New 2-Edge-Balanced Graphs from Bipartite Graphs

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Abstract: Let G be a graph of order n satisfying that there exists $\lambda \in \mathbb{Z}^+$ for which every graph of order n and size t is contained in exactly λ distinct subgraphs of the complete graph K_n isomorphic to G. Then G is called t-edge-balanced and λ the *index* of G. In this article, new examples of 2-edge-balanced graphs are constructed from bipartite graphs and some further methods are introduced to obtain more from old. © 2015 Wiley Periodicals, Inc. J. Combin. Designs 24: 343–351, 2016

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1. INTRODUCTION

Consider a graph G of order n with the property that there exists $\lambda \in \mathbb{Z}^+$ for which every graph of order n and size t is contained in exactly λ distinct subgraphs of K_n isomorphic to G. We call such a graph G *t-edge-balanced* and λ its *index*. One can generalize this problem to obtain graphical t-designs. We refer the reader to other studies [3–7] for history and known results on t-edge-balanced graphs and graphical t-designs.

Although no example of infinite families of t-edge-balanced graphs for $t \geq 3$ is known, few examples of 2-edge-balanced graphs have been constructed. Alltop [1] shows that the graph of order $2k-3$ containing a cycle of length k and $k-3$ isolated vertices is 2-edge-balanced with index $\lambda = (2k-6)!/(k-3)!$ for $k \geq 3$. This gives rise to a graphical 2-$(\binom{2k-3}{2}, k, (2k-6)!/(k-3)!)$ design for all $k \geq 3$. Caliskan and Chee [2] show that some trees with certain properties are 2-edge-balanced and there are infinite families of integer-valued polynomials each of which results in infinite families of such trees.

In this article, new examples of infinite families of 2-edge-balanced graphs are constructed from bipartite graphs and some further methods are introduced to obtain more examples from old.

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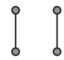
G	H_1	 H_2
$\lvert\mathrm{Aut}(G)\rvert$	$2(n-3)!$	$8(n-4)!$

FIGURE 1. Isomorphism classes of graphs of size two.

2. PRELIMINARIES

We assume the reader is familiar with the basic definitions related to combinatorial t-designs and note that graphs are all simple throughout the article.

Let K_n be the complete graph on n vertices with the set of vertices V and set of edges E, then the action of the symmetric group S_n on V naturally induces an action on E. A t-design $(X, \mathcal{B})$ with $X = E$ and $\mathcal{B}$ a collection of graphs of certain size k closed under the action of S_n is called *graphical*. In particular, $\mathcal{B}$ is a union of orbits of S_n on k-subsets of E. A graphical t-$(\binom{n}{2}, k, \lambda)$ design $(X, \mathcal{B})$ such that $\mathcal{B}$ contains a single orbit represented by G is equivalent to G being a graph of order n and size k that is t-edge-balanced with index λ.

Let G and H be graphs of order n. Alltop [1] discusses how to compute $\lambda_{H:G}$, the number of distinct subgraphs of K_n isomorphic to G each of which contains H, as follows.

Lemma 2.1 (Alltop [1]). *If G contains $n_{H:G}$ distinct subgraphs isomorphic to H, then*

$$\lambda_{H:G} = n_{H:G}\frac{|\mathrm{Aut}(H)|}{|\mathrm{Aut}(G)|}.$$

There are only two isomorphism classes of graphs of order n and size two. See Figure 1 (Isolated vertices are not shown in graph drawings in Figure 1). Thus, G is 2-edge-balanced if and only if $\lambda_{H_1:G} = \lambda_{H_2:G}$. By Lemma 2.1, this is equivalent to

$$\frac{n_{H_2:G}}{n_{H_1:G}} = \frac{|\mathrm{Aut}(H_1)|}{|\mathrm{Aut}(H_2)|} = \frac{n-3}{4}.$$

Theorem 2.1 (Alltop [1]). *A graph G of order n is 2-edge-balanced if and only if*

$$\frac{n_{H_2:G}}{n_{H_1:G}} = \frac{n-3}{4}.$$

For a given graph G of order n and size k, let d_G be the list of nonzero vertex degrees $d(v_1)d(v_2)\ldots d(v_n)$ written in nonincreasing order, then d_G is called the *degree sequence* of G. To simplify, we write the degree sequence d_G in the form $d_1^{s_1}d_2^{s_2}\ldots d_r^{s_r}$, where $r \leq n$ and $d_i \neq d_j$ whenever $i \neq j$. We also compute that

$$\sum_{i=1}^{r} s_i d_i = 2k \text{ and } n_{H_1:G} + n_{H_2:G} = \binom{k}{2}. \tag{1}$$

Let us assume that $d_1^{s_1}d_2^{s_2}\ldots d_r^{s_r}$ is the degree sequence of a 2-edge-balanced graph G of order n and size k, then $n_{H_1:G} = \sum s_i\binom{d_i}{2}$. Now consider a graph G' of order n with

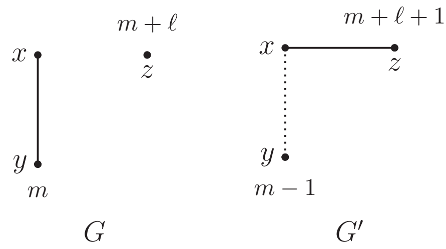


FIGURE 2. $G \overset{\Phi(xyz)}{\longrightarrow} G'$, $m \in \mathbb{Z}^+$, $\ell \in \mathbb{Z}$.

$d_{G'} = d_G$, then the size of G' is equal to k and $n_{H_1:G'} = n_{H_1:G}$. This implies that G' is also 2-edge-balanced by (1).

Theorem 2.2. *Let G be 2-edge-balanced. If a graph has the same order and degree sequence as G, then it is 2-edge-balanced.*

A *2-switch* is the replacement of a pair of edges xy and zw in a graph by the edges yz and wx, given that yz and wx did not appear in the original graph.

Theorem 2.3 (Berge [8]). *If G and H are two simple graphs with the same vertex set V, then they have the exact same degree sequences if and only if there is a sequence of 2-switches that transforms G into H.*

We use 2-switches below in Section 3.3 (by Theorems 2.2 and 2.3) to obtain new 2-edge-balanced graphs from old.

Let G be a graph with the set of vertices V and edges E. If $x, y, z \in V$, $xy \in E$ and $xz \notin E$, then we define the operation $\Phi(xyz)$ on G that removes the edge xy and adjoins xz. We call this new graph as G'. See Figure 2 (In figures throughout the article, we have the convention that only the edges related to the specified operation are shown in graph drawings and the dotted edges represent the edges that no longer exist in the current graph. Numbers next to vertices represents the vertex degrees). If $d(z) - d(y) = \ell$, where $\ell \in \mathbb{Z}$, then

$$n_{H_1:G'} - n_{H_1:G} = \ell + 1.$$

Let us further define a second operation Ψ on G. If $x, y, z, w \in V$, $xy \in E$, and $zw \notin E$, then define $\Psi(xyzw)$ on G that removes the edge xy and adjoins zw (see Figure 3). If $d(x) - 2 = d(y) = d(z) = d(w)$, then

$$n_{H_1:G'} = n_{H_1:G}.$$

3. NEW 2-EDGE-BALANCED GRAPHS

In what follows two constructions which yield new 2-edge-balanced graphs are discussed. Both use bipartite graphs and give rise to new infinite families of graphical 2-designs. We further introduce some methods including 2-switches to convert these graphs into other 2-edge-balanced graphs.

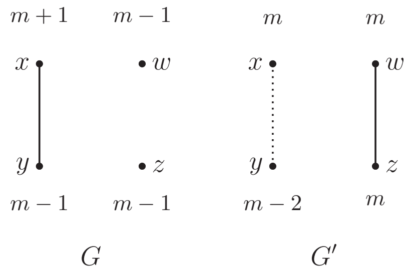


FIGURE 3. $G \overset{\Psi(xywz)}{\longrightarrow} G'$, $m \in \mathbb{Z}^+ \setminus \{1\}$.

3.1. Construction 1

Let $m > 1$ be an integer and G an m-regular graph on $2m$ vertices. We define G' as $G' = G \cup \{v\}$, where v is an isolated vertex. Note that G' is of order $2m + 1$. A simple computation shows that $n_{H_1:G'} = 2m\binom{m}{2} = m^2(m-1)$ and $n_{H_2:G'} = \binom{m^2}{2} - n_{H_1:G'} = \frac{1}{2}m^2(m-1)^2$. Since

$$\frac{n_{H_2}}{n_{H_1}} = \frac{m^2(m-1)^2}{2m^2(m-1)} = \frac{m-1}{2} = \frac{(2m+1)-3}{4}, \tag{2}$$

it follows by Theorem 2.1 that G' is 2-edge-balanced. We present this result in the following theorem.

Theorem 3.1. *Let $m > 1$, then an m-regular graph on $2m$ vertices together with an isolated vertex is 2-edge-balanced.*

Now consider the complete bipartite graph $G = K_{m,m}$. Since G is m-regular on $2m$ vertices with $|Aut(G)| = 2m!m!$, adjoining G an isolated vertex results in a 2-edge-balanced graph with index

$$\lambda = \frac{(2m-2)!}{(m-2)!(m-1)!}.$$

See G_1 in Figure 4 for an example with $m = 3$.

Corollary 3.1. *Let $G = K_{m,m}$ be the complete bipartite graph for $m > 1$, then $G' = G \cup \{v\}$, where v is an isolated vertex, is 2-edge-balanced.*

Thus, the existence of graphical 2-designs on $2m^2 + m$ points follows from Corollary 3.1.

Corollary 3.2. *There is a graphical $2-(2m^2+m, m^2, \frac{(2m-2)!}{(m-2)!(m-1)!})$ design for any $m > 1$.*

3.2. Construction 2

For $m > 1$, let $G = K_{m-1,m+1}$ be a complete bipartite graph, then we adjoin a single new vertex v and an edge e connecting v to one of the vertices of G with valency $m - 1$. We call this new graph of order $2m + 1$ as G'. See G_3 in Figure 4 for an example of G'

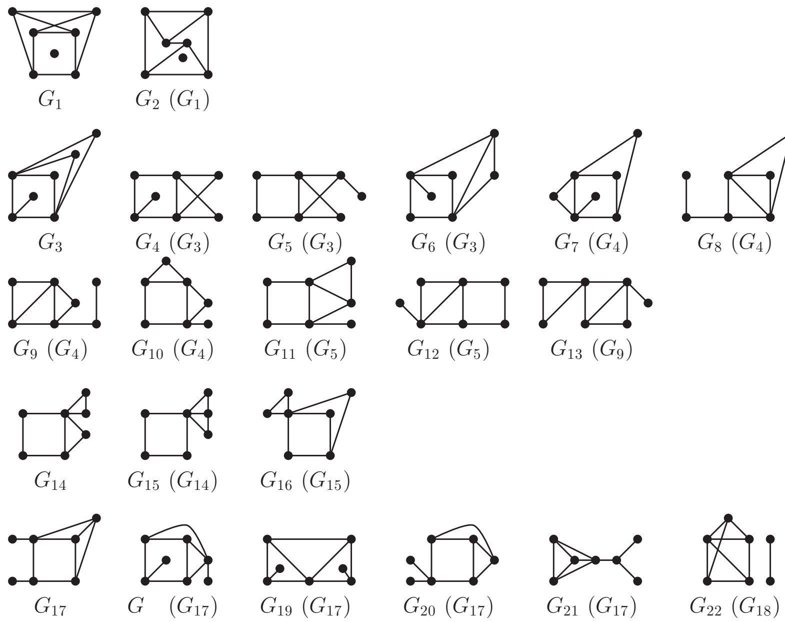


FIGURE 4. 2-Edge-balanced graphs on 7 vertices and 9 edges.

with $m = 3$. It follows for G' that $n_{H_1:G'} = (m-1)m^2$ and $n_{H_2:G'} = \frac{1}{2}m^2(m-1)^2$, then it results in that

$$\frac{n_{H_2:G'}}{n_{H_1:G'}} = \frac{m^2(m-1)^2}{2m^2(m-1)} = \frac{m-1}{2} = \frac{(2m+1)-3}{4}.$$

This implies that G' is 2-edge-balanced. Moreover, we compute that

$$|Aut(G')| = (m-1)!m!$$

and obtain its index as

$$\lambda = \frac{2m(2m-2)!}{(m-1)!(m-2)!}.$$

Theorem 3.2. *G' is 2-edge-balanced with index $\lambda = \frac{2m(2m-2)!}{(m-1)!(m-2)!}$.*

Corollary 3.3. *There is a graphical $2-(2m^2+m, m^2, \frac{2m(2m-2)!}{(m-1)!(m-2)!})$ design for any $m > 1$.*

3.3. 2-Switches and Further Methods

In this section, we discuss how to obtain more 2-edge-balanced graphs from old. Once we construct some 2-edge-balanced graphs on what some 2-switches are available, Theorem 2.3 points out the existence of other 2-edge-balanced graphs whenever the original graphs can be converted to nonisomorphic graphs by a sequence of 2-switches. Since the degree sequence of the original graph is fixed by a 2-switch, one can apply 2-switches

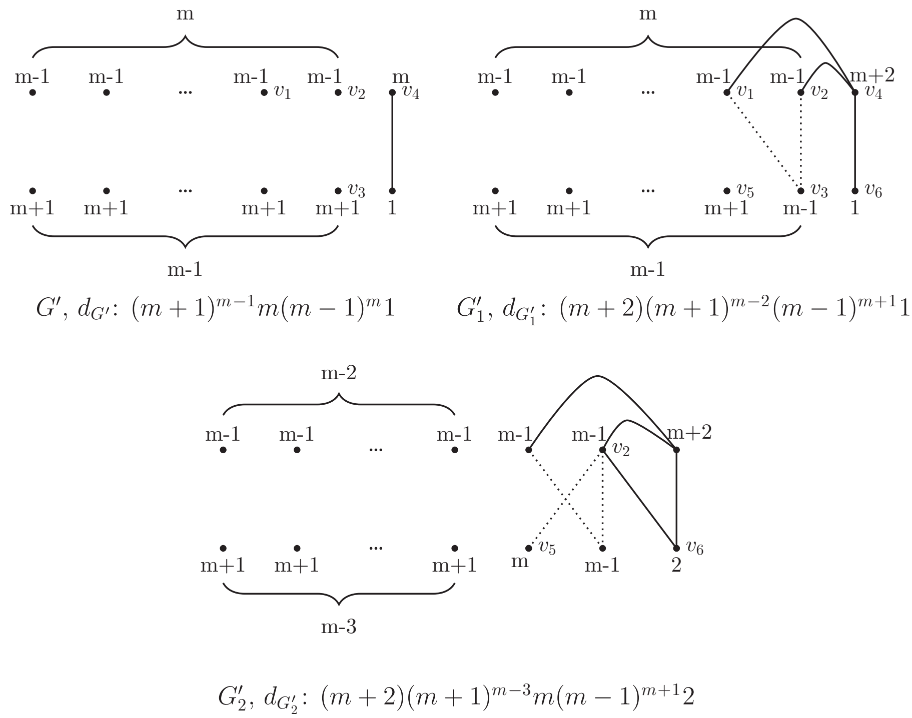


FIGURE 5. $n_{H_1:G'} - n_{H_1:G'_2} = m - 3$.

(where applicable) on 2-edge-balanced graphs that are constructed in Sections 3.1 and 3.2. See Figure 4 for all possible examples of graphs on 7 vertices and 9 edges obtained from G_1 and G_3 by a sequence of 2-switches. In Figure 4, the label "G_i (G_j)" means that the graph G_i is obtained from the graph G_j by a single 2-switch.

In the next, we introduce further methods to construct more examples from the graphs constructed above. Let us start with the graph G' (for $m > 2$) introduced in Section 3.2. See Figure 5 for the following discussion. The complete bipartite graph $K_{m-1,m+1}$ has the degree sequence $(m+1)^{m-1}(m-1)^{m+1}$, then the graph G' has the degree sequence $(m+1)^{m-1}m(m-1)^m 1$. We first pick two vertices—say v_1 and v_2—of degree $m-1$ and a vertex—say v_3—of degree $m+1$, then remove the edges v_1v_3 and v_2v_3 and adjoin new edges v_1v_4 and v_2v_4, where v_4 is the only vertex of degree m on G'. We call this new graph as G'_1 and its degree sequence is $(m+2)(m+1)^{m-2}(m-1)^{m+1}1$. Next step is to pick a vertex—say v_5—of degree $m+1$, then remove the edge v_2v_5 and adjoin a new edge v_2v_6, where v_6 is the only vertex of degree 1 in G'_1. The degree sequence of this new graph G'_2 is equal to $(m+2)(m+1)^{m-3}m(m-1)^{m+1}2$. We compute that

$$n_{H_1:G'} - n_{H_1:G'_2} = m - 3.$$

The size of the graph G' is preserved in G'_2, so we need to change the structure of G'_2 and increase the number graphs H_1 in G'_2 by $m-3$. If $m = 3$, we are done. We need to apply the operation Φ once or twice for $m \in \{4, 5, 6\}$ (see Figure 6). We apply the operations $\Phi(x_1z_1z_2)$, $\Phi(x_0y_0z_0)$ and $\Phi(z_0y_1z_1)$ (in this order) on the specified vertices

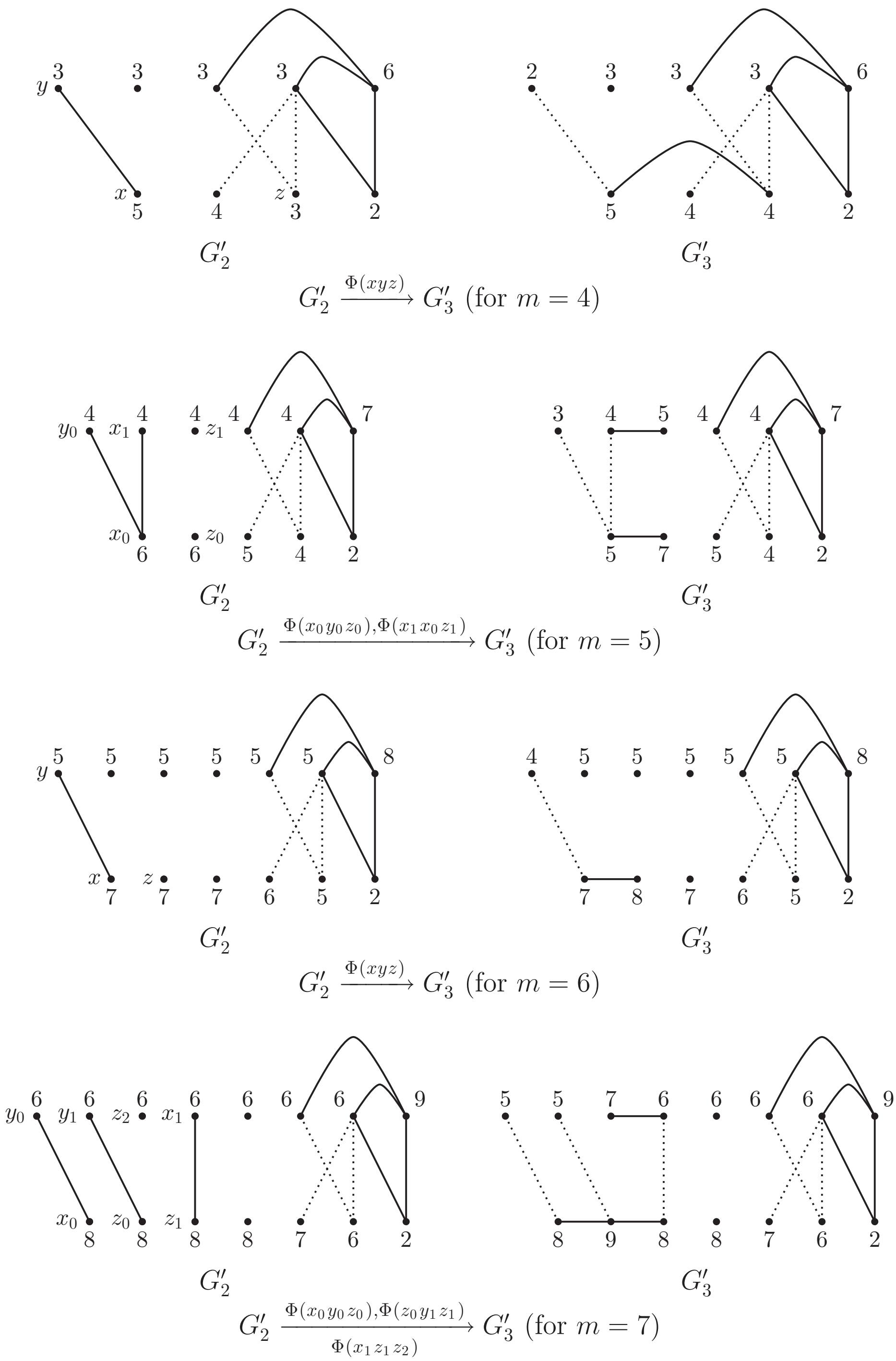


FIGURE 6. $G'_2 \to G'_3$, $m \in \{4, 5, 6, 7\}$.

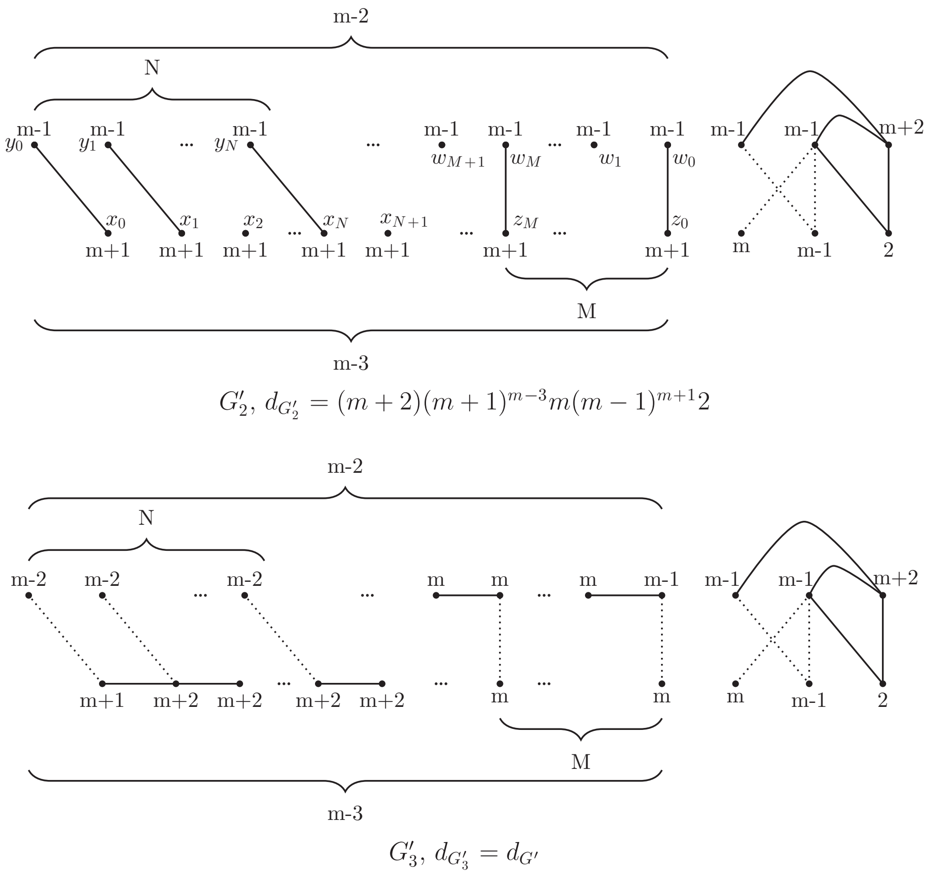


FIGURE 7. $G_2' \xrightarrow[\Phi(w_j z_j w_{j+1})]{\Phi(x_i y_i x_{i+1})} G_3'$, $0 \leq i \leq N$ and $0 \leq j \leq M$ (for $m \geq 8$).

for $m = 7$. Each of first two increases $n_{H_1:G_2'}$ by three and the latest decreases it by two. For $m \geq 8$, we apply $N + M$ many Φ operations, where

$$N = \lceil \frac{m-3}{3} \rceil \text{ and } M = 3(N+1) - m,$$

to increase the number of H_1 by $m - 3$ (see Figure 7). Each operation $\Phi(x_i y_i x_{i+1})$, $0 \leq i \leq N$, increases its number by three and each of $\Phi(w_j z_j w_{j+1})$, $0 \leq j \leq M$, decreases it by only one. Note that operations $\Phi(x_i y_i x_{i+1})$ and $\Phi(w_j z_j w_{j+1})$ do not use a common vertex or edge, since

$$N + M < m - 3,$$

for all $m \geq 8$.

We discuss above how to change the structure of the graph G_2' to obtain G_3' (for $m > 2$), which has the same order and degree sequence as the graph G'. Thus, we have the following result by Theorem 2.2.

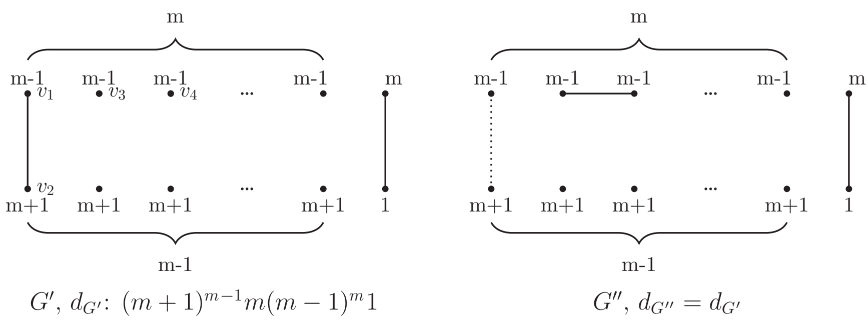


FIGURE 8. $G' \xrightarrow{\Psi(v_1v_2v_3v_4)} G''$.

Theorem 3.3. *G'_3 is 2-edge-balanced.*

To illustrate this method for $m = 3$, we let $G' = G_3$ (see Figure 4), then obtain that $G'_2 = G_{14}$.

For the sake of obtaining all the graphs shown in Figure 4, we now introduce another method to construct (in general) other examples of 2-edge-balanced graphs from the graph G' (of Section 3.2) for all $m > 2$. This method, specifically, is to construct the graph G_{17} from G_3 in Figure 4. It consists only of the operation Ψ as shown in Figure 8. This operation fixes the number of graphs H_1 without changing the size of the graph, so the graph G'' is 2-edge-balanced.

Theorem 3.4. *G'' is 2-edge-balanced.*

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