unstable and center linear subspaces
$N_\epsilon(x)$	The epsilon neighborhood of a point x in the given phase space
$\mathcal{C}^k$	The set of k times continuously differentiable functions

LIST OF ACRONYMS AND ABBREVIATIONS

IVP	Initial Value Problem
ODEs	Ordinary Differential Equations

1. INTRODUCTION

This thesis introduces the phenomenon of synchronization in coupled chaotic dynamical systems. By synchronization, we mean a notion of strong correlations between coupled systems. Dynamical systems have synchronization if the distance of their states goes to zero when time is sufficiently large. Intuitively, synchronization refers to the tendency to have the same dynamical behavior. In this sense, we can categorize synchronization in real-world examples. From biology point of view, heart beating is an example of synchronized cells. Parkinson's disease and epilepsy are also related to synchronization phenomena. From the mathematical point of view, we want to understand these phenomena more concretely. Suppose that we have identical systems, then we can define their synchronization, and it is called complete synchronization. On the other hand, if we took totally different kinds of systems and coupled them, we can define another kind of synchronization called generalized synchronization. Chaotic dynamics is one of the scientific revolutions of the twentieth century related to unpredictability—initiated by Henri Poincaré in the late-nineteenth century, and we know that chaotic behavior typically occurs when we measure a positive Lyapunov exponent—we will discuss some details of chaotic dynamics in Section 5. In this thesis, we will provide the necessary conditions for the existence of generalized synchronization between chaotic systems.

In particular, consider two non-linear systems

$$\begin{aligned}\dot{x} &= f(x)\\ \dot{y} &= g(y) + \alpha(x-y)\end{aligned}$$

where $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$.

We take the case of two different coupled systems with α coupling and investigate the critical coupling value for the existence of generalized synchronization using the

auxiliary system approach. For a master system, we used the Lorenz system, and for a slave system, we used the Rössler system and we calculated the critical coupling value for synchronization. One step further is the three-coupled systems.
Consider the following equations

$$\begin{aligned}\dot{x} &= f(x) + \alpha(z - x)\\ \dot{y} &= g(y) + \beta(x - y)\\ \dot{z} &= h(z) + \gamma(y - z)\end{aligned}$$

where $f, g, h : \mathbb{R}^n \to \mathbb{R}^n$ are $\mathcal{C}^\infty$ and $\alpha, \beta, \gamma \in \mathbb{R}$. We have three master systems and these systems are a slave system also. Numerically we calculate the criticality for the existence of generalized synchronization by using the auxiliary system approach. We used three different chaotic systems for our simulations, Lorenz system and two Rössler systems. The parameters of the first Rössler system $a = 0.2,\ b = 0.2,\ c = 5.7$ and other one is $a = 0.2,\ b = 0.2,\ c = 20$. We investigated and illustrated the generalized synchronization phenomena with respect to the these parameter values. The direction of the thesis is as follows. Sections 2, 3, and 4 briefly give the background of non-linear dynamical systems theory. Section 5 will discuss the chaos phenomena, and in chapters we study synchronization scenarios between coupled systems.

2. ORDINARY DIFFERENTIAL EQUATIONS

This chapter aims to develop some elementary knowledge of ordinary differential equations. The definitions and examples here illustrate some basic ideas for finding the differential equations' solutions. Later, we introduce the existence and uniqueness theorem as the criteria for this aim. The first couple of examples related to the existence of solutions and later examples aimed to give the reader the taste of specific topics, e.g., equilibria, periodic solutions that we will often return to throughout this thesis.

2.1 Basics of ODEs

Consider a differential equation

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_0) &= x_0 \end{aligned} \tag{2.1.1}$$

with $f : U \to \mathbb{R}^n$. Here $U \subseteq \mathbb{R} \times \mathbb{R}^n$ is an open set[1] and (t_0, x_0) is a initial value for system (2.1.1). This type of differential equation is called *Initial Value Problem* (IVP).

Definition 2.1.1 (Solution of an IVP)**.** Consider an initial value problem

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{2.1.2}$$

with $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and $f \in \mathcal{C}^1$[2]. By a solution we mean a function

$$\begin{aligned} x &: \mathcal{I} \to \mathbb{R}^n \\ t &\mapsto x\,(t) \end{aligned} \tag{2.1.3}$$

where $\mathcal{I} \subset \mathbb{R}$ is an interval containing 0 that satisfies system (2.1.2).

[1]See, Definition 8.1.2.
[2]See, Definition 8.1.4.

The purpose of this section is asking the questions such as, does this system have a solution or not. If it has, it is unique or infinite? Here are a few examples for each of these possibilities.

Example 2.1.1 (Unique solution)**.** Consider

$$\begin{aligned} \dot{x} &= ax, \quad a \in \mathbb{R} \\ x(0) &= x_0. \end{aligned} \tag{2.1.4}$$

The function $x(t) : \mathbb{R} \to \mathbb{R}$, where $x(t) = e^{at}x_0$ is a solution of system (2.1.4) (see, Figures 2.1 and 2.2). It follows from Theorem 2.2.1 (proved later) that this solution is unique.

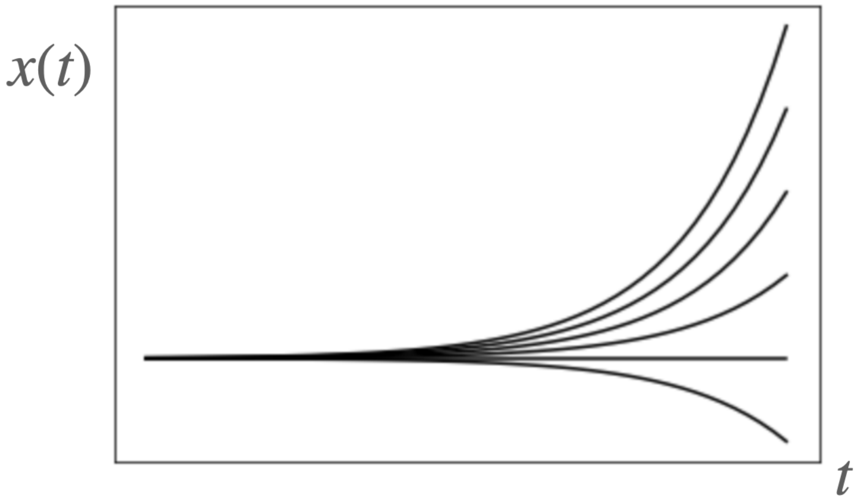

Figure 2.1 The solution graphs for $\dot{x} = ax$, for $a > 0$. Each curve represents a particular solution.

Example 2.1.2 (No solution)**.** Consider $\dot{x} = f(x)$ where

$$f(x) = \begin{cases} 1 & \text{when } x < 0 \\ -1 & \text{when } x \geq 0 \end{cases}$$

Consider a solution at $x(0) = 0$. Such a solution must decrease since $\frac{dx(0)}{dt} = -1$, but for all negative values of x, solutions must increase. This cannot happen.

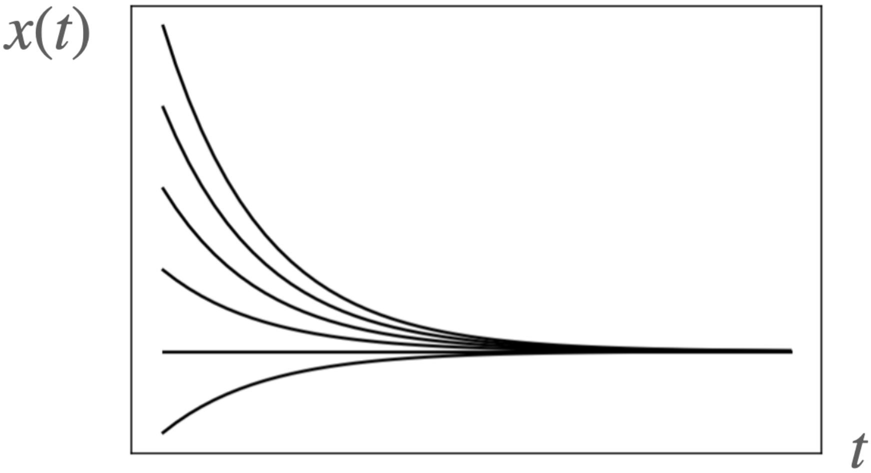

Figure 2.2 The solution graphs for $\dot{x} = ax,$ for $a < 0$.

Example 2.1.3 (Infinitely many solutions)**.** Take $\dot{x} = |x|^{\frac{1}{2}}$ with $x(0) = 0$. First, $x(t) = 0$ is clearly a solution. Then we can see that $x(t)$ defined by

$$x(t) = \begin{cases} 0, & \text{when } t \leq 1 \\ (t-1)^2/4, & \text{when } t > 1 \end{cases}$$

is a solution. For every t value we have another solution, since $t \in \mathbb{R}$, we have infinitely many solutions.

This example is taken from Van Strien (2018, p. 11). From these two examples we can see that, to ensure existence and uniqueness of solutions, there must be some concrete conditions on the function $f(x)$. In the first example, the function $f(x)$ is not continuous at the point 0. In the second example the function $f(x)$ is not differentiable at the point 0. We will see in Theorem 2.2.1 that if we add differentiability (in fact, the weaker condition of locally Lipchitz[3]) on the function $f(x)$ that gives us the Existence and uniqueness of a solution.

2.2 Existence and Uniqueness of Solutions of ODEs

In this section, we focus on the elementary theorem of differential equations, the existence and uniqueness theorem. Let f be a $\mathcal{C}^1$ function on $\mathbb{R}^n$, and consider an

[3]See, Definition 8.1.6.

IVP

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x_0 \end{aligned} \tag{2.2.1}$$

In Definition 2.1.1, we defined the solution of system (2.2.1). In geometrical sense, the solution $x(t)$ of system (2.2.1) is a curve, and at every time t_0, the vector $f(x(t_0))$ is tangent to this curve at the point $x(t_0) \in \mathbb{R}^n$. As we mentioned before, non-linear differential equations may not have (unique) solutions at certain initial conditions. The following fundamental theorem gives criteria for the existence and uniqueness of a solution.

Theorem 2.2.1 (Existence and uniqueness of solutions of ODEs)**.** *Consider $\dot{x} = f(x)$, $x(0) = x_0$ with $f : U \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^n$ is open. Suppose f be a $\mathcal{C}^1$ function. Then, there exists a unique function $x : (-h, h) \to \mathbb{R}^n$ such that it is a solution for system (2.2.1).*

Sketch of the proof. Suppose that $\mathcal{I}$ be a compact interval in $\mathbb{R}$. Let $C(\mathcal{I}, \mathbb{R}^n)$ be the space of continuous functions defined on $\mathcal{I}$ to $\mathbb{R}^n$. We know that, the space $C(\mathcal{I}, \mathbb{R}^n)$ is a Banach space (see, Definition 8.1.8) with a norm $\|u(t)\| = \max_{t\in\mathcal{I}} |u(t)|$[4] where $u(t) \in C(\mathcal{I}, \mathbb{R}^n)$. By integration

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

is equivalent to

$$x(t) - x(0) = \int_0^t f(x(s))ds.$$

Then, it follows that finding a solution of IVP turns out to find the fixed point x of the operator $F : C(\mathcal{I}, \mathbb{R}^n) \to C(\mathcal{I}, \mathbb{R}^n)$ such that

$$F(u(t)) = x_0 + \int_0^t f(u(s))ds$$

[4]Here, $|u(t)| = \sqrt{\sum_{i=1}^n (u_i^2)}$ where $u \in \mathbb{R}^n$ and $t \in \mathcal{I}$.

on Banach space $C(\mathcal{I}, \mathbb{R}^n)$. One can show that F is a contraction map on Banach space $C(\mathcal{I}, \mathbb{R}^n)$. Then, by Banach fixed point theorem (see, Theorem 8.1.1), the operator F has a unique fixed point. $\square$

Let us introduce the concept of the flow of a system of differential equations.

Definition 2.2.1 (Flow of a system of ODEs)**.** Consider IVP (2.2.1) and let $\phi_t(x_0)$ be the solution at the initial condition $x_0 \in \mathbb{R}^n$. Then, the function $\phi(t, x_0) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\phi(t, x_0) := \phi_t(x_0)$ is called the flow of system $\dot{x} = f(x)$.

The following definition is the orbit of a point $x_0 \in \mathbb{R}^n$.

Definition 2.2.2 (Orbits)**.** Let $x_0 \in \mathbb{R}^n$. The set

$$\{\phi(t, x_0) : t \in \mathbb{R}\}$$

is called the orbit through x_0.

Definition 2.2.3 (Equilibria)**.** If the set $\{\phi(t, x_0) : t \in \mathbb{R}\} = \{x_0\}$ then, we say x_0 is an equilibrium point.

Definition 2.2.4 (Periodic orbits)**.** We say $\phi_t(x)$ is a periodic orbit of the system $\dot{x} = f(x)$ if there exists a constant $T > 0$ and a nonequilibrium point $x \in \mathbb{R}^n$ such that $\phi_T(x) = x$.

Remark 2.2.1. *It follows that $\phi_{t+T}(x) = \phi_t(x)$ for every $t \in \mathbb{R}$. The minimum value of T is called the period of the solution.*

We already know the equilibrium solution must satisfy the equation $\frac{\partial \phi}{\partial t}(t, x_0) = f(\phi(t, x))$. Since $\phi(t, x_0) = x_0$ then $\dot{x}_0 = 0 = f(x_0)$. Hence, if we have the equilibrium solution then, we have $f(x_0) = 0$. Moreover, we can take this equality as a definition of an equilibrium solution.

3. LINEAR SYSTEMS

In this thesis, our aim is studying generalized synchronization. To be more precise, consider the system

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} (x_j - x_i) \quad \text{for } i \in \{1, 2, \cdots, N\}$$

where $x_i, x_j \in \mathbb{R}^n, \alpha \in \mathbb{R}$ and $f_i \in \mathcal{C}^{\infty}(\mathbb{R}^n)$. In our case, we take two or three non-linear coupled systems, and these are the main models for our synchronization phenomena that we introduce later. To examine the behavior of above system, we use the mainly linearization technique, and we have to know the properties of linear systems for this manner. Linear systems will be our reference to analyzing stability, and they play a key role in examining the behavior of solutions of non-linear systems. Let us begin to present the linear system of differential equations

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$, A is n by n matrix and

$$\dot{x} = \frac{dx}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}.$$

The massive portion of this chapter is concerned the computation of a matrix exponential (see, Definition 3.1.2) and its computational properties (see, Proposition 3.1.3). Then we focus on the stability of linear systems and introduce the spaces E^s, E^u, E^c, and finally, we define the concept of sinks and sources.

3.1 Linear Operators and the Matrix Exponential

In order to define matrix exponential we should define the concept of convergence in the linear spaces. Consider a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. The operator norm of T defined by

$$\|T\| = \max_{|x|\leq 1} \big|T(x)\big|$$

where $|\cdot|$ is Euclidean norm on $\mathbb{R}^n$. More precisely, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.

Remark 3.1.1. *We denote the linear space (or vector space, see, Definition 8.1.7) as $L(\mathbb{R}^n)$.*

Let us define the concept of convergence on linear spaces $L(\mathbb{R}^n)$.

Definition 3.1.1 (Convergence of linear operators)**.** Suppose that $T_k \in L(\mathbb{R}^n)$ for $k = 1, 2, \cdots$. The sequence of operators T_k are called convergent to $T \in L(\mathbb{R}^n)$ when $k \to \infty$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $k \geq N$ we have $\|T_k - T\| < \epsilon$.

Lemma 3.1.1. *Take $S, T \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then the followings hold.*

(i) $\big|T(x)\big| \leq \|T\| \; |x|$

(ii) $\|TS\| \leq \|T\| \|S\|$

(iii) $\big\|T^k\big\| \leq \|T\|^k$ *for all* $k = 0, 1, 2, \cdots$.

Proof. As it is possible to find the proof in any standard books, we do not prove it here. For proof see in Ref. Perko (2014, p. 11). □

Theorem 3.1.2. *Consider $T \in L(\mathbb{R}^n)$ and let t_0 be positive real number. Then,*

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!} \tag{3.1.1}$$

is absolutely (see, Definition 8.1.9) and uniformly convergent (see, Definition 8.1.10) for all $|t| < t_0$.

Proof. Take $\|T\| = a$. By Lemma 3.1.1, for $|t| < t_0$, we have

$$\frac{\left\|T^k t^k\right\|}{k!} \leq \frac{\|T\|^k t_0^k}{k!} \leq \frac{a^k t_0^k}{k!}.$$

On the other hand,

$$\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_0}.$$

By Weierstrass M-test[5]

$$\frac{\left\|T^k t^k\right\|}{k!}$$

is absolutely and uniformly convergent for all $|t| < t_0$. □

Let A be a real $n \times n$ matrix. Suppose that, the linear operator T is represented by matrix A. Then, the exponential of the linear operator A is absolutely convergent. More precisely:

Definition 3.1.2 (The exponential of a matrix)**.** We define the exponential of an $n \times n$ matrix A by

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

where $t \in \mathbb{R}$.

Here are some properties of the matrix exponential.

Proposition 3.1.3. *Let $A,\ B,\ T\ \in \mathbb{R}^{n \times n}$. Suppose that matrix T be invertible. Then, the following statements hold.*

(i) If $B = T^{-1}AT$ then, $e^B = T^{-1}e^A T$
(ii) If $AB = BA$ then, $e^{A+B} = e^A e^B$
(iii) $e^{-A} = (e^A)^{-1}$

Proof. See, Van Strien (2018, p. 43). □

[5]See, Proposition 8.1.2.

3.2 Stability of Linear Systems

In this section we introduce the concept of stability, and later in Section 4.2, we will see that if we have hyperbolic equilibria, then the flow of non-linear system and the flow of its linearized system conjugate with each other (see, Theorem 4.2.2). Before we give the theorem, we will focus on some fundamental definitions of linear systems, and immediately after, we give the definition of stability of linear systems.

Take a matrix $A \in \mathbb{R}^{n\times n}$. The eigenvalues of the matrix A is denoted by λ_i for $i \in \{1, 2, \cdots, n\}$. It can be calculated from $\det(A - \lambda_i\ I) = 0$. Consider the non-zero vectors v_j such that $v_j \in \text{Null}(A - \lambda_i\ I)$.[6] Then, vector v_j are called associated eigenvectors with eigenvalues λ_i where $j \in \{1, 2, \cdots, n\}$. Here I is the $n \times n$ identity matrix. Also the vectors v such that $v \in \text{Null}(A - \lambda I)^k$ but $v \notin \text{Null}(A - \lambda I)^{k-1}$ is called generalized eigenvectors of order k where $k \in \mathbb{N}$.

Definition 3.2.1. Let $A \in \mathbb{R}^{n\times n}$, we say A is hyperbolic if all of its eigenvalues have non-zero real parts.

Definition 3.2.2. Let $A \in \mathbb{R}^{n\times n}$, we say equilibrium point x_0 is hyperbolic if all of eigenvalues of matrix A have non-zero real parts.

In here, we define the stable unstable and center subspaces, E^s, E^u and E^c of the autonomous linear system

$$\dot{x} = Ax, \tag{3.2.1}$$

where $A \in \mathbb{R}^{n\times n}$. Let $w_j = u_j + iv_j$ be a generalized eigenvector of A associated to an eigenvalue $\lambda_j = a_j + ib_j$, where $1 \le j \le n$.

Definition 3.2.3. Let $w_j = u_j + iv_j \in \mathbb{C}$ be the generalized eigenvector and $\lambda_j =$

[6] The null space of matrix A denoted as $\text{Null}(A)$. By definition, the vector $x \in \text{Null}(A)$ if
$Ax = 0$.

$a_j + ib_j \in \mathbb{C}$ be the corresponding eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$. Then,

$$E^s := \text{span}\{u_j, v_j : a_j < 0\}$$
$$E^u := \text{span}\{u_j, v_j : a_j > 0\}$$
$$E^c := \text{span}\{u_j, v_j : a_j = 0\}$$

for all $1 \leq j \leq n$.

Remark 3.2.1. *The spaces E^s, E^u and E^c are called stable, unstable and center subspaces of system (3.2.1) respectively.*

Example 3.2.1. Find the stable, unstable and center subspaces of the linear system (3.2.1) with the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution. Consider $\dot{x} = Ax$ then $\dot{x_1} = x_1$ and $\dot{x_2} = -x_2$. Let us calculate the eigenvalues and the corresponding eigenvectors of the matrix A. That is $\det(A - \lambda_i I) = 0$ for $i = 1, 2$. Hence, we have $\lambda_1 = 1$ and its associative eigenvector is $v_1 = e_1$. Likewise, $\lambda_2 = -1$ and its corresponding eigenvector is $v_2 = e_2$. Then, by the definition of the spaces E^s, E^u

$$E^s = \text{span}\{e_2\}$$
$$E^u = \text{span}\{e_1\}$$

where $e_1 = (1, 0),\ e_2 = (0, 1)$ that is the standard basis of $\mathbb{R}^2$.

Example 3.2.2. [7] Consider the linear system (3.2.1). Take the matrix A as

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{3.2.2}$$

The eigenvalues of the matrix A is $\lambda_1 = \lambda_2 = i$ and $\lambda_3 = 2$. The eigenvectors of A is $(0, 1, 0) \pm i(1, 0, 0)$ associated to eigenvalues λ_1, λ_2. Likewise, the eigenvectors of A

[7] This example and figure are taken from Perko (2014, p. 52).

is $(0, 0, 1)$ associated to eigenvalue λ_3. Then, by definition the center and unstable subspaces are

$$E^c = \text{span}\{(0, 1, 0), (1, 0, 0)\}$$
$$E^u = \text{span}\{(0, 0, 1)\}.$$

Hence, $(x_1, x_2)-$ plane is the center subspace E^c and x_3 axis is the unstable subspace E^u (see, Figure 3.1).

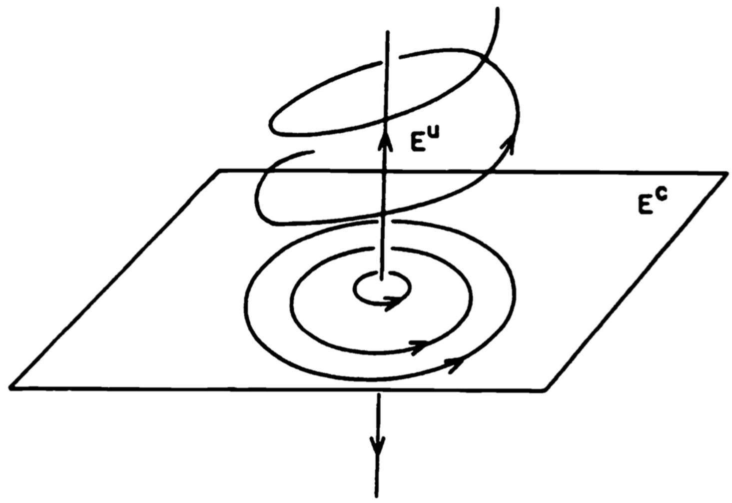

Figure 3.1 The center E^c and unstable E^u spaces of the linear system (3.2.2).

Theorem 3.2.1. *Let $A \in \mathbb{R}^{n\times n}$. Thus,*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c$$

where E^s, E^u and E^c are the stable, unstable and center subspaces of the linear system (3.2.1). Moreover, E^s, E^u and E^c are invariant[8] *with respect to the flow of (3.2.1), i.e., $e^{At}E \subseteq E$ where $E \in \{E^s, E^u, E^c\}$.*

Proof. See, Perko (2014, p. 55). □

Now, we can define the sinks and sources of n-dimensional linear systems.

[8]See, Definition 8.2.2.

Definition 3.2.4 (Sinks)**.** Let x^* be the equilibrium solution for the linear system (3.2.1). If all of the eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$ have negative real parts, then we say x^* is sink for the linear system (3.2.1).

And likewise we can define the source as follows:

Definition 3.2.5 (Sources)**.** Let x^* be the equilibrium solution for the linear system (3.2.1). If all of the eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$ have positive real parts, then we say the x^* is source for the linear system (3.2.1).

Proposition 3.2.2. *Consider linear system (3.2.1) again. If $x_0 \in E^s$ then, $e^{At}x_0 \in E^s$ for every $t \in \mathbb{R}$ and $\lim_{t\to\infty} e^{At}x_0 = 0$. Likewise, if $x_0 \in E^u$ then, $e^{At}x_0 \in E^u$ for every $t \in \mathbb{R}$ and $\lim_{t\to-\infty} e^{At}x_0 = 0$.*

Proof. See, Perko (2014, p. 58). □

4. STABILITY OF THE EQUILIBRIA

In this section we discuss the stability of the equilibrium points of the non-linear system

$$\dot{x} = f(x).$$

We will see that the stability of hyperbolic equilibrium point x^* of this system is determined by the sign of the real parts of eigenvalues of its linearized system (see, Theorem 4.2.3). However, the stability of non-hyperbolic points is generally more difficult to determine. In the following we introduce Lyapunov function (see, Theorem 4.3.1) as one way to do that. Before we start let us define an equilibrium point's stability, instability, and asymptotic stability in the following definitions.

4.1 Basic Definitions

Suppose that $x^* \in \mathbb{R}^n$ is an equilibrium point for system

$$\dot{x} = f(x). \tag{4.1.1}$$

Let $\phi(t, x)$ be the flow of system (4.1.1).

Definition 4.1.1 (Stability of an equilibrium point)**.** The point x^* is a *stable* equilibrium if for every $\epsilon > 0$ there exists $\delta > 0$ such that $N_\delta(x^*) \subseteq N_\epsilon(x^*)$[9] and for every initial value $x_0 \in N_\delta(x^*)$ the flow $\phi(t, x_0)$ remains in the neighbourhood $N_\epsilon(x^*)$ for all $t \geq 0$.

Definition 4.1.2 (Instability of an equilibrium point)**.** The equilibrium x^* is called *unstable* if it is not stable.

[9] To make a emphasize of the radius, the $\epsilon-$neighbourhood of the equilibrium point x^* is denoted as $N_\epsilon(x^*)$ (see, Definition 8.1.3).

Definition 4.1.3 (Asymptotic stability of an equilibrium point)**.** If we have a stable equilibrium point x^* and for all $x_0 \in N_\delta(x^*)$ we have also $\lim_{t\to\infty} \phi(t, x_0) = x^*$, then x^* is called *asymptotically stable.*

4.2 Linearization

Let us define concept of the linearization of a given system.

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^k$ function. Take $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then,

$$f(x_1, x_2, \cdots, x_n) = (f_1(x), f_2(x), \cdots, f_n(x))$$

where function $f_i : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^k$, for all $i = 1, 2, \cdots, n$.

Theorem 4.2.1 (Taylor's theorem)**.** *Let f be a $\mathcal{C}^k$ function at the origin that we described above. Assume that we have non-negative integers α_i for every $i = 1, \cdots, n$. Consider following definitions $|\alpha| := |\alpha_1 + \alpha_2 + \cdots + \alpha_n|$ with $\alpha! := \alpha_1!\ \alpha_2!\ \cdots\ \alpha_n!$ and $x^\alpha := x_1^{\alpha_1}\ x_2^{\alpha_2}\ \cdots\ x_n^{\alpha_n}$. Then, for $|\alpha| = k$ and $i = 1, 2, \cdots, n$ there exists a function $R_{i,\alpha} : \mathbb{R}^n \to \mathbb{R}^n$ such that we have*

$$f(x) = \sum_{|\alpha|\leq k} \sum_{i=1}^{n} \frac{D^\alpha f_i(0)}{\alpha!}\ x^\alpha + \sum_{|\alpha|=k} \sum_{j=1}^{n} R_{j,\alpha}(x)\ x^\alpha$$

where $\lim_{x\to 0} R_{i,\alpha}(x) = 0$.

Proof. As it is possible to find the proof in any standard books, we do not prove it here. □

Example 4.2.1 (Taylor's theorem for $n = 1$)**.** Consider $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^k$. Then, at the origin we have

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(k)}(0)\frac{x^k}{k!}$$

and

$$\mathcal{R}(x) = R(x)\ x^k$$

where $\lim_{x\to 0} R(x) = 0$.

Consider a system

$$\dot{x} = f(x) \tag{4.2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that the equilibrium point of this system is $x^* = 0$. By Taylor's theorem we can write this system

$$\dot{x} = Df(0)x + R(x)$$

where $\lim_{x\to 0} R(x) = 0$ and $Df(0)$ is $n \times n$ matrix. More precisely,

$$Df(0) = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} & \cdots & \frac{\partial f_1(0)}{\partial x_n} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} & \cdots & \frac{\partial f_2(0)}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n(0)}{\partial x_1} & \frac{\partial f_n(0)}{\partial x_2} & \cdots & \frac{\partial f_n(0)}{\partial x_n} \end{pmatrix}.$$

Remark 4.2.1. *The matrix $Df(0)$ is Jacobian matrix of the function $f : \mathbb{R}^n \to \mathbb{R}^n$ at the origin.*

Definition 4.2.1. Consider system (4.2.1) again, $Df(0)$ is called the linear part of the system at the origin, and

$$\dot{x} = Df(0)x$$

is called the linearized system at the origin.

A *homeomorphism* is a function that continuous, one-to-one, onto, and the inverse is also continuous. Consider two systems of differential equations on $\mathbb{R}^n$ that is $\dot{x} = f(x)$ and $\dot{y} = g(y)$ where $f, g \in \mathcal{C}^1$. The systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are equivalent if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h(\phi_f(t, x)) = \phi_g(t, h(x))$$

where $\phi_f(t, x)$ and $\phi_g(t, h(x))$ is the flow of the systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ respectively.

Now, we turn on to classify the stability of the equilibrium points of the non-linear systems, and the following theorem is the criteria for stability of these systems.

Theorem 4.2.2 (Linearization theorem (Hartman–Grobman theorem)). *Consider the system*

$$\dot{x} = f(x) \tag{4.2.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and f is $\mathcal{C}^k$. Suppose that system (4.2.2) has hyperbolic equilibrium point $x^* \in \mathbb{R}^n$. Then, the flow of non-linear system (4.2.2) is conjugate to the flow of its linearized system in the neighbourhood of x^*.

Proof. See, Hirsch, Smale and Devaney (2013, p. 168). □

Note that the following theorem and the Hartman-Grobman theorem are practical for us. The Example 4.2.2 shows how useful to determine the stability of non-linear systems more efficiently by using the following theorem.

Theorem 4.2.3. *Suppose $x^* \in \mathbb{R}^n$ is an equilibrium point of system (4.2.2). If all of the eigenvalues of the linearized system at x^* have negative real parts then, equilibrium point x^* is asymptotically stable.*

Proof. See, Wiggins (2003, p. 11). □

Example 4.2.2. Consider the non-linear system

$$\dot{x_1} = -x_1 - x_2^3$$
$$\dot{x_2} = -x_2 + x_1^2$$

The origin is an equilibrium point for this system. The linearized system at the origin

$$\begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since the equilibrium point origin has negative real parts, the corresponding stable subspace E^s is union of x_1 and x_2 axis. Thus, by using Theorem 4.2.3, the origin is asymptotically stable.

Let us briefly introduce the non-autonomous systems. Consider $A(t) \in \mathbb{R}^{n\times n}$ and it is continuous with respect to the time variable t. Then, we have following theorem.

Theorem 4.2.4 (Existence and uniqueness theorem for non-autonomous linear system)**.** *Consider the following linear non-autonomous system*

$$\begin{aligned} \dot{x} &= A(t)x \\ x(t_0) &= x_0 \end{aligned} \tag{4.2.3}$$

where t defined on the interval $[a,b]$. Then, system (4.2.3) has unique solution on the interval $[a,b]$.

Proof. See, Hirsch, Smale and Devaney (2013, p. 401). □

Consider the autonomous non-linear system $\dot{x} = f(x)$ again where f is $\mathcal{C}^1$. Let $x(t)$ be the solution of this system defined on $t \in [a,b]$. Now, fix the time variable $t_0 \in [a,b]$ with $x(t_0) = x_0$. For every $t \in [a,b]$ let us define

$$A(t) := Df(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}$$

where $Df(x)$ is the Jacobian matrix of $f(x)$ at the point $x(t) \in \mathbb{R}^n$. We already know that $f \in \mathcal{C}^1$ function then, for every $x(t) \in \mathbb{R}^n$ the Jacobian $A(t) \in \mathbb{R}^{n\times n}$ is continuous family of matrices. Consider the non-autonomous linear system

$$\dot{u}(t) = A(t)u(t) = Df(x(t))u(t). \tag{4.2.4}$$

The above equation is called variational equation along the solution $x(t)$. Moreover, if $u(t)$ is the solution of equation (4.2.4) at $u(t_0) = u_0$ then, the function

$$x(t) + u(t)$$

is the good approximation of $y(t)$ which is the solution of the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ y(t_0) &= x_0 + u_0 \end{aligned} \tag{4.2.5}$$

where u_0 is small enough.

Example 4.2.3. Consider the non-linear system of differential equations

$$\dot{x_1} = x_1 + x_2^3$$
$$\dot{x_2} = -x_2 + x_1^2$$

One of the equilibrium solution for this system is $x^* = (0, 0)$. It follows that the Jacobian matrix for this system at x^* is $Df(x^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, the variational equation $\dot{u} = Au$ corresponding to the equation $\dot{u} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u$. This is an autonomous linear system and we can solve this system explicitly. Thus, the solution of the variational equation is $u(t) = (x_1(0)e^t, x_2(0)e^{-t})$ where the initial values $x_1(0)$ and $x_2(0)$ are given. Then, by system (4.2.5) it follows that if we sufficiently close to the origin the solution of the non-linear equation is approximately close to $u(t)$.

4.3 Lyapunov Functions

The stability of an equilibrium point is obvious when we have the hyperbolic equilibrium point. On the other hand, if we have a non-hyperbolic equilibrium point then, this case is more complicated to analyze. In this section, we introduce an alternative method and give some conditions on the stability of an equilibrium point for a given system.

The set of all initial conditions with solutions tend to the equilibrium point is called *basin of attraction of an equilibrium point*, i.e.,

$$\{y \in \mathbb{R}^n : \lim_{t \to \infty} x(t) = x^* \text{ where } x(t_0) = y \text{ for every } t_0 \in \mathbb{R}\}$$

here x^* is equilibrium point of the system $\dot{x} = f(x)$.

Let x^* be an equilibrium point of the system $\dot{x} = f(x)$. Suppose that the function $L : U_{x^*} \to \mathbb{R}$ be differentiable on an open set U_{x^*} in $\mathbb{R}^n$ that contains x^*. Consider the function

$$\dot{L}(x) = DL_x(f(x)).$$

The operator DL_x is stands for the derivative of the function $L(f)$ with respect to the point $x = (x_1, x_2, \cdots, x_n) \in U_{x^*}$.

Remark 4.3.1. *More precisely, by chain rule we can calculate the derivative of the function L as the following $\dot{L}(x) = \frac{d}{dx}L(x(t)) = DL_x \frac{dx(t)}{dt} = DL_x(f(x))$.*

Here is a method that we can investigate the stability of a non-hyperbolic equilibria.

Theorem 4.3.1. *Let x^* be the equilibrium point for the system $\dot{x} = f(x)$. Let $L : U_{x^*} \to \mathbb{R}^{\geq 0}$ be a differentiable function on an open set U_{x^*}. Suppose that*

(i) $L(x^*) = 0$.
(ii) If $\dot{L} \leq 0$ for all $x \in U_{x^*} - \{x^*\}$ then, x^* is stable.
(iii) If $\dot{L} < 0$ for all $x \in U_{x^*} - \{x^*\}$ then, x^* is asymptotically stable.

Proof. See, Hirsch, Smale and Devaney (2013, p. 196). □

Function L that satisfies (i) and (ii) is called *Lyapunov function* of the equilibrium point x^*. If the condition (iii) happens then we say L is *strict* Lyapunov function of x^*. Now, as an example of how to find Lyapunov function let us introduce the Lorenz system (4.3.1).

Example 4.3.1 (Lorenz system)**.** Consider the system

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= -\beta z + xy, \end{aligned} \tag{4.3.1}$$

where, β, σ and ρ are real positive constants.

Let us calculate the equilibrium points of Lorenz equations. From the first equation we have $y = x$. Then eliminating y from second and third equations, we obtain

$$\begin{aligned} x(\rho - 1 - z) &= 0 \\ -\beta z + x^2 &= 0. \end{aligned} \tag{4.3.2}$$

Take $x = 0$. Then, we have $y = 0$ and $z = 0$. On the other hand, $z = \rho - 1$. It follows that, we have $x = \pm\sqrt{\beta(\rho - 1)}$. Note that these expressions only valid for x and y are only real and it follows $\rho \geq 1$. Thus we have three equilibrium points $p_1, p_2 = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1)$ and $p_3 = (0, 0, 0)$. Now consider the Lorenz system as $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$. The linearized Lorenz system is

$$Df(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (\rho - z) & -1 & -x \\ y & x & -\beta \end{pmatrix}.$$

The corresponding matrix of the equilibrium point $p_3 = (0, 0, 0)$ is

$$Df(0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \tag{4.3.3}$$

Consider the eigenvalues of the linearized Lorenz system above. Take the parameters as $\sigma = 0$, $\beta = 8/3$, $\rho = 0$. The eigenvalues of (4.3.3) are $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -8/3$. Thus, the matrix $Df(0)$ is non-hyperbolic. So far we have seen theorems about hyperbolic equilibrium cases and we determined the stability of them. However, the linearized matrix at p_3 is not hyperbolic, then it follows that we must use a method that is valid for a non-hyperbolic case. Let us use Theorem 4.3.1 that we introduced to determine the stability of Lorenz system.

Consider the function $L(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. By the definition of Lyapunov function it must satisfy Theorem 4.3.1. Then, by Theorem 4.3.1 the function L must have positive real values. We can easily see that the function $L(x) \geq 0$ for all $x, y, z \in \mathbb{R}$. Moreover, at the equilibrium point x^* the function L must also satisfy $L(x^*) = 0$. Since $L(p_3) = 0$ but $L(p_1)$ and $L(p_2)$ different then zero it follows that, the assumption of theorem does not hold and the only candidate is the point p_3 to determine the stability.

Let us calculate the derivative of $L(x, y, z)$

$$\dot{L}(x, y, z) = \frac{2}{\sigma}x\dot{x} + 2y\dot{y} + 2z\dot{z}.$$

The corresponding values of $\dot{x}, \dot{y}, \dot{z}$ implies

$$\begin{aligned}
\dot{L} &= \frac{2}{\sigma}x(\sigma(y - x)) + 2yx(\rho - z) - y + 2z(-\beta z + xy) \\
\dot{L} &= 2xy - 2x^2 + 2\rho xy - 2xyz - 2\beta z^2 + 2xyz \\
\dot{L} &= 2xy - 2x^2 + 2\rho xy - 2\beta z^2 \\
\dot{L} &= -2x^2 - 2\beta z^2
\end{aligned}$$

where $\rho = -1$ and $\beta > 0$.

Since $\dot{L} < 0$ also holds by using Theorem 4.3.1, the point $p_3 = (0, 0, 0)$ is asymptotically stable equilibrium for system (4.3.1).

5. CHAOS THEORY

In the previous chapter, we have already introduced the Lorenz system, and we examined the behavior of the equilibrium solution near the origin. The Lorenz system is the most famous chaotic differential equation formulated in 1963 by Edward Norton Lorenz, and it is the system of differential equations that the simplified atmosphere model. In this chapter, we focus on the fundamental definitions of chaos theory. We define the sensitive dependence on initial conditions (see, Definition 5.1.1) in the following the chaotic invariant set (see, Definition 5.1.3) and attractor (see, Definition 5.1.4). The Rössler equations will be another famous chaotic system of differential equations, and we will examine the equilibrium points of Rössler system also. Finally, we will give the concept of Lyapunov exponents and determine chaotic behavior by itself. Later on, the Lorenz and Rössler equations will be examples of our numerical calculations of Lyapunov exponents and synchronization phenomena.

5.1 Fundamental Definitions

Consider an autonomous $\mathcal{C}^k$ systems of differential equations on $\mathbb{R}^n$

$$\dot{x} = f(x). \tag{5.1.1}$$

Suppose that $\phi_t(x)$ be the flow of system (5.1.1). Suppose further that, the set $\Lambda \subseteq \mathbb{R}^n$ is compact and invariant with respect to the flow $\phi_t(x)$. Then, we have following definitions.

Definition 5.1.1 (Sensitive dependence on initial conditions (SDIC))**.** The flow $\phi_t(x)$ have sensitive dependence on initial conditions on the set Λ if there exists $\epsilon > 0$ such that for all $x \in \Lambda$ and any neighborhood $N(x)$ of x, there exist $y \in N(x)$ and $t > 0$ such that we have $\left\|\phi_t(x) - \phi_t(y)\right\| > \epsilon$.

Intuitively, this definition basically says that for any point x in Λ we can find at least one point y arbitrarily close to x such that after some time x and y diverge from each other.

Let us introduce the notion of topological transitivity. In the following we are going to examine what is the concept of chaos and as well as the strange attractor.

Definition 5.1.2 (Topological transitivity)**.** Let Λ be closed and invariant set with respect to the flow $\phi_t(x)$. The set Λ is topologically transitive if for every two open sets $U_1, U_2 \subseteq \Lambda$ there exists time value $t \in \mathbb{R}$ such that $\phi_t(U_1) \cap U_2 \neq \emptyset$.

Definition 5.1.3 (Chaotic invariant set)**.** [10] [11] The compact and invariant set Λ is called chaotic if the following statements hold

(i) $\phi_t(x)$ has sensitive dependence on initial condition on Λ.
(ii) Λ is topologically transitive with respect to the flow $\phi_t(x)$.

Example 5.1.1. Consider

$$\dot{x} = ax, \quad x \in \mathbb{R} \tag{5.1.2}$$

with $a > 0$. The flow of the system (5.1.2) is

$$\phi_t(x) = xe^{at}. \tag{5.1.3}$$

Then, by the equation (5.1.3) we can say the following,

(i) $\phi_t(x)$ is topologically transitive on $(-\infty, 0)$ and $(0, \infty)$ but these two intervals are not a compact set.
(ii) Take two different arbitrary points $x_1, x_2 \in \mathbb{R}$. Then,

$$\left|\phi_t(x_1) - \phi_t(x_2)\right| = e^{at}|x_1 - x_2|\,.$$

[10]This definition is taken from Wiggins (2003, p. 736).
[11] Some people add following additional requirement to this definition.
$(iii)^*$ The periodic orbits of $\phi_t(x)$ are dense in Λ.

We can conclude that the flow $\phi(t,x) = xe^{at}$ has a sensitive dependence on initial conditions on $\mathbb{R}$. This means, the distance between two arbitrary points are growing when $t > 0$.

Remark 5.1.1. *The flow $\phi_t(x) = e^{at}$ has no periodic orbits.*

Definition 5.1.4 (Attractor)**.** [12] Suppose that $\dot{x} = f(x)$ is the system of differential equations with respect to the flow $\phi_t(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$.

(i) The set Λ is compact and invariant.
(ii) There exists an open set U such that $\Lambda \subseteq U$ and the set U is invariant with respect to the flow $\phi_t(x)$. Moreover, we have $\bigcap_{t\geq 0} \overline{\phi_t(U)} = \Lambda$.
(iii) The set Λ is topologically transitive.

If the statements (i), (ii), (iii) are hold then, we say the set Λ is an attractor.

Definition 5.1.5 (Strange attractor)**.** The set $\Lambda \subseteq \mathbb{R}^n$ is strange attractor if Λ is chaotic.

In the next section we are going to give a definition of the Lyapunov exponent. We will see that the positive Lyapunov exponents have been a standard way to detecting when a dynamical system is chaotic. But before we do that let us introduce the Rössler equations.

Example 5.1.2 (Rössler equations)**.** Consider the system of equations

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{5.1.4}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^3$ with all parameters $a, b, c \in \mathbb{R}$. The chaotic behaviour can see in the case that $a = 0.2,\ b = 0.2,\ \text{ and } c = 5.7$.

[12] Unfortunately there is no common accepted definition of an attractor in mathematics, some people choose condition (i) and (ii) as a definition of attractor. Moreover, if the set Λ also meets condition (iii), then it is called a transitive attractor.

5.2 Lyapunov Exponents

In this section we focus on Lyapunov exponents which is the criteria for detecting the chaotic behaviour.

Consider the system of differential equations

$$\dot{x} = f(x) \tag{5.2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f \in \mathcal{C}^k$. Take the solution $\phi_t(x) = \phi(t, x) = x(t)$ at the initial value $\phi(0, x) = x_0$. Consider the linearization of (5.2.1) at $x(t)$

$$\dot{u} = Df(x(t))u, \quad u \in \mathbb{R}^n. \tag{5.2.2}$$

Let $X(t, x_0)$ be the fundamental solution matrix[13] of the system (5.2.2). Let $e \neq 0$ in $\mathbb{R}^n$. The expression

$$\frac{\|X(t, x_0)e\|}{\|e\|}$$

is called the coefficient of expansion in the direction e along the orbit through x_0.

Definition 5.2.1 (Lyapunov exponent)**.** The Lyapunov exponent in the direction of e through the orbit of x_0 is

$$\lambda(x_0, e) = \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|X(t, x_0)e\|}{\|e\|}.$$

We also define $\lambda(x_0, 0) = -\infty$ (see, Figure 5.1).

Now, let us give a Lemma about matrix 2-norm or Euclidian norm from Linear Algebra.

Lemma 5.2.1. *Let $A \in \mathbb{R}^{n\times n}$. Consider matrix norm induced by Euclidian vector norm $\|A\| = \max \frac{\|Ax\|}{\|x\|}$ where $x \neq 0$. Then, $\|A\|$ is largest eigenvalue of the matrix $A^\top A$.*

[13]For every $u_0 \in \mathbb{R}^n$ we can always find a unique solution $u(t) = X(t, x_0)u_0$ of the linear initial value problem $\dot{u} = Df(x)u$ with $u(0) = u_0$.

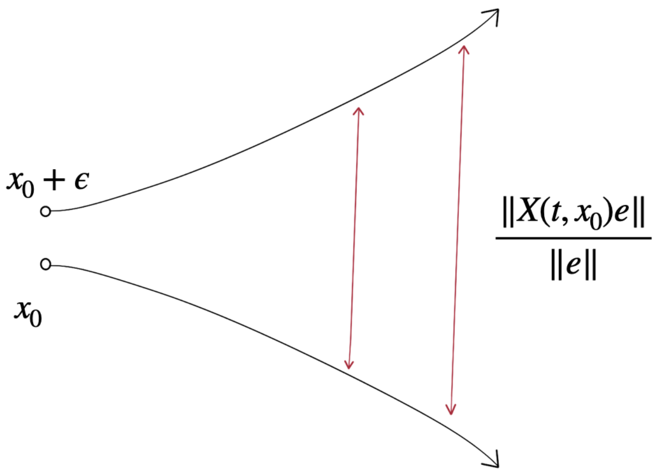

Figure 5.1 The expansion rate for infinitesimally close orbits to $x(t)$.

Proof. For proof, see, Meyer (2000, p. 281). □

Then, by Definition 5.2.1 and Lemma 5.2.1 if $\alpha(t)$ is the largest eigenvalue of $X^\top X$ then we have

$$\lambda_{\max} = \limsup_{t\to\infty} \frac{1}{t} \log \alpha(t). \tag{5.2.3}$$

Example 5.2.1. Consider the linear system

$$\dot{x} = ax, \quad x \in \mathbb{R},$$

where $a \in \mathbb{R}$. We can evaluate the above equation at $x = 0$, $x > 0$ and $x < 0$. The fundamental solution matrix is given by

$$X(t) = e^{at}.$$

For each case we have one Lyapunov exponent and we can see it is the value of a. Thus, if $a > 0$ trajectories of separate exponentially when $t \to \infty$.

6. SYNCHRONIZATION

In this chapter, we are going to consider the two or three coupled non-linear systems. First, we briefly examine the notion of synchronization and discuss the synchronization scenarios between two coupled non-linear systems. In the following, we introduce the generalized synchronization (See, Definition 6.2.1) and give an example of how to determine generalized synchronization (see, Theorem 6.2.1). Finally, we will discuss the basic results for the synchronization of chaotic systems.

6.1 Complete Synchronization

Let us give the definition of complete and generalized synchronization. Take a system

$$\begin{aligned} \dot{x_1} &= f(x_1) + \alpha(x_2 - x_1) \\ \dot{x_2} &= f(x_2) + \alpha(x_1 - x_2) \end{aligned} \tag{6.1.1}$$

where $x_1, x_2 \in \mathbb{R}^n, \alpha \in \mathbb{R}$ and $f \in \mathcal{C}^2$ function on $\mathbb{R}^n$. Let us define $M := \{(x_1, x_2) \in \mathbb{R}^{2n} : x_1 = x_2\}$. We say system (6.1.1) has complete synchronization if there exists an open set N such that $M \subseteq N$ and for every initial value $(x_1(0), x_2(0)) \in N$ we have

$$\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0.$$

Remark 6.1.1. *Here the norm $\|.\|$ is the Euclidian norm on the space $\mathbb{R}^n$.*

6.2 Generalized Synchronization

Consider two non-linear systems

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= g(y, h(x)) \end{aligned} \tag{6.2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^{m+k} \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^k$ with $f, g, h \in \mathcal{C}^\infty$.

Definition 6.2.1 (Generalized synchronization (GS))**.** System (6.2.1) has generalized synchronization if the following two properties hold.

(i) There exists an unique function $F : \mathbb{R}^n \to \mathbb{R}^m$ with an invariant manifold $M = \{(x, y) : y = F(x)\}$ and a neighborhood $N \subseteq N(\tilde{x}_0) \times N(\tilde{y}_0) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ such that $M \subseteq N$.
Moreover,

(ii) For any $(x_0, y_0) \in N$[14] the corresponding flow $\Phi = (\phi_f, \phi_g)$[15] of system (6.2.1) converges to the manifold M when $t \to \infty$.

Remark 6.2.1. *If the manifold M is graph of an identity function then system has complete synchronization.*

Theorem 6.2.1. *System (6.2.1) have generalized synchronization if and only if for all $(x_0, y_0) \in N \subseteq N(\tilde{x}_0) \times N(\tilde{y}_0)$ and for every $y_1, y_2 \in N(\tilde{y}_0)$ we have*

$$\lim_{t\to\infty} \big\| y(t, x_0, y_1) - y(t, x_0, y_2) \big\| = 0. \tag{6.2.2}$$

Proof. Let $\phi_f^t : \mathbb{R}^n \to \mathbb{R}^n$ be the flow of system $\dot{x} = f(x)$ and $\phi_g^t : \mathbb{R}^{m+k} \to \mathbb{R}^m$ be the flow of system $\dot{y} = g(y, h(x))$. Consider flow $\Phi(\phi_f, \phi_f)$ of system (6.2.1). Our aim is construct a map $F : \mathbb{R}^n \to \mathbb{R}^m$. To do this, take a point $x_0 \in N(x)$ and determine the corresponding point $y_0 \in N(y)$ such that we have $y_0 = F(x_0)$. Consider point $(\phi_f^{-t}(x_0), y_0)$, after time t has evaluated we have $(x_0, \phi^t(y_0))$ and it is close to the set M. Define $\tilde{F} := \lim_{t\to\infty} \phi_g^t(\phi_f^{-t}(x_0), y_0)$. Then, by (6.2.2), we have $\Big\| \phi_g^t(\phi_f^{-t}(x_0), y_{10}) - \phi_g^t(\phi_f^{-t}(x_0), y_{20}) \Big\| \to 0$ for all $y_{10}, y_{20} \in N(y)$. Thus, $\tilde{F}(x_0, y_0)$ is independent from any arbitrary $y_0 \in N(y)$. We can define the synchronization manifold M by function $F(x_0) := \tilde{F}(x_0, y_0)$. ∎

[14] The point x_0 is the initial value at $t = 0$ for the system $\dot{x} = f(x)$ and also y_0 is the initial value of the system $\dot{y} = g(y, h(x))$ at $t = 0$.

[15] The function $\phi_f : \mathbb{R}^n \to \mathbb{R}^n$ is the flow of system $\dot{x} = f(x)$ and likewise $\phi_g : \mathbb{R}^{m+k} \to \mathbb{R}^m$ is the flow of system $\dot{y} = g(y, h(x))$.

Now, we are going to consider an example to investigate the generalized synchronization phenomena.

Consider two non-linear coupled system

$$
\begin{aligned}
\dot{x} &= f(x) \\
\dot{y} &= g(y) + \alpha(x - y)
\end{aligned} \tag{6.2.3}
$$

where $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{C}^{\infty}(\mathbb{R}^n)$. To detect the generalized synchronization, we will use the auxiliary system approach[16] (see Figure 6.1), this method mainly induces system the generalized synchronization to complete synchronization phenomena. Let us consider the auxiliary system

$$
\dot{\tilde{y}} = g(\tilde{y}) + \alpha(x - \tilde{y}) \tag{6.2.4}
$$

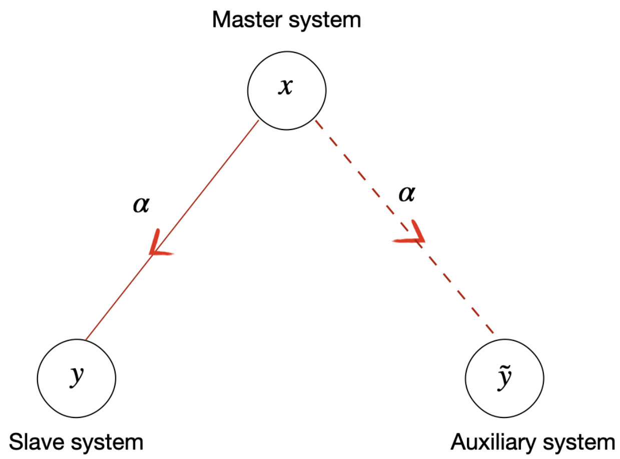

Figure 6.1 Illustration of the auxiliary system approach for the generalized synchronization: state x drives the state y and $\tilde{y}$ and also state $\tilde{y}$ is exact copy of driven state y. If we see the complete synchronization between y and $\tilde{y}$ then we say x and y have generalized synchronization.

that is the exact copy of the system $\dot{y} = g(y) + \alpha(x - y)$. Consider a new variable $z := y - \tilde{y}$. Then, subtracting of y and $\tilde{y}$ we get

$$
\begin{aligned}
\dot{z} &= \dot{y} - \dot{\tilde{y}} \\
\dot{z} &= g(y) - g(\tilde{y}) - \alpha z.
\end{aligned} \tag{6.2.5}
$$

[16]Further details can be found: Abarbanel, Rulkov and Sushchik (1996, pp.4528–4535).

Now, our aim is to find sufficient conditions for $\alpha \in \mathbb{R}$ so that we can have

$$\lim_{t\to\infty} z(t) = 0.$$

Let us linearize equation (6.2.5) at the point $z = 0$. By the Taylor expansion at $y = \tilde{y}$ we have

$$g(\tilde{y}) = g(y) - Dg(y)(y - \tilde{y}) + R(y) \tag{6.2.6}$$

where $R(y(t))$ stands for high order terms and it satisfy $\lim_{t\to\infty} R(y) \to 0$. Now, we use equation (6.2.6) to rewrite equation (6.2.5)

$$\dot{z} = [Dg(y) - \alpha]z. \tag{6.2.7}$$

Let us define new variable

$$w = e^{\alpha t} z(t). \tag{6.2.8}$$

Consider the derivative of w

$$\begin{aligned}
\dot{w} &= \alpha e^{\alpha t} z + e^{\alpha t}\dot{z} \\
\dot{w} &= \alpha w + [Dg(y(t)) - \alpha]e^{\alpha t} z \qquad (6.2.9)\\
\dot{w} &= Dg(y(t))w.
\end{aligned}$$

The last equation is variational equation along the solution $y(t)$. Suppose that $X(t, y_0)$ be the fundamental matrix[17] solution of (6.2.9) then, we have $w(t) = X(t, y_0)w_0$ for any arbitrary initial condition $w_0 \in \mathbb{R}^n$. Suppose that Λ be the maximal Lyapunov exponent for $y(t)$. Then, the Lyapunov exponent of $y(t)$

$$\Lambda = \limsup_{t\to\infty} \frac{1}{t} \ln \frac{\|X(t, y_0)w_0\|}{\|w_0\|}.$$

By definition of limsup, there exist a $\tau > 0$ such that we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{\|X(t, y_0)w_0\|}{\|w_0\|} := \lim_{t\to\infty} \left(\sup_{\tau\geq t} \left\{ \frac{1}{t} \ln \frac{\|X(t, y_0)w_0\|}{\|w_0\|} \right\} \right).$$

[17]Consider a smooth system $\dot{x} = f(x)$ with $x(0) = x_0$ where $x \in \mathbb{R}^n$. Let $x(t)$ be the solution of this system. By a fundamental solution matrix we mean that a function of a matrix such that for any initial value $w_0 \in \mathbb{R}^n$ we have unique solution $w(t) = X(t, x_0)w_0$ of the initial value problem $\dot{w} = Df(x(t))w$ and $w(0) = 0$.

Then, we have

$$\sup_{\tau \geq t} \frac{1}{t} \ln \frac{\|X(t, y_0)w_0\|}{\|w_0\|} < \Lambda + \epsilon$$

where $\epsilon > 0$. Thus,

$$\ln \frac{\|X(t, y_0)w_0\|}{\|w_0\|} < (\Lambda + \epsilon)t, \ \forall t > \tau$$

$$\|X(t, y_0)w_0\| < e^{(\Lambda+\epsilon)t}\|w_0\| , \ \forall \tau t > .$$

By above inequality, there exists a constant $C := \|w_0\|$ such that

$$\|w(t)\| \leq Ce^{(\Lambda+\epsilon)t}, \ \forall t > \tau. \tag{6.2.10}$$

From equation (6.2.8) and (6.2.10) we obtain that

$$\|z(t)\| \leq Ce^{(\Lambda+\epsilon-\alpha)t}, \ \forall t > \tau. \tag{6.2.11}$$

Since $\epsilon > 0$ is arbitrary we can take the critical α as the following

$$\alpha_c = \Lambda(y(t)).$$

Thus, we observe synchronization when we have $\alpha \geq \alpha_c$.

Example 6.2.1. Let us consider two identical Rössler system with parameters $a = 0.2$, $b = 0.2$ and $c = 5.7$. They are driven by Lorenz system with parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ via (x, y, z)-components where $\alpha \in \mathbb{R}$ is coupling constant. We used the auxiliary approach for determine the generalized synchronization. The numerical results showed that the critical coupling for this system is $\alpha_c = 0.07$.

For a systematic analysis, we consider the distance E between two Rössler systems for varying coupling strengths. Consider x_1 and x_2 as the representation of states of different Rössler systems. Define

$$E = \frac{1}{T}\sqrt{\sum_{t=0}^{T}\left|x_1(t) - x_2(t)\right|^2} \tag{6.2.12}$$

where $T \in \mathbb{N}$. We will use for the synchronization error.

For given $\alpha = 0.05$ (see, Figure 6.2) complete synchronization is not happened between the Rössler and auxiliary Rössler therefore there is no generalized synchronization between the Lorenz and the Rössler systems. If a coupling constant larger than the critical one $\alpha = 0.07$, i.e., $\alpha = 0.08$ (see, Figure 6.3) then, the Rössler system and the copy of the Rössler system have complete synchronization and we can observe the generalized synchronization between master and slave systems.

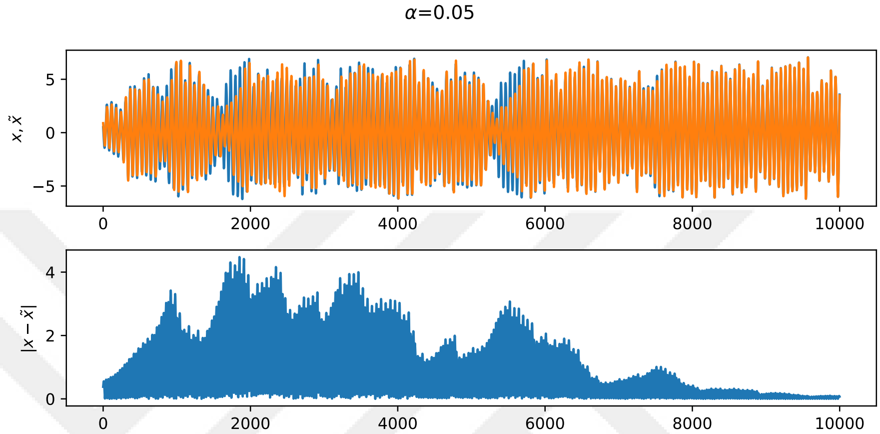

Figure 6.2 Trajectories of Rössler and copy of Rössler system. If given α value smaller than the critical coupling constant $\alpha_c = 0.07$ then, there is no synchronization.

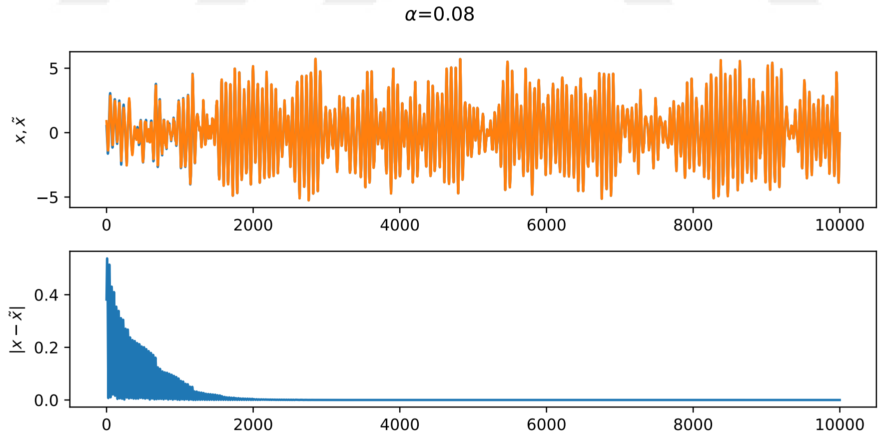

Figure 6.3 Trajectories Rössler and copy of Rössler system. If given α value larger than the critical coupling constant $\alpha_c = 0.07$ then, synchronization is obtained.

Figure 6.4 shows the synchronization diagram of E against the coupling strength α. We can conclude that, after the critically $\alpha_c = 0.07$ we have always $E \approx 10^{-17}$. Since we have same E value for every $\alpha \geq 0.07$ it corresponds to the zero value, i.e.,

$10^{-17} \approx 0$, for such system scenario.

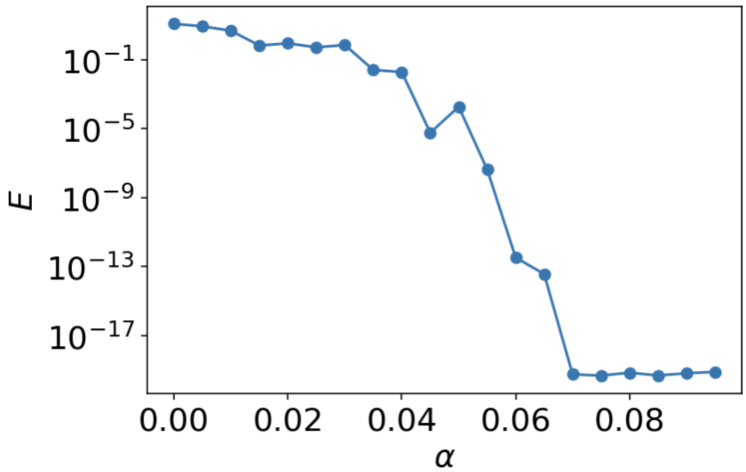

Figure 6.4 Error calculation between Rössler and copy of Rössler. The criticality for two coupled system in Example 6.2.1 is approximately $\alpha_c \approx 0.07$.

6.3 Generalized Synchronization on Graphs

In the simplest case for generalized synchronization application on complex networks, after two coupled systems, is three coupled systems. We will examine this scenario by using auxiliary system approach. Consider three non-linear coupled system

$$\begin{aligned} \dot{x} &= f(x) + \alpha(z - x) \\ \dot{y} &= g(y) + \beta(x - y) \\ \dot{z} &= h(z) + \gamma(y - z) \end{aligned} \tag{6.3.1}$$

where $f, g, h : \mathbb{R}^n \to \mathbb{R}^n$ are $\mathcal{C}^\infty$ and $\alpha, \beta, \gamma \in \mathbb{R}$. To investigate synchronization we use the auxiliary approach one by one. Figure 6.5 shows the diagram of a such system.

Example 6.3.1. Consider two Rössler systems (see, system (5.1.4)) with respect to parameters $a = 0.2,\ b = 0.2,\ c = 5.2$ and $a = 0.2,\ b = 0.2,\ c = 20$ driven by Lorenz system (see, system (4.3.1)) with parameters $\sigma = 28,\ \beta = 10\,\rho = 8/3$ via $-(x, y, z)$ components with $\alpha \in \mathbb{R}$ coupling constant, i.e., $\alpha = \beta = \gamma$. Illustration 6.5 is exactly the same coupling direction scenario for this example, i.e., Lorenz system

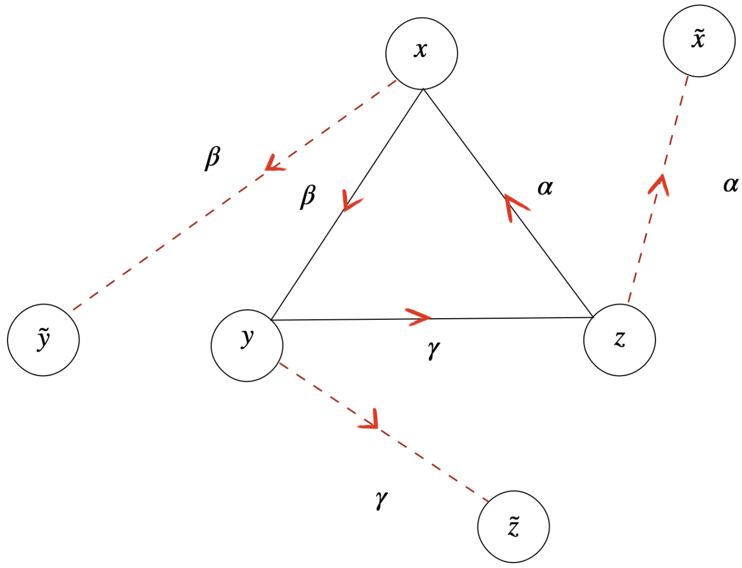

Figure 6.5 Generalized synchronization: representation of auxiliary approach of the three coupled non-linear system.

drives Rössler system ($c = 5.7$), Rössler system ($c = 5.7$) drives a another Rössler system ($c = 20$) and finally Rössler system ($c = 20$) drives Lorenz system.

We used the auxiliary system approach the detect generalized synchronization. The numerical calculations showed that generalized synchronization exists at the critical coupling $\alpha_c \approx 1.19$. If we zoom in values $\alpha \in (0, 0.09)$ then we can investigate that critical constant for the Rössler systems ($c = 5.7$) is $\alpha_c = 0.018$. When we give smaller values such as $\alpha = 0.016$, there is no complete synchronization between the Rössler systems ($c = 5.7$). Likewise, we can observe that the criticality for second Rössler systems ($c = 20$) is $\alpha_c = 0.079$. Below the criticality we do not obtain synchronization (see, Figure 6.7).

It follows that the Lorenz systems are the last system that we can examine a state of complete synchronization. It seems the transient time t for the Lorenz system is large for such a system scenario. After a couple of realizations for this numerics, the criticality may reduce. However, this is beyond the scope of this thesis. We can conclude that, for Runge–Kutta integration scheme for time value 100000 with 0.1 time step we have the coupling constant $\alpha_c = 1.19$, if given α value is larger or equal than criticality we can observe the state of complete synchronization for each

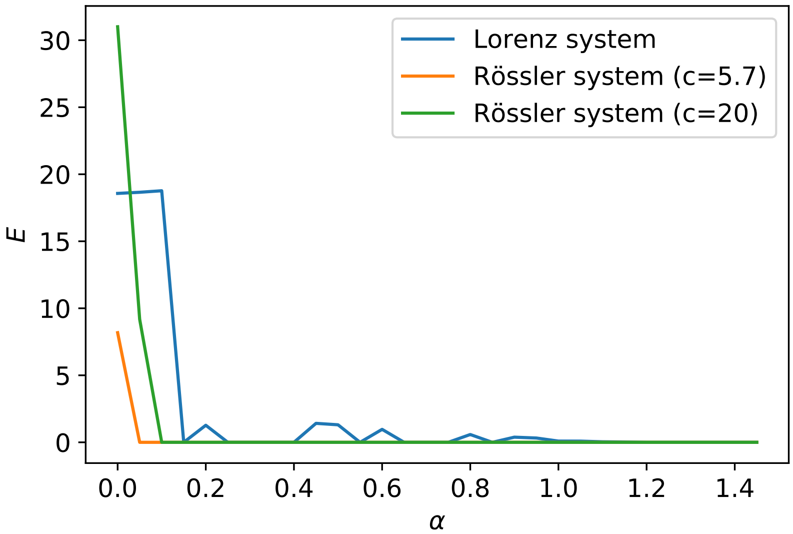

Figure 6.6 Error Calculation of Three Coupled Systems for $0 \leq \alpha \leq 1.14$.

sub-system. After Lorenz gets in complete synchrony at $\alpha \approx 1.19$, the whole system (Lorenz, Rössler ($c = 5.7$), and Rössler ($c = 20$)) has generalized synchronization (see, Figure 6.6).

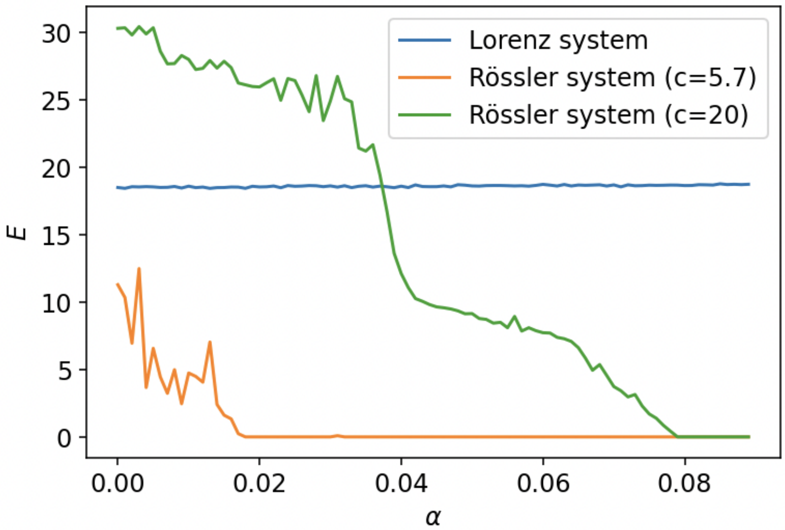

Figure 6.7 Error Calculation of Three Coupled Systems for $0 \leq \alpha \leq 0.09$.

7. CONCLUSION

Overall, generalized synchronization is an important topic in science, and it has many applications in most of the engineering, medical, and biological fields. As science and engineering disciplines grow, the use of synchronization grows as new mathematical problems are encountered and new mathematical skills are required. In this respect, synchronization will play a significant role in applied mathematics. The broad utility of synchronization to mathematics reflects the deep connection between the dynamical systems and science. The synchronization phenomenon has been studied with two coupled systems since 1990. Concerning this, we took two different non-linear coupled systems where $\mathcal{C}^{\infty}(\mathbb{R}^n)$ and calculated the critical constant for the generalized synchronization. It turns out that the critical coupling value is exactly the Lyapunov exponent of the driven system. We saw that the analytical calculations coincide with the numerical calculations using the auxiliary system approach. Later, we focus on the next step, three coupled systems, which is a new perspective for synchronization cases till now. More precisely, we took three different systems and coupled them so that; each system is the driver state and the driven state both for our specific example. We saw there exists a generalized synchronization between these three systems. The idea of how we investigated the generalized synchronization was the following. First, we took the whole system as a partition of three different systems (see, Figure 6.5). For every case, we used the auxiliary approach one by one. Thus, by this approach, if we have complete synchronization between the driven system and a copy of the driven (auxiliary system), we say the main system has generalized synchronization. If we proceed by this order for every partition, it follows that the system gets synchrony in a generalized sense. After accomplishing these steps, we have seen numerical results using the auxiliary system approach and illustrate the synchronization time series. Synchronization is strongly related to many other concepts. It is also crucial to many mathematical areas, including computational mathematics, chaos theory, dynamical systems, and many other disciplines. The auxiliary systems approach is a technique used mainly in detecting generalized synchronization. However, this is not the only way of investigating synchronization. As we mentioned before, the definition of generalized synchronization is basically a geometrical problem, finding the functional relationship between two or more (different) systems. At this point, there are many missing theoretical aspects in the notion of synchronization. However, this topic has had significant attention because of its broad application in many scientific fields and still continues growing since 1990, and there might be more theoretical mathematics studies in the future for synchronization phenomena.

8. PRELIMINARIES

This section will give basic definitions and theorems of Analysis and Geometry that are required and used in this thesis.

8.1 Preliminaries From Analysis

Definition 8.1.1 (Open balls)**.** Consider a metric space (X, d). Let $\epsilon > 0$ and $x_0 \in X$. We define

$$B_\epsilon(x_0) = \{x \in X : d(x, x_0) < \epsilon\}.$$

The above set is called an open ball of radius ϵ centered at the point $x_0 \in X$.

Definition 8.1.2 (Open sets)**.** We say a set $U \subseteq X$ is an open set if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

Definition 8.1.3 (Neighborhood of a point)**.** The set $N_\epsilon(x)$ is called a neighborhood (or ϵ-neighborhood) of x if there exists an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subseteq N_\epsilon(x)$. Sometimes we simply denote the neighborhood of x as $N(x)$.

Definition 8.1.4. A function is continuously differentiable or $\mathcal{C}^1$ if all of its partial derivatives exist and are continuous. Inductively, a function is called $\mathcal{C}^k$, if its partial derivatives are $\mathcal{C}^{k-1}$.

Definition 8.1.5 (Lipschitz functions)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We say a function $f : X \to Y$ is Lipschitz if there exist $K \geq 0$ such that

$$d_Y(f(u_1), f(u_2)) \leq K d_X(u_1, u_2) \tag{8.1.1}$$

holds for all $u_1, u_2 \in X$.

Definition 8.1.6 (Locally Lipschitz)**.** We say f is locally Lipschitz if for every $x \in X$, there exists a neighbourhood $N(x)$ such that $f : N(x) \to Y$ is a Lipschitz function.

Remark 8.1.1. *Here the neighborhood $N(x)$ of x is defined on the metric that induced from the metric space (X, d_X).*

Definition 8.1.7 (Vector space(Linear space))**.** The set X over the field $\mathbb{R}$ is a vector space if X be closed set under the operators addition and scalar multiplication, i.e., for every $x_1, x_2 \in X$ and $c_1, c_2 \in \mathbb{R}$ we have $c_1x_1 + c_2x_2 \in X$.

Remark 8.1.2. *In our case we usually take the set $X = \mathbb{R}^n$ where $n \in \{1, 2, \cdots, N\}$.*

Definition 8.1.8 (Banach space)**.** We say the set $X \subseteq \mathbb{R}^n$ is a Banach space if it is a complete vector space with a norm equipment.

Theorem 8.1.1 (Banach fixed point theorem)**.** *Suppose that (X, d) be complete metric space. Consider $f : (X, d) \to (X, d)$ with $\lambda \in (0, 1)$ such that*

$$d(f(x), f(y)) \leq \lambda d(x, y) \text{ for all } x, y \in X.$$

Then, f has a fixed point $p \in X$, i.e., $f(p) = p$ and for every $x_0 \in X$ the sequence $x_{n+1} := f(x_n)$ converges to the fixed point p.

Proof. See, e.g., Van Strien (2018, p. 21). □

Definition 8.1.9 (Absolute convergence)**.** Suppose that $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. The series is absolutely convergent if and only if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Definition 8.1.10 (Uniform convergence)**.** — Let (X, d_X) be a metric space. Let $X \subseteq \mathbb{R}$ and suppose that $f_n : X \to \mathbb{R}$ for $n = 1, 2, \cdots$. Then, we say f_n converges uniformly to $f : X \to \mathbb{R}$ if for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$ and for all $x \in X$ we have $d_X(f_n(x), f(x)) < \epsilon$.

Proposition 8.1.2 (Weierstrass M-test)**.** *Let $X \subseteq \mathbb{R}$ and (X, d_X) be a metric space. Suppose that the sequence of functions $(f_n)_{n=1}^{\infty}$ be bounded and continuous functions on the space X such that $\sum_{n=1}^{\infty} \|f_n\|_{\infty}$ is convergent. Then the series $\sum_{n=1}^{\infty}(f_n)$ converges uniformly to some function f on X, and the function f is also continuous.*

Proof. See, Rudin (2018, p. 148). □

8.2 Preliminaries From Geometry

Definition 8.2.1 (Manifolds)**.** Consider a topological space (X, τ_X). The set X is a manifold if for every $x \in X$ there exists an open neighborhood U_x that contains x such that $f : U_x \to \mathbb{R}^n$ is a homeomorphism for some natural number n.

Definition 8.2.2 (Invariance condition for manifolds)**.** Consider $X \subseteq \mathbb{R}^n$ and take the flow ϕ of the system $\dot{x} = f(x)$. The set X is called invariant with respect to the flow ϕ if for every $x \in X$, the flow $\phi(t, x)$ in the set X, for all $t \in \mathbb{R}$.

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