Invariant manifolds of homoclinic orbits and the dynamical consequences of a super-homoclinic: A case study in $\mathbb{R}^4$ with $\mathbb{Z}_2$-symmetry and integral of motion

Sajjad Bakrani [a,b,*], Jeroen S.W. Lamb [a], Dmitry Turaev [a]

[a] *Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom*
[b] *Faculty of Engineering and Natural Sciences, Kadir Has University, 34083 Istanbul, Turkey*

Abstract

We consider a $\mathbb{Z}_2$-equivariant flow in $\mathbb{R}^4$ with an integral of motion and a hyperbolic equilibrium with a transverse homoclinic orbit Γ. We provide criteria for the existence of stable and unstable invariant manifolds of Γ. We prove that if these manifolds intersect transversely, creating a so-called super-homoclinic, then in any neighborhood of this super-homoclinic there exist infinitely many multi-pulse homoclinic loops. An application to a system of coupled nonlinear Schrödinger equations is considered.
© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

Keywords: Homoclinic; Super-homoclinic; Invariant manifold; Coupled Schrödinger equations

Contents

* Corresponding author at: Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom.
  E-mail addresses: s.bakrani-balani@imperial.ac.uk (S. Bakrani), jeroen.lamb@imperial.ac.uk (J.S.W. Lamb), d.turaev@imperial.ac.uk (D. Turaev).

1. Introduction

1.1. Background

Consider a Hamiltonian system (or more generally, a system with a smooth first integral) defined for $x \in \mathbb{R}^{2n}$, $n \geq 2$ with an integral H, and a hyperbolic equilibrium O at the origin. An orbit $\Gamma = \{x(t) : t \in \mathbb{R}\}$ of this system is said to be '*homoclinic to O*' or '*a homoclinic loop*' if it belongs to both stable and unstable invariant manifolds of O, or equivalently, $x(t) \to O$ as $t \to \pm\infty$. Existence of homoclinic orbits for systems with a smooth first integral is known to be a robust phenomenon. This is due to the fact that the n-dimensional stable and unstable invariant manifolds of O lie in the same $(2n-1)$-dimensional level $H = \text{constant}$, and they may intersect transversely in that level along the homoclinic orbits. A natural question which arises here is the possible dynamics near homoclinic orbits in the level $H(x) = H(O)$.

When $x(t)$ oscillates as it converges to O (this happens when the leading, i.e. the nearest to the imaginary axis, eigenvalues of the linear part of the system at O are complex), the dynamics

in $H^{-1}(O)$ is highly non-trivial [9,6,17,19,18,7,3]. On the other hand, when the leading eigenvalues are real and $\|x(t)\|$ decays to zero monotonically as $t \to \pm\infty$, the generic dynamics in $H^{-1}(O)$ are very simple. Thus, the only orbits staying in a small neighborhood of a finite bunch of generic homoclinic loops to a saddle with real leading eigenvalues are only the homoclinic loops themselves and the point O [26,29].

In this case, to have interesting behavior, we need degeneracies or symmetries. It was shown in [24] (based on an earlier work [12]) that symmetries can lead to the emergence of the so-called super-homoclinic orbits which, in turn, serve as limits of infinite series of multi-pulse homoclinic loops. Namely, as shown in [24,28], in certain situations, the homoclinic loops or bunches of homoclinic loops can have stable or unstable invariant manifolds; the super-homoclinics correspond to the intersection of these manifolds.

In the non-conservative setting, super-homoclinic orbits and the non-trivial dynamics associated with them were discovered and studied by Homburg [14]. Eleonsky et al. [12] spotted super-homoclinic orbits in the numerical investigation of an electromagnetic field in a nonlinear medium. Barrientos et al. [5] found super-homoclinics near a homoclinic loop to a saddle-focus in the context of reversible systems, similar to the structure described for the Hamiltonian case by Belyakov and Shilnikov in [6]. Chawanya and Ashwin [8] built an example of a heteroclinic network that possesses a super-homoclinic in the sense of an orbit which connects sub-networks. In general, super-homoclinic orbits may potentially appear in heteroclinic networks, especially if the network undergoes a chaotic behavior, see e.g. [20].

In this paper, we consider the simplest case of $\mathbb{Z}_2$-symmetry which results in the emergence of stable and unstable invariant manifolds for homoclinic loops, enabling the creation of super-homoclinic orbits, and describe the multi-pulse homoclinics associated to them.

1.2. Problem setting and results

Consider a $\mathcal{C}^\infty$-smooth 4-dimensional system of differential equations

$$\dot{x} = X(x), \quad x \in \mathbb{R}^4, \tag{1.1}$$

with a $\mathcal{C}^\infty$-smooth first integral $H : \mathbb{R}^4 \to \mathbb{R}$, i.e.

$$\nabla H(x) \cdot X(x) \equiv 0. \tag{1.2}$$

Assumption 1. X has a hyperbolic equilibrium state O at the origin.

By (1.2), we have $H'(0)X'(0) \equiv 0$. Since $X'(0)$ is nonsingular by Assumption 1, the linear part of H at O vanishes.

Assumption 2. The quadratic part of H at O is a nondegenerate quadratic form.

It is easy to see (see e.g. [2]) that when Assumptions 1 and 2 are satisfied, system (1.1) near O can be brought to the following form by a linear transformation:

$$\dot{u} = -Au + o(|u|, |v|), \quad \dot{v} = A^T v + o(|u|, |v|), \tag{1.3}$$
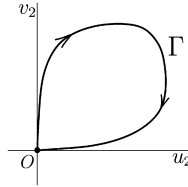
Fig. 1. The transverse homoclinic loop Γ in the invariant plane $\{u_1 = v_1 = 0\}$.

where $u = (u_1, u_2) \in \mathbb{R}^2$, $v = (v_1, v_2) \in \mathbb{R}^2$ and A is a matrix whose eigenvalues have positive real parts. Moreover, the first integral takes the form:

$$H = \langle v, Au \rangle + o\left(u^2 + v^2\right), \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$.

Assumption 3. System (1.3) is invariant with respect to the symmetry

$$(u_1, v_1) \leftrightarrow (-u_1, -v_1). \tag{1.5}$$

Assumption 3 implies that the plane $\{u_1 = v_1 = 0\}$ is invariant with respect to the flow of system (1.3).

Note that we can always assume that H is invariant with respect to symmetry (1.5), i.e.

$$H(-u_1, u_2, -v_1, v_2) = H(u_1, u_2, v_1, v_2). \tag{1.6}$$

Otherwise, $\widetilde{H}(u_1, u_2, v_1, v_2) := \frac{1}{2}[H(u_1, u_2, v_1, v_2) + H(-u_1, u_2, -v_1, v_2)]$ can be taken as the first integral.

The equilibrium state O is a saddle with 2-dimensional stable and unstable invariant manifolds $W^s(O)$ and $W^u(O)$ which are tangent at O to the u-plane and v-plane respectively. Both the invariant manifolds lie in the 3-dimensional level $\{H = 0\}$ and may intersect transversely in that level, producing a number of homoclinic loops. We consider the following specific case:

Assumption 4. In the invariant plane $\{u_1 = v_1 = 0\}$, there exists a homoclinic loop Γ of the transverse intersection of $W^s(O)$ and $W^u(O)$ (see Fig. 1).

Since the action of this symmetry commutes with the linear part of system (1.3), the matrix A is diagonal and takes the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

for some positive real numbers λ_1 and λ_2. Without loss of generality, let $\lambda_1 \leq \lambda_2$.

Assumption 5. $\lambda_2 \neq 2\lambda_1$.

This is not a technical assumption. Indeed, we will see that the cases $\lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$ are dynamically different.

Let $\mathcal{U}$ be a sufficiently small neighborhood of $\Gamma \cup \{O\}$ in the zero level-set $\{H = 0\}$. The main issue which is addressed in this paper is giving a complete description of dynamics in $\mathcal{U}$.

Definition 1. Let $\mathcal{A} = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, where Γ_i are homoclinic to an equilibrium O, and $m \geq 1$. Consider a sufficiently small open neighborhood $\mathcal{U}$ of $\mathcal{A} \cup \{O\}$ in the energy level of O. The local stable (resp. unstable) set of $\mathcal{A}$, denoted by $W^s_{\text{loc}}(\mathcal{A}, \mathcal{U})$ (resp. $W^u_{\text{loc}}(\mathcal{A}, \mathcal{U})$), is the union of $\mathcal{A}$ itself and the set of the points in $\mathcal{U}$ whose forward (resp. backward) orbits lie in $\mathcal{U}$ and their ω-limit sets (resp. α-limit sets) coincide with $\mathcal{A} \cup \{O\}$. We may use the notations $W^s_{\text{loc}}(\mathcal{A})$ and $W^u_{\text{loc}}(\mathcal{A})$ for the stable and unstable sets of $\mathcal{A}$ when the neighborhood $\mathcal{U}$ is clear from the context.

By this definition, the local stable and unstable sets of Γ always contain Γ. Note that these sets lie in the zero-level set $\{H = 0\}$. Denote by $W^s_{\mathcal{U}}(O)$ (resp. $W^u_{\mathcal{U}}(O)$) the set of the points in $W^s_{\text{glo}}(O)$ (resp. $W^u_{\text{glo}}(O)$) whose forward (resp. backward) orbits lie entirely in $\mathcal{U}$. Obviously,

$$W^s_{\mathcal{U}}(O) \cap W^s_{\text{loc}}(\Gamma) = W^u_{\mathcal{U}}(O) \cap W^u_{\text{loc}}(\Gamma) = \Gamma.$$

1.2.1. Dynamics near a single homoclinic orbit

Our first result is the following:

Theorem A1. *Under Assumptions 1-5, the forward (backward) orbit of a point in $\mathcal{U}$ lies entirely in $\mathcal{U}$ if and only if it belongs to $W^s_{\mathcal{U}}(O) \cup W^s_{loc}(\Gamma)$ (resp. $W^u_{\mathcal{U}}(O) \cup W^u_{loc}(\Gamma)$).*

By this theorem, to understand the dynamics near the homoclinic orbit Γ, we need to study the local stable and unstable sets of this orbit. This is done in the following two theorems.

Theorem A2. *If $\lambda_2 < 2\lambda_1$ and Assumptions 1-4 hold, then $W^s_{loc}(\Gamma) = W^u_{loc}(\Gamma) = \Gamma$. Thus, by Theorem A1, the forward and backward orbits of any point in $\mathcal{U}$ either leave $\mathcal{U}$ or converge to O.*

The next theorem describes the local stable and local unstable sets of Γ when $2\lambda_1 < \lambda_2$. The formulation uses a specific choice of coordinates near the equilibrium O. We introduce this coordinate system in Section 2 (see normal form (2.12)). For this choice of coordinates, system (1.3) keeps its form and its invariance with respect to symmetry (1.5). Moreover, the first integral takes the form

$$H(u_1, u_2, v_1, v_2) = \lambda_1 u_1 v_1 - \lambda_2 u_2 v_2 + o\left(u^2 + v^2\right), \tag{1.7}$$

and satisfies (1.6). The local stable and unstable, as well as the local strong stable and strong unstable, invariant manifolds of O are straightened (i.e. $W^s_{\text{loc}}(O) = \{v_1 = v_2 = 0\}$, $W^u_{\text{loc}}(O) = \{u_1 = u_2 = 0\}$, $W^{ss}_{\text{loc}}(O) = \{u_1 = v_1 = v_2 = 0\}$, $W^{uu}_{\text{loc}}(O) = \{u_1 = u_2 = v_1 = 0\}$), and the loop Γ leaves O along v_2-axis toward positive v_2 and enters O along u_2-axis toward positive u_2 (see Fig. 2).

Take a small $\delta > 0$ and consider two small 2-dimensional cross-sections to the loop Γ inside the level $\{H = 0\}$: $\Pi^s = \{u_2 = \delta\} \cap \{H = 0\}$ and $\Pi^u = \{v_2 = \delta\} \cap \{H = 0\}$ (see Fig. 2). On each
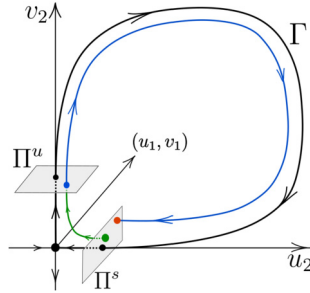
Fig. 2. This figure shows the positions of the cross-sections Π^s and Π^u to the homoclinic loop Γ. The green and blue curves correspond to the maps T^{loc} and T^{glo}, respectively. Namely, T^{loc} maps the green point on Π^s to the blue point on Π^u and then T^{glo} maps the blue point to the red point on Π^s. The red point is the image of the green point by the Poincaré map $T = T^{\text{glo}} \circ T^{\text{loc}}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

of the cross-sections Π^s and Π^u, the variables u_2 and v_2 are uniquely determined by (u_1, v_1) (see Lemma 2.12). This allows us to parametrize Π^s and Π^u by the (u_1, v_1)-coordinates.

Orbits which lie in $\mathcal{U}$ define a Poincaré map T that takes a subset of Π^s to Π^s. This map can be written as a composition of a local map T^{loc} from a subset of Π^s to Π^u which is defined by the flow inside the δ-neighborhood of O, and a global map T^{glo} from Π^u to Π^s which is defined by the flow near the global piece of Γ outside the δ-neighborhood of O (see Fig. 2). Since the flight time from Π^u to Π^s is bounded, the global map T^{glo} is a diffeomorphism. Consider the points $M^s = \Gamma \cap \Pi^s$ and $M^u = \Gamma \cap \Pi^u$ (note that both points correspond to $(0, 0)$ in $\Pi^{s,u}$). The Taylor expansion of T^{glo} at M^u has the form

$$T^{\text{glo}}(u_1, v_1) = (au_1 + bv_1 + o(u_1, v_1), \, cu_1 + dv_1 + o(u_1, v_1)), \tag{1.8}$$

for some $a, b, c, d \in \mathbb{R}$. Since the local unstable manifold of O corresponds to $\{u_1 = 0\}$, and the local stable manifold corresponds to $\{v_1 = 0\}$, the transversality assumption (see Assumption 4) is equivalent to $d \neq 0$.

Theorem A3. *Let $2\lambda_1 < \lambda_2$ and Assumptions 1-4 hold. Suppose that system (1.3) near the equilibrium O is brought to the form (2.12) and let b, c and d in (1.8) be non-zero.*

(i) *If $cd > 0$, then $W^s_{loc}(\Gamma) = \Gamma$. If $cd < 0$, then $W^s_{loc}(\Gamma)$ is a $\mathcal{C}^1$-smooth 2-dimensional invariant manifold which is tangent to $W^s_{glo}(O)$ at every point of Γ.*

(ii) *If $bd < 0$, then $W^u_{loc}(\Gamma) = \Gamma$. If $bd > 0$, then $W^u_{loc}(\Gamma)$ is a $\mathcal{C}^1$-smooth 2-dimensional invariant manifold which is tangent to $W^u_{glo}(O)$ at every point of Γ.*

To stress the smoothness of $W^s_{\text{loc}}(\Gamma)$ and $W^u_{\text{loc}}(\Gamma)$, we further call them local stable and unstable invariant manifolds.

We remark a parallel to the case of a general homoclinic loop, considered in detail by Homburg [14], see also [27,22,25], for which the existence of stable/unstable manifolds depends on the sign of a certain coefficient in the Poincaré map (the so-called separatrix value [23]).
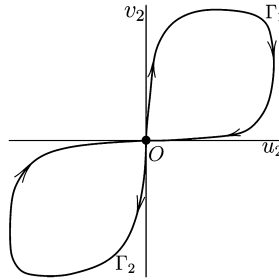
Fig. 3. A pair of transverse homoclinic loops ($\Gamma_1 \cup \Gamma_2$) in the invariant plane $\{u_1 = v_1 = 0\}$.

1.2.2. Dynamics near a homoclinic figure-eight

Next, we consider the existence of a pair of homoclinic loops in the invariant plane $\{u_1 = v_1 = 0\}$:

Assumption 6. There exist two homoclinic loops Γ_1 and Γ_2 of transverse intersection of $W^s(O)$ and $W^u(O)$ in the invariant plane $\{u_1 = v_1 = 0\}$ such that they leave and enter O along opposite directions (see Fig. 3).

Such scenario happens generically, when the level-set $\{H = 0\}$ is compact. Let $\mathcal{V}$ be a small neighborhood of $\Gamma_1 \cup \{O\} \cup \Gamma_2$ in the level-set $\{H = 0\}$ and denote by $W^s_{\mathcal{V}}(O)$ (resp. $W^u_{\mathcal{V}}(O)$) the set of the points in $W^s_{\text{glo}}(O)$ (resp. $W^u_{\text{glo}}(O)$) whose forward (resp. backward) orbits lie entirely in $\mathcal{V}$. Then

Theorem B1. *Under Assumptions 1-3, 5 and 6, the forward (resp. backward) orbit of a point in $\mathcal{V}$ lies entirely in $\mathcal{V}$ if and only if it belongs to $W^s_{\mathcal{V}}(O) \cup W^s_{loc}(\Gamma_1) \cup W^s_{loc}(\Gamma_1 \cup \Gamma_2) \cup W^s_{loc}(\Gamma_2)$ (resp. $W^u_{\mathcal{V}}(O) \cup W^u_{loc}(\Gamma_1) \cup W^u_{loc}(\Gamma_1 \cup \Gamma_2) \cup W^u_{loc}(\Gamma_2)$).*

The next two theorems give analogues of Theorems A2 and A3 for the case of homoclinic figure-eight.

Theorem B2. *If $\lambda_2 < 2\lambda_1$, and Assumptions 1-3 and 6 hold, then $W^s_{loc}(\Gamma_1 \cup \Gamma_2) = W^u_{loc}(\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2$. Thus, by Theorem B1, the forward and backward orbits of any point in $\mathcal{V}$ either leave $\mathcal{V}$ or converge to O.*

Consider the cross-sections $\Pi^s_1 = \{u_2 = \delta\} \cap \{H = 0\}$ and $\Pi^u_1 = \{v_2 = \delta\} \cap \{H = 0\}$ on Γ_1, and $\Pi^s_2 = \{u_2 = -\delta\} \cap \{H = 0\}$ and $\Pi^u_2 = \{v_2 = -\delta\} \cap \{H = 0\}$ on Γ_2 (see Fig. 7). We can choose (u_1, v_1)-coordinates on each of these cross-sections (see Lemma 2.12). Let T_i, T^{loc}_i and T^{glo}_i be the associated maps along Γ_i. For $i = 1, 2$, consider the points $M^{s,u}_i = \Gamma_i \cap \Pi^{s,u}_i$, and let a_i, b_i, c_i and d_i be the corresponding Taylor coefficients of T^{glo}_i, as in (1.8).

Theorem B3. *Assume $2\lambda_1 < \lambda_2$ and Assumptions 1-3 and 6. Suppose that system (1.3) near the equilibrium O is brought to the form (2.12) and let b_i, c_i and d_i ($i = 1, 2$) be non-zero.*

(i) *If $c_1 d_1 > 0$ and $c_2 d_2 > 0$, then $W^s_{loc}(\Gamma_1 \cup \Gamma_2)$ is a $\mathcal{C}^1$-smooth 2-dimensional invariant manifold which is tangent to $W^s_{glo}(O)$ at every point of $\Gamma_1 \cup \Gamma_2$.*

Fig. 4. The orbit Γ (brown) is homoclinic to the saddle equilibrium O. The super-homoclinic orbit $\mathcal{S}$ (blue) is homoclinic to $\Gamma \cup \{O\}$.

(ii) If $b_1 d_1 < 0$ and $b_2 d_2 < 0$, then $W_{loc}^u (\Gamma_1 \cup \Gamma_2)$ is a $\mathcal{C}^1$-smooth 2-dimensional invariant manifold which is tangent to $W_{glo}^u (O)$ at every point of $\Gamma_1 \cup \Gamma_2$.
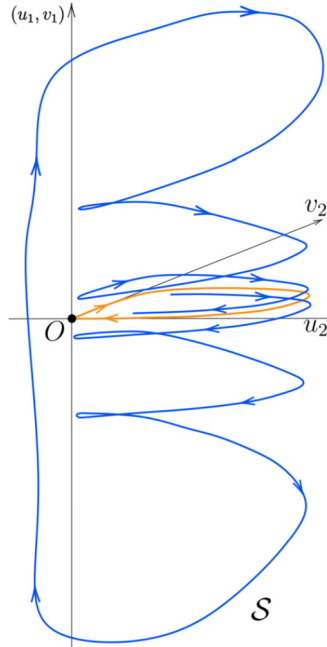
(iii) Otherwise, we have $W_{loc}^s (\Gamma_1 \cup \Gamma_2) = W_{loc}^u (\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2$.

Let $0 < \gamma = \frac{\lambda_1}{\lambda_2} \leq 1$, and consider a one-parameter family $\{X_\gamma\}$ of the vector fields of the form (1.3) that satisfy the assumptions stated above. In particular, suppose that X_γ possesses a homoclinic orbit Γ_γ which persists as γ varies. Then, according to our results, when $\gamma > 0.5$, there is no dynamics near the homoclinic orbit Γ_γ in its energy level, while when $\gamma < 0.5$, depending on how the global map behaves (i.e. what the coefficients a, b, c and d are), there may exist stable and unstable invariant manifolds to the homoclinic loop Γ_γ. This lets us to conjecture that saddle periodic orbits can be born in the level $H = 0$ as γ crosses $\frac{1}{2}$. This question requires a further investigation.

1.2.3. Dynamics near a super-homoclinic orbit

Coming back to the case of the single homoclinic loop Γ, we consider the case in which both $W_{loc}^s (\Gamma)$ and $W_{loc}^u (\Gamma)$ are non-trivial. Notice that, according to Theorem A3, in order for these two manifolds to coexist, we require $cd < 0$ and $bd > 0$. Continuing these two local manifolds by the flow of the system gives the global stable and unstable invariant manifolds of Γ, denoted by $W_{glo}^s (\Gamma)$ and $W_{glo}^u (\Gamma)$, respectively. These manifolds lie in the 3-dimensional level $\{H = 0\}$ which means that it would be reasonable if we assume that they intersect transversely in that level. Any orbit at this intersection is bi-asymptotic, or in other words, homoclinic to the union of Γ and the equilibrium O, i.e. converges to $\Gamma \cup \{O\}$ as $t \to \pm\infty$. We refer to such an orbit as 'homoclinic to homoclinic' or 'super-homoclinic' orbit (See Fig. 4).

Assumption 7. There exists a super-homoclinic orbit $\mathcal{S}$ of the transverse intersection of $W^s_{\text{glo}}(\Gamma)$ and $W^u_{\text{glo}}(\Gamma)$.

Theorem C1. *Under Assumptions 1-4 and 7, there exist infinitely many multi-pulse homoclinic loops in a small neighborhood of $\mathcal{S} \cup \Gamma \cup \{O\}$ (i.e. the closure of $\mathcal{S}$).*

A similar result holds for a *homoclinic to homoclinic figure-eight*:

Assumption 8. There exists a super-homoclinic orbit $\mathcal{S}$ of the transverse intersection of $W^s_{\text{glo}}(\Gamma_1 \cup \Gamma_2)$ and $W^u_{\text{glo}}(\Gamma_1 \cup \Gamma_2)$.

Theorem C2. *Under Assumptions 1-3, 5, 6 and 8, there exist infinitely many multi-pulse homoclinic loops in a small neighborhood of $\mathcal{S} \cup \{O\} \cup \Gamma_1 \cup \Gamma_2$ (i.e. the closure of $\mathcal{S}$).*

The multi-pulse homoclinic orbits in Theorem C1 (and Theorem C2) refer to homoclinic loops that intersect the cross-section Π^s (resp. $\Pi^s_1 \cup \Pi^s_2$) in n points. We call such orbits n-pulse homoclinic. We prove that the existence of super-homoclinic orbits implies the existence of n-pulse homoclinic orbits for arbitrarily large n.

1.3. Coupled nonlinear Schrödinger equations

The coupled nonlinear Schrödinger equation (CNLSE) is one of the basic models for light propagation. This equation also has various applications in different branches of physics since it appears as a universal model of behavior near a threshold of instability (see e.g. [16]). In this section, we discuss this equation as an application of our theory.

The CNLSE is written as

$$
\begin{aligned}
i\Psi_t + \Psi_{xx} + 2\left(\alpha |\Psi|^2 + |\Phi|^2\right)\Psi = 0, \\
i\Phi_t + \Phi_{xx} + 2\left(|\Psi|^2 + \beta |\Phi|^2\right)\Phi = 0,
\end{aligned}
\tag{1.9}
$$

where Ψ and Φ are complex-valued functions of (t, x). We consider the case where α and β are positive real constants. We consider the steady-state solutions of (1.9) which are of the form

$$
\Psi(t, x) = e^{i\omega_1^2 t}\psi(x), \qquad \Phi(t, x) = e^{i\omega_2^2 t}\phi(x),
$$

for some real valued functions ψ and ϕ. By a rescaling, we can assume $\omega_1 = 1$ and $\omega_2 = \omega$ ($\omega > 0$). Thus, the stationary solutions of CNLSE satisfy

$$
\psi_{xx} = \psi - 2\left(\alpha\psi^2 + \phi^2\right)\psi, \qquad \phi_{xx} = \omega^2\phi - 2\left(\psi^2 + \beta\phi^2\right)\phi.
$$

Define $\psi_1(x) = \psi$, $\psi_2(x) = \psi_x$, $\phi_1(x) = \phi$ and $\phi_2(x) = \phi_x$. Then,

$$
\begin{aligned}
\dot{\psi}_1 = \frac{\partial H}{\partial \psi_2} = \psi_2, \qquad & \dot{\psi}_2 = -\frac{\partial H}{\partial \psi_1} = \psi_1 - 2\left(\alpha\psi_1^2 + \phi_1^2\right)\psi_1, \\
\dot{\phi}_1 = \frac{\partial H}{\partial \phi_2} = \phi_2, \qquad & \dot{\phi}_2 = -\frac{\partial H}{\partial \phi_1} = \omega^2\phi_1 - 2\left(\psi_1^2 + \beta\phi_1^2\right)\phi_1,
\end{aligned}
\tag{1.10}
$$

where $H = \frac{1}{2}\left[\psi_2^2 + \phi_2^2 - \psi_1^2 - \omega^2\phi_1^2 + \alpha\psi_1^4 + 2\psi_1^2\phi_1^2 + \beta\phi_1^4\right]$. This system is Hamiltonian with two degrees of freedom. Diagonalizing the linear part reduces this system to

$$
\begin{aligned}
\dot{u}_1 &= -u_1 + E_1(u,v), & \dot{v}_1 &= +v_1 + \frac{1}{2}E_1(u,v), \\
\dot{u}_2 &= -\omega u_2 + E_2(u,v), & \dot{v}_2 &= +\omega v_2 - \frac{\omega}{2}E_2(u,v),
\end{aligned}
\tag{1.11}
$$

where E_1 and E_2 are cubic functions of (u,v), and transforms the Hamiltonian H to the form $H = u_1 v_1 - \omega u_2 v_2 + O\left(\|(u,v)\|^4\right)$. This system is invariant with respect to symmetry (1.5) and the symmetry $(u_2, v_2) \leftrightarrow (-u_2, -v_2)$. Therefore, assuming $1 \le \omega \ne 2$, system (1.11) meets all Assumptions 1-3 and 5. In addition, it possesses a pair of homoclinic solutions (homoclinic figure-eight):

$$
u_1(x) = 0, \quad u_2(x) = \frac{\kappa\omega e^{\omega x}}{\sqrt{\beta}\cosh^2(\omega x)}, \quad v_1(x) = 0, \quad v_2(x) = \frac{\kappa\omega^2 e^{-\omega x}}{2\sqrt{\beta}\cosh^2(\omega x)}, \quad (\kappa = \pm 1).
\tag{1.12}
$$

These solutions correspond to the following solutions of (1.9):

$$
\Psi(t,x) = 0, \qquad \Phi(t,x) = \pm\frac{\omega e^{i\omega^2 t}}{\sqrt{\beta}\cosh(\omega x)}.
\tag{1.13}
$$

We consider the case where the homoclinic figure-eight (1.12) is transverse, i.e. Assumption 6 is met. Therefore, the dynamics near this homoclinic figure-eight in the level $\{H = 0\}$ can be analyzed by Theorems B2 and B3. For $\omega < 2$, Theorem B2 implies that both forward and backward orbits of any point close to the homoclinic figure-eight leave a small neighborhood of it (in the level $\{H = 0\}$) unless it lies on the stable or unstable invariant manifolds of O. For the case of $\omega > 2$, in order to apply Theorem B3, one needs to reduce system (1.11) to normal form (2.12) and compute the coefficients a_i, b_i, c_i and d_i ($i = 1, 2$).

System (1.11) is reversible with respect to the linear involution $u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_2$. In general, a system $\dot{x} = f(x)$ on $\mathbb{R}^n$ is said to be reversible with respect to an involution R, i.e. a diffeomorphism on $\mathbb{R}^n$ with the property $R^2 = \mathrm{id}$, if $dR \circ f = -f \circ R$. It is easily seen that when $x(t)$ is a solution, so does $R \circ x(-t)$. The reversibility of system (1.11) implies

Proposition 1.1. *For* $A = \Gamma_1, \Gamma_2, \Gamma_1 \cup \Gamma_2$, *the manifold* $W_{loc}^s(A)$ *is non-trivial if and only if* $W_{loc}^u(A)$ *is non-trivial.*

Reducing system (1.11) to normal form (2.12) preserves the invariance of the system with respect to the symmetry $(u_2, v_2) \leftrightarrow (-u_2, -v_2)$. This implies that the loops Γ_1 and Γ_2 are symmetric, and $a_1 = a_2 = a$, $b_1 = b_2 = b$, $c_1 = c_2 = c$ and $d_1 = d_2 = d$. Because of the symmetry, Proposition 1.1 implies

Proposition 1.2. *Simultaneously, all the manifolds* $W_{loc}^u(\Gamma_1)$, $W_{loc}^s(\Gamma_1)$, $W_{loc}^u(\Gamma_2)$ *and* $W_{loc}^s(\Gamma_2)$ *are either trivial or non-trivial.*

Concerning the coefficients a, b, c and d, notice that the Hamiltonian structure of the equations implies that the map T^{glo} is area- and orientation-preserving; hence $ad - bc = 1$. Therefore,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so the reversibility implies $b = -c$. Recall also that the transversality condition implies $d \neq 0$.

It follows from Theorem B3 that if $bd > 0$, then the local unstable invariant manifold of each of the loops Γ_1 and Γ_2 is non-trivial, while the local unstable invariant manifold of the homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$ is trivial (i.e. coincides with $\Gamma_1 \cup \{O\} \cup \Gamma_2$). In contrast, when $bd < 0$, the local unstable invariant manifold of the homoclinic figure-eight is non-trivial, while the local unstable invariant manifold of each of the loops Γ_1 and Γ_2 is trivial. The same conclusion holds for the corresponding stable manifolds. This analysis together with Propositions 1.1 and 1.2 yields

Proposition 1.3. *Let $\omega > 2$ and suppose that the coefficients b, c and d are non-zero. Then, one (and only one) of the following two scenarios holds:*

(i) *The manifolds $W_{loc}^u (\Gamma_1 \cup \Gamma_2)$ and $W_{loc}^s (\Gamma_1 \cup \Gamma_2)$ are non-trivial, i.e. $bd = -cd < 0$.*
(ii) *The manifolds $W_{loc}^u (\Gamma_1)$, $W_{loc}^s (\Gamma_1)$, $W_{loc}^u (\Gamma_2)$ and $W_{loc}^s (\Gamma_2)$ are non-trivial, i.e. $bd = -cd > 0$.*

To figure out which of the scenarios above happens for CNLSE, one needs to find the corresponding coefficients a, b, c and d. For any particular values of α, β and ω, this can be done numerically by solving the linearization of system (1.10) along the solution (1.13). It is also easy to show that as $\omega \to \infty$, the coefficient d changes sign infinitely many times. When $d = 0$, we have $b^2 = 1$, so crossing $d = 0$ leads to a change of sign of bd. Thus, both cases of the above proposition are realized in CNLSE. In either case, there are non-trivial local stable and unstable invariant manifolds of the homoclinic orbits in the CNLSE. Globalizing these manifolds, we conjecture that they intersect transversely along some super-homoclinic orbits:

Conjecture 1.4. *The coupled nonlinear Schrödinger equations given by (1.9) possess transverse super-homoclinic orbits.*

Our conjecture is based on the fact that the stable and unstable manifolds of the homoclinic loops are 2-dimensional manifolds lying in the same compact 3-dimensional energy level, and - by Poincaré recurrence - come infinitely close to each other. Hence, they are likely to intersect transversely along super-homoclinic orbits. Moreover, numerical evidence [11,10,30,31] points to the existence of infinitely many multi-pulse homoclinic orbits in the CNLSE. This supports our conjecture since, by Theorems C1 and C2, the existence of these multi-pulse homoclinics might be a bi-product of the existence of super-homoclinic orbits.

1.4. The methods and organization of the paper

The standard approach for investigating the dynamics near homoclinic orbits is to study the Poincaré maps along these orbits. The main difficulty in dealing with these maps is that the Poincaré map along a homoclinic orbit is a singular map defined on a non-trivial domain. Let Γ be a homoclinic orbit, Σ be a small cross-section to it, T be the Poincaré map defined on some domain $\mathcal{D} \subset \Sigma$, and M be the intersection point of Γ and Σ. The point M does not belong to the domain $\mathcal{D}$, however, it is in the closure of this set. The domain $\mathcal{D}$ in our case consists of several (at least two) connected components each with non-empty interior, while the set $\mathcal{D} \cup \{M\}$ is connected, but $\mathcal{D} \cup \{M\}$ does not contain an open neighborhood of M in Σ. The point M is a

singularity for the Poincaré map T: as $z \to M$ for $z \in \mathcal{D}$, we have $\|dT(z)\| \to \infty$, so, extending the map T to $\mathcal{D} \cup \{M\}$ by defining $T(M) = M$ and turning M to the fixed point of T, does not remove this singularity.

Due to these properties, the smooth theory of Hadamard-Perron cannot be applied directly to study the invariant manifolds of the Poincaré maps along homoclinics. Our approach for investigating the invariant manifolds is applying the method of Shilnikov cross-maps [23,13]. To describe this method, assume that the Poincaré map T is written by

$$\overline{x} = f(x, y),$$
$$\overline{y} = g(x, y)$$

where f and g are some functions defined on $\mathcal{D}$ such that

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(x,y) \to (0,0)} g(x, y) = (0, 0),$$

and the point M corresponds to $(x, y) = (0, 0)$. Suppose that y in the second equation can be solved in terms of $(x, \overline{y})$, i.e. $y = G(x, \overline{y})$ for some function G. This leads us to introduce the map $T^\times$ defined by

$$\overline{x} = F(x, \overline{y}),$$
$$y = G(x, \overline{y}),$$

where $F(x, \overline{y}) = f(x, G(x, \overline{y}))$. In other words, the Poincaré map T takes (x, y) to $(\overline{x}, \overline{y})$ if and only if the cross-map $T^\times$ maps $(x, \overline{y})$ to $(\overline{x}, y)$. Denote the domain of $T^\times$ by $\mathcal{D}^\times$. Note that the point $M = (x, \overline{y}) = (0, 0)$ is not in the domain $\mathcal{D}^\times$, however, it lies in the closure of $\mathcal{D}^\times$.

The advantage of dealing with the cross-map $T^\times$ over the Poincaré map T is that despite $\|dT(x, y)\| \to \infty$ as $(x, y) \to M$, the limit $\lim_{(x,\overline{y}) \to M} T^\times(x, \overline{y})$ may exist. This property, if the (x, y) coordinates system on Σ is chosen appropriately, enables us to extend the cross-map $T^\times$ to an open neighborhood of M smoothly. Then, according to Theorem 8 (see Appendix D), if *extended* $T^\times$ satisfies certain properties, the Poincaré map T possesses an invariant manifold that contains the ω-limit points of every forward orbit of the domain. This is exactly the procedure that we follow in this paper to prove the existence of invariant manifolds of the Poincaré maps along the homoclinic orbits.

In order to obtain necessary estimates for the Poincaré map, we first bring our system near the equilibrium state O to a normal form. Notice that our system is not necessarily linearizable. Indeed, since the spectrum of the linear part of the system is $\{-\lambda_2, -\lambda_1, \lambda_1, \lambda_2\}$, for some $0 < \lambda_1 \leq \lambda_2$, resonances prevent the linearization. In this paper, we follow the approach of Ovsyaninkov and Shilnikov [21] (see also [28] and [23]) for the normal form reduction. In contrast to the standard normal form approach in which the system near a hyperbolic equilibrium reduces to a polynomial vector field that consists of only resonant terms up to some order (see e.g. [15] and [4]), in the approach of [21], some non-resonant terms remain in the normal form, while some resonant terms of low-orders are eliminated.

Once the system is brought to a normal form, we need to investigate the behavior of the orbits near the equilibrium state O. This enables us to compute the Poincaré maps. To do this, we apply the method of successive approximations (more specifically, Shilnikov's method of solving boundary value problems, see [23]) to estimate the flow near the equilibrium O.

This paper is organized as follows: Section 2 is dedicated to the study of the local map T^{loc}. In Section 2.1, we define this map and its domain precisely. Then, in Section 2.2, we bring our system near the equilibrium state O to a normal form. In Section 2.3, we investigate the behavior of the orbits near the equilibrium state O. Finally, in Section 2.4, we analyze the domain and the behavior of the local map.

In Section 3, we use the results of Section 2 to study the dynamics near the homoclinic orbits. In Section 3.1, we introduce some notations. In Section 3.2, we study the dynamics near the homoclinic orbit Γ when $\lambda_2 < 2\lambda_1$. Theorem A2 is proved in this section. The dynamics near Γ when $2\lambda_1 < \lambda_2$ is studied in Section 3.3. We prove Theorem A3 in this section. Theorem A1 is also proved in these two sections. The case of homoclinic figure-eight is studied in Section 3.4. The proofs of Theorems B1, B2 and B3 are provided in this section. Finally, we discuss the case of superhomoclinics in Section 3.5. We prove Theorems C1 and C2 in this section.

Most of the technical lemmas and calculations are postponed to appendices. We also give a brief introduction to the method of cross-maps in Appendix D.

2. Analysis near the equilibrium state O

2.1. Set-up and notations

Our approach for studying the dynamics near the homoclinic loop Γ (and homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$) is based on the study of the behavior of the corresponding Poincaré map(s). As was mentioned earlier, the Poincaré map T along the homoclinic loop Γ can be written as the composition of a global and a local map. This section is dedicated to the study of the behavior of the local map T^{loc}. To this end, we first need to choose appropriate coordinates near the equilibrium state O of system (1.3). This is done in Section 2.2 below. We consider three different cases of $\lambda_1 = \lambda_2$, $\lambda_1 < \lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$, and introduce a specific normal form for each case. In Section 2.3, we employ the Shilnikov technique for solving boundary value problems to compute the flow near the equilibrium O. This allows us to find an approximation for the local map. Finally, in Section 2.4, we study the behavior of this map and investigate some of its properties.

In comparison to the global map, the local map has more complicated behavior. Indeed, T^{glo} is a diffeomorphism and can be approximated by its Taylor polynomial while the local map T^{loc} is a singular map with a non-trivial domain.

Let us now give a more precise meaning to the above terminologies. Recall the cross-sections Π^s and Π^u. In all of the normal forms considered in Section 2.2, the local stable and local unstable as well as the local strong stable and local strong unstable invariant manifolds of O are straightened. Therefore, the homoclinic loop Γ intersects Π^s and Π^u at $M^s = (0, \delta, 0, 0)$ and $M^u = (0, 0, 0, \delta)$, respectively. As it is proved later (see Section 2.4.1), we can choose a (u_1, v_1)

Fig. 5. The ϵ-ball around M^s in Π^s and the ϵ_u-ball around M^u in Π^u are shown by green and red colors, respectively. The domain $\mathcal{D}$ of the Poincaré map T is the set of the points (u_{10}, v_{10}) in the green ball whose forward orbits intersect Π^u at $(u_{1\tau}, v_{1\tau})$ in the red ball (see relation (2.1)).

coordinate-system on each of these cross-sections. Both M^s and M^u correspond to $(0,0)$ in this coordinate-system.

Consider a point (u_{10}, v_{10}) on Π^s close to M^s (e.g. the green point in Fig. 2) whose forward orbit goes along the homoclinic loop Γ, after a certain time τ it crosses Π^u at a point $(u_{1\tau}, v_{1\tau})$ (e.g. the blue point in Fig. 2), and after a finite time it comes back to Π^s at a point $(\overline{u}_{10}, \overline{v}_{10})$ (e.g. the red point in Fig. 2). Obviously, $\tau \to \infty$ as $(u_{10}, v_{10}) \to M^s$. Let $\mathcal{D} \subset \Pi^s$ be the set of all such points (u_{10}, v_{10}) that satisfy

$$\|(u_{10}, v_{10})\| < \epsilon \quad \text{and} \quad \|(u_{1\tau}, v_{1\tau})\| < \epsilon_u, \tag{2.1}$$

for some sufficiently small constants $0 < \epsilon \le \epsilon_u < \delta$ (see Fig. 5). It is trivial that $M^s \notin \mathcal{D}$. When $\mathcal{D} \ne \emptyset$, we define the Poincaré map $T : \mathcal{D} \to \Pi^s$ by $(u_{10}, v_{10}) \longmapsto (\overline{u}_{10}, \overline{v}_{10})$. The local map $T^{loc} : \mathcal{D} \to \Pi^u$ is defined by

$$(u_{10}, v_{10}) \longmapsto (u_{1\tau}, v_{1\tau}). \tag{2.2}$$

The global map is defined on the ϵ_u-ball $\mathcal{B}_{\epsilon_u}$ in Π^u centered at M^u, i.e. $T^{glo} : \mathcal{B}_{\epsilon_u} \to \Pi^s$, and its restriction to $T^{loc}(\mathcal{D}) \subset \mathcal{B}_{\epsilon_u}$ is

$$(u_{1\tau}, v_{1\tau}) \longmapsto (\overline{u}_{10}, \overline{v}_{10}). \tag{2.3}$$

Obviously, $T = T^{glo} \circ T^{loc}$.

Not every orbit starting from Π^s goes along Γ and intersects the cross-section Π^u. Trivial examples are the orbits that start at $W^s_{\text{loc}}(O) \cap \Pi^s$. Other examples are the orbits that go along the other branch of $W^u_{\text{loc}}(O)$ (negative side of v_2-axis). Consider a cross-section $\Sigma = \{v_2 = -\delta\} \cap \{H = 0\}$ to the negative branch of $W^u_{\text{loc}}(O)$. It will be shown that (u_1, v_1)-coordinates can be chosen on this cross-section. Then

Definition 2. We denote by $\mathbb{D}$ the set of the points (u_{10}, v_{10}) on Π^s close to M^s whose forward orbits go along the negative branch of $W^u_{\text{loc}}(O)$, and after a certain time τ they cross Σ at $(u_{1\tau}, v_{1\tau})$ such that (2.1) holds (see Fig. 6).

For the case of homoclinic figure-eight, we define the domains $\mathcal{D}$ and $\mathbb{D}$ for each loop:

Notation 2.1. For $i = 1, 2$, we denote by $\mathcal{D}^i$ and $\mathbb{D}^i$ the corresponding domains $\mathcal{D} \subset \Pi^s_i$ and $\mathbb{D} \subset \Pi^s_i$ of the loop Γ_i, respectively.
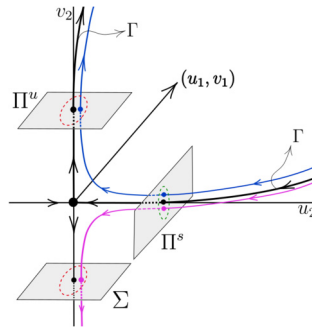
Fig. 6. The domain $\mathcal{D}$ of the Poincaré map is defined as the set of the points on Π^s close to M^s that go along the homoclinic loop Γ and intersect $\Pi^u = \{v_2 = \delta\} \cap \{H = 0\}$ at points close to M^u. For instance, the blue point on Π^s belongs to $\mathcal{D}$. Similarly, we define $\mathbb{D}$ as the set of the points on Π^s close to M^s that go along Γ until they get close to O and then go along the negative side of v_2-axis and intersect the cross-section $\Sigma = \{v_2 = -\delta\} \cap \{H = 0\}$ at points close to the point of the intersection of Σ and v_2-axis (e.g. the pink point on Π^s belongs to $\mathbb{D}$).
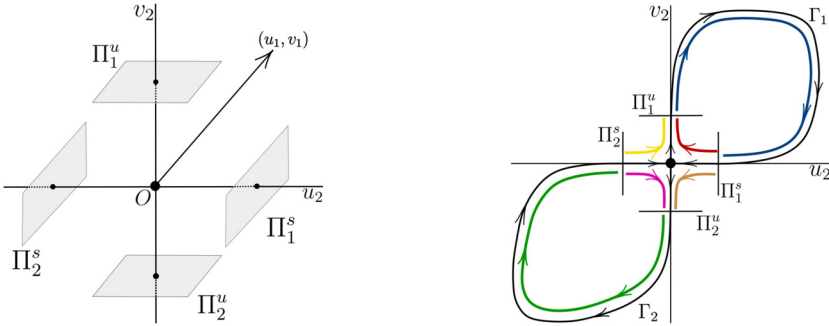
Fig. 7. (left) The positions of the cross-sections Π_1^s, Π_1^u, Π_2^s and Π_2^u are shown. (right) Γ_1 and Γ_2 are homoclinic orbits. The blue, brown, green, yellow, red and pink curves correspond to the maps T_1^{glo}, T_{12}^{glo}, T_2^{glo}, T_{21}^{loc}, T_1^{loc} and T_2^{loc}, respectively. The Poincaré maps T_1 (along Γ_1) and T_2 (along Γ_2) are defined by $T_1 = T_1^{\text{glo}} \circ T_1^{\text{loc}}$ and $T_2 = T_2^{\text{glo}} \circ T_2^{\text{loc}}$, respectively.

An orbit starting from $\mathcal{D}^1 \subset \Pi_1^s$ (resp. $\mathcal{D}^2 \subset \Pi_2^s$) goes along Γ_1 (resp. Γ_2) and intersects Π_1^u (resp. Π_2^u), while an orbit which starts from $\mathbb{D}^1 \subset \Pi_1^s$ (resp. $\mathbb{D}^2 \subset \Pi_2^s$) goes along the negative (resp. positive) side of v_2-axis and intersects Π_2^u (resp. Π_1^u).

We introduced the Poincaré, local and global maps along a single homoclinic loop above. For the case of homoclinic figure eight, we also define these maps for each loop:

Notation 2.2. We denote by T_i, T_i^{loc} and T_i^{glo} the Poincaré, local and global maps along Γ_i $(i = 1, 2)$, respectively (see Fig. 7).

To study the case of a homoclinic figure-eight, we consider two extra local maps:

Definition 3. We define the map $T_{12}^{\text{loc}} : \mathbb{D}^1 \subset \Pi_1^s \to \Pi_2^u$ $(T_{21}^{\text{loc}} : \mathbb{D}^2 \subset \Pi_2^s \to \Pi_1^u)$ by $(u_{10}, v_{10}) \mapsto (u_{1\tau}, v_{1\tau})$ where $(u_{10}, v_{10}) \in \mathbb{D}^1$ $(\in \mathbb{D}^2)$ and $(u_{1\tau}, v_{1\tau}) \in \Pi_2^u$ $(\in \Pi_1^u)$ (see Fig. 7).

2.2. Choice of coordinates near the equilibrium state O

This section is dedicated to finding suitable coordinate systems near the equilibrium state O. As it was mentioned above, we consider three different cases of $\lambda_1 = \lambda_2$, $\lambda_1 < \lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$, and for each case we bring system (1.3) into a particular normal form. The proofs of the results stated below are postponed to Appendix A. We start with the following:

Lemma 2.3. *Consider system (1.3) and first integral (1.4). There exists a $\mathcal{C}^\infty$-smooth change of coordinates which brings system (1.3) to the form*

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + f_{11}(u_1, u_2, v_1, v_2)u_1 + f_{12}(u_1, u_2, v_1, v_2)u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + f_{21}(u_1, u_2, v_1, v_2)u_1 + f_{22}(u_1, u_2, v_1, v_2)u_2, \\
\dot{v}_1 &= +\lambda_1 v_1 + g_{11}(u_1, u_2, v_1, v_2)v_1 + g_{12}(u_1, u_2, v_1, v_2)v_2, \\
\dot{v}_2 &= +\lambda_2 v_2 + g_{21}(u_1, u_2, v_1, v_2)v_1 + g_{22}(u_1, u_2, v_1, v_2)v_2,
\end{aligned}
\tag{2.4}
$$

where the functions f_{ij}, g_{ij} are $\mathcal{C}^\infty$-smooth and vanish at the origin, i.e.

$$
f_{ij}(0,0,0,0) = g_{ij}(0,0,0,0) = 0,
\tag{2.5}
$$

and transforms first integral (1.4) to

$$
H = \lambda_1 u_1 v_1 - \lambda_2 u_2 v_2.
\tag{2.6}
$$

Moreover, system (2.4) remains invariant with respect to symmetry (1.5). In particular,

$$
f_{12}(0, u_2, 0, v_2) \equiv 0, \quad g_{12}(0, u_2, 0, v_2) \equiv 0.
\tag{2.7}
$$

The statement of Lemma 2.3 holds for arbitrary $0 < \lambda_1 \leq \lambda_2$. However, we will particularly use this normal form for analyzing the case $\lambda_1 = \lambda_2$.

Lemma 2.4. *Consider system (1.3) and first integral (1.4), and assume $\lambda_1 < \lambda_2$. There exists a $\mathcal{C}^\infty$-smooth change of coordinates which brings system (1.3) to the form*

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + f_{11}(u_1, v) u_1 + f_{12}(u_1, u_2, v) u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + f_{21}(u_1, v) u_1 + f_{22}(u_1, u_2, v) u_2, \\
\dot{v}_1 &= +\lambda_1 v_1 + g_{11}(u, v_1) v_1 + g_{12}(u, v_1, v_2) v_2, \\
\dot{v}_2 &= +\lambda_2 v_2 + g_{21}(u, v_1) v_1 + g_{22}(u, v_1, v_2) v_2,
\end{aligned}
\tag{2.8}
$$

where the functions f_{ij}, g_{ij} are $\mathcal{C}^\infty$-smooth and satisfy the identities

$$
\begin{aligned}
f_{11}(0, v) &\equiv 0, \quad f_{11}(u_1, 0) \equiv 0, \quad f_{12}(u, 0) \equiv 0, \quad f_{21}(0, v) \equiv 0, \quad f_{22}(0, v) \equiv 0, \\
g_{11}(u, 0) &\equiv 0, \quad g_{11}(0, v_1) \equiv 0, \quad g_{12}(0, v) \equiv 0, \quad g_{21}(u, 0) \equiv 0, \quad g_{22}(u, 0) \equiv 0.
\end{aligned}
\tag{2.9}
$$

This change of coordinates transforms first integral (1.4) to

$$H = \lambda_1 u_1 v_1 \left[1 + H_1(u, v)\right] - \lambda_2 u_2 v_2 \left[1 + H_2(u, v)\right], \tag{2.10}$$

where H_1 and H_2 are $\mathcal{C}^\infty$ functions that vanish at O. We can write (2.10) as

$$H = \lambda_1 u_1 v_1 \left[1 + o(1)\right] - \lambda_2 u_2 v_2 \left[1 + o(1)\right]. \tag{2.11}$$

Moreover, normal form (2.8) and first integral (2.10) remain invariant with respect to symmetry (1.5). In particular, (2.7) holds.

The statement of Lemma 2.4 holds for arbitrary $\lambda_1 < \lambda_2$. However, we will particularly use this normal form to analyze the local dynamics near O when $\lambda_1 < \lambda_2 < 2\lambda_1$. The normal form that is used for analyzing the case $2\lambda_1 < \lambda_2$ is given by the following:

Lemma 2.5. *Consider system (1.3) and first integral (1.4) and assume $2\lambda_1 < \lambda_2$. Let q be the largest integer such that $q\lambda_1 < \lambda_2$. There exists a $\mathcal{C}^q$-smooth change of coordinates which brings system (1.3) to the form*

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + f_{11}(u_1, v) u_1 + f_{12}(u_1, u_2, v) u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + f_{22}(u_1, u_2, v) u_2, \\
\dot{v}_1 &= +\lambda_1 v_1 + g_{11}(u, v_1) v_1 + g_{12}(u, v_1, v_2) v_2, \\
\dot{v}_2 &= +\lambda_2 v_2 + g_{22}(u, v_1, v_2) v_2,
\end{aligned}
\tag{2.12}
$$

where f_{ij} and g_{ij} are $\mathcal{C}^{q-1}$-smooth and satisfy identities (2.9). This change of coordinates transforms first integral (1.4) to

$$H = \lambda_1 u_1 v_1 \left[1 + H_1(u, v)\right] - \lambda_2 u_2 v_2 \left[1 + H_2(u, v)\right] + u_2 v_1^2 H_3(u, v) + v_2 u_1^2 H_4(u, v), \tag{2.13}$$

where H is $\mathcal{C}^q$, and H_1, H_2, H_3 and H_4 are some $\mathcal{C}^{q-1}$, $\mathcal{C}^q$, $\mathcal{C}^{q-2}$ and $\mathcal{C}^{q-2}$ functions, respectively, such that $H_1(O) = H_2(O) = 0$. Moreover, system (2.12) and first integral (2.13) remain invariant with respect to symmetry (1.5). In particular, (2.7) holds.

Remark 2.6. For simplicity, we can write (2.13) as

$$H = \lambda_1 u_1 v_1 \left[1 + o(1)\right] - \lambda_2 u_2 v_2 \left[1 + o(1)\right] + u_2 v_1^2 O(1) + v_2 u_1^2 O(1).$$

A common structure of all of normal forms (2.4), (2.8) and (2.12) is that the local stable and unstable as well as the local strong stable and strong unstable invariant manifolds of the equilibrium O are straightened, i.e. $W^s_{\text{loc}} = \{v = 0\}$, $W^u_{\text{loc}} = \{u = 0\}$, $W^{ss}_{\text{loc}} = \{u_1 = v_1 = v_2 = 0\}$ and $W^{uu}_{\text{loc}} = \{u_1 = u_2 = v_1 = 0\}$. For the particular case of normal form (2.12), the local extended stable and extended unstable invariant manifolds of O are straightened too, i.e. $W^{sE}_{\text{loc}} = \{v_2 = 0\}$ and $W^{uE}_{\text{loc}} = \{u_2 = 0\}$.

2.3. Trajectories near the equilibrium state O

In this section, we estimate the solutions of systems (2.4), (2.8) and (2.12) near the equilibrium state O by using the technique of successive approximations.

Consider the system

$$
\begin{aligned}
\dot{u}_i &= -\lambda_i u_i + F_i (u_1, u_2, v_1, v_2) \\
\dot{v}_i &= +\lambda_i v_i + G_i (u_1, u_2, v_1, v_2)
\end{aligned}
\qquad (i = 1, 2)
\qquad (2.14)
$$

where F_1, F_2, G_1 and G_2 and their first derivatives vanish at the origin. By [23] (Theorem 2.9), for given $\tau \geq 0$ and sufficiently small u_{10}, u_{20}, $v_{1\tau}$ and $v_{2\tau}$ there exists a unique solution $(u_1(t), u_2(t), v_1(t), v_2(t))$ of system (2.14) such that

$$
u_1(0) = u_{10}, \quad u_2(0) = u_{20}, \quad v_1(\tau) = v_{1\tau}, \quad v_2(\tau) = v_{2\tau}. \qquad (2.15)
$$

The dependence of this solution on each of the variables τ, u_{10}, u_{20}, $v_{1\tau}$ and $v_{2\tau}$ is as smooth as the original system (2.14).

The following lemmas estimate the solutions of systems (2.4), (2.8) and (2.12) that satisfy boundary condition (2.15). We prove these lemmas in Appendix B.

Lemma 2.7. *Let* $\lambda = \lambda_1 = \lambda_2$. *There exists* $M > 0$ *such that for any sufficiently small* $\delta > 0$, *and any* u_{10}, u_{20}, $v_{1\tau}$ *and* $v_{2\tau}$, *where* $\max\{|u_{10}|, |u_{20}|, |v_{1\tau}|, |v_{2\tau}|\} \leq \delta$, *the solution* $(u(t), v(t))$ *of system (2.4) that satisfies boundary condition (2.15) can be written as*

$$
\begin{aligned}
u_1(t) &= e^{-\lambda t} u_{10} + \xi_1(x), & u_2(t) &= e^{-\lambda t} u_{20} + \xi_2(x), \\
v_1(t) &= e^{-\lambda(\tau-t)} v_{1\tau} + \zeta_1(x), & v_2(t) &= e^{-\lambda(\tau-t)} v_{2\tau} + \zeta_2(x),
\end{aligned}
\qquad (2.16)
$$

where $x = (t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})$, $t \in [0, \tau]$, $\max\{|\xi_1|, |\xi_2|\} \leq M e^{-\lambda t} \delta^2$ *and* $\max\{|\zeta_1|, |\zeta_2|\} \leq M e^{-\lambda(\tau-t)} \delta^2$. *We can also write*

$$
\begin{aligned}
u_1(t) &= e^{-\lambda t} \left[u_{10} + O\left(\delta^2\right) \right], & u_2(t) &= e^{-\lambda t} \left[u_{20} + O\left(\delta^2\right) \right], \\
v_1(t) &= e^{-\lambda(\tau-t)} \left[v_{1\tau} + O\left(\delta^2\right) \right], & v_2(t) &= e^{-\lambda(\tau-t)} \left[v_{2\tau} + O\left(\delta^2\right) \right].
\end{aligned}
\qquad (2.17)
$$

Lemma 2.8. *There exists* $M > 0$ *such that for any sufficiently small* $\delta > 0$, *and any* u_{10}, u_{20}, $v_{1\tau}$ *and* $v_{2\tau}$, *where* $\max\{|u_{10}|, |u_{20}|, |v_{1\tau}|, |v_{2\tau}|\} \leq \delta$, *the solution* $(u(t), v(t))$ *of system (2.8) that satisfies boundary condition (2.15) can be written as*

$$
\begin{aligned}
u_1(t) &= e^{-\lambda_1 t} u_{10} + \xi_1(x), & u_2(t) &= e^{-\lambda_2 t} u_{20} + \xi_2(x), \\
v_1(t) &= e^{-\lambda_1(\tau-t)} v_{1\tau} + \zeta_1(x), & v_2(t) &= e^{-\lambda_2(\tau-t)} v_{2\tau} + \zeta_2(x),
\end{aligned}
\qquad (2.18)
$$

where $x = (t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})$, $t \in [0, \tau]$, *and*

$$
\begin{aligned}
|\xi_1| &\leq M \left[e^{-\lambda_1 t} \delta |u_{10}| + e^{-\lambda_1(\tau-t)-\lambda_2 t} \delta |v_{1\tau}| \right], & |\xi_2| &\leq M e^{-\lambda_2 t} \delta^2, \\
|\zeta_1| &\leq M \left[e^{-\lambda_1(\tau-t)} \delta |v_{1\tau}| + e^{-\lambda_2(\tau-t)-\lambda_1 t} \delta |u_{10}| \right], & |\zeta_2| &\leq M e^{-\lambda_2(\tau-t)} \delta^2.
\end{aligned}
$$

We can also write

$$u_1(t) = e^{-\lambda_1 t} u_{10} [1 + O(\delta)] + e^{-\lambda_1(\tau-t)-\lambda_2 t} O(\delta v_{1\tau}), \qquad u_2(t) = e^{-\lambda_2 t} \left[u_{20} + O\left(\delta^2\right) \right],$$

$$v_1(t) = e^{-\lambda_1(\tau-t)} v_{1\tau} [1 + O(\delta)] + e^{-\lambda_2(\tau-t)-\lambda_1 t} O(\delta u_{10}), \quad v_2(t) = e^{-\lambda_2(\tau-t)} \left[v_{2\tau} + O\left(\delta^2\right) \right].$$

$$\tag{2.19}$$

Lemma 2.9. *There exists $M > 0$ such that for any sufficiently small $\delta > 0$, and any u_{10}, u_{20}, $v_{1\tau}$ and $v_{2\tau}$, where $\max\{|u_{10}|, |u_{20}|, |v_{1\tau}|, |v_{2\tau}|\} \leq \delta$, the solution $(u(t), v(t))$ of system (2.12) that satisfies boundary condition (2.15) can be written in the form (2.18), where $t \in [0, \tau]$ and*

$$|\xi_1| \leq M \left[e^{-\lambda_1 t} \delta |u_{10}| + e^{-\lambda_1(\tau+t)} \delta |v_{1\tau}| \right], \qquad\qquad |\xi_2| \leq M e^{-\lambda_2 t} \delta^2,$$

$$|\zeta_1| \leq M \left[e^{-\lambda_1(\tau-t)} \delta |v_{1\tau}| + e^{-\lambda_1(2\tau+t)} \delta |u_{10}| \right], \qquad |\zeta_2| \leq M e^{-\lambda_2(\tau-t)} \delta^2.$$

Remark 2.10. For simplicity, we can write the solution given by Lemma 2.9 as

$$u_1(t) = e^{-\lambda_1 t} u_{10} [1 + O(\delta)] + e^{-\lambda_1(\tau+t)} O(\delta v_{1\tau}), \qquad u_2(t) = e^{-\lambda_2 t} \left[u_{20} + O\left(\delta^2\right) \right],$$

$$v_1(t) = e^{-\lambda_1(\tau-t)} v_{1\tau} [1 + O(\delta)] + e^{-\lambda_1(2\tau-t)} O(\delta u_{10}), \quad v_2(t) = e^{-\lambda_2(\tau-t)} \left[v_{2\tau} + O\left(\delta^2\right) \right].$$

$$\tag{2.20}$$

2.4. Local maps and their properties

In this section, we use the results of the previous two sections to study the local maps for each of systems (2.4), (2.8) and (2.12). Recall (2.2) and write

$$T^{\text{loc}}(u_{10}, v_{10}) = (u_{1\tau}, v_{1\tau}) = \left(\eta_1(u_{10}, v_{10}), \eta_2(u_{10}, v_{10}) \right), \tag{2.21}$$

where η_1 and η_2 are some functions. In the previous section, for each of systems (2.4), (2.8) and (2.12), we have approximated the unique solution (u^*, v^*) which satisfies boundary conditions (2.15) (see Lemmas 2.7, 2.8 and 2.9). We write this solution as

$$
\begin{aligned}
u_1^*(t) &= u_1^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau}), \\
u_2^*(t) &= u_2^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau}), \\
v_1^*(t) &= v_1^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau}), \\
v_2^*(t) &= v_2^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau}),
\end{aligned}
\tag{2.22}
$$

to emphasize that it explicitly depends on t, τ, u_{10}, u_{20}, $v_{1\tau}$ and $v_{2\tau}$. This solution represents an orbit which at $t = 0$ is at the point $(u_{10}, u_{20}, v_{10}, v_{20})$ and at $t = \tau$ is at the point $(u_{1\tau}, u_{2\tau}, v_{1\tau}, v_{2\tau})$.

To study the map T^{loc}, we consider the case in which $u_{20} = v_{2\tau} = \delta$, i.e. the points $(u_{10}, u_{20}, v_{10}, v_{20})$ and $(u_{1\tau}, u_{2\tau}, v_{1\tau}, v_{2\tau})$ belong to Π^s and Π^u, respectively. Evaluating the first equation of (2.22) at $t = \tau$ and the last two equations of (2.22) at $t = 0$ gives

$$u_{1\tau} = u_1^* (\tau, \tau, u_{10}, \delta, v_{1\tau}, \delta),$$
$$v_{10} = v_1^* (0, \tau, u_{10}, \delta, v_{1\tau}, \delta), \tag{2.23}$$
$$v_{20} = v_2^* (0, \tau, u_{10}, \delta, v_{1\tau}, \delta),$$

which is an implicit relation between u_{10}, v_{10}, v_{20}, $u_{1\tau} = \eta_1 (u_{10}, v_{10})$, $v_{1\tau} = \eta_2 (u_{10}, v_{10})$ and τ. On the other hand, τ and v_{20} can be expressed as functions of (u_{10}, v_{10}). This allows us to approximate the functions η_1 and η_2.

Notation 2.11. Hereafter, we use the following notation: $\gamma = \frac{\lambda_1}{\lambda_2}$.

2.4.1. Choice of coordinates on the cross-sections

We point out here that we can always choose (u_1, v_1)- coordinate system on each of our cross-sections:

Lemma 2.12. *In each of the cases $\lambda_1 = \lambda_2$, $\lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$, for any arbitrary point (u_1, u_2, v_1, v_2) of each of the cross-sections Π^s, Π^u, Π_1^s, Π_1^u, Π_2^s, Π_2^u and Σ, the variables u_2 and v_2 are uniquely determined by (u_1, v_1).*

Proof. We only prove the statement for Π^s and Π^u. The proof for the other cross-sections is the same.

Consider system (2.4) and suppose $\lambda_1 = \lambda_2$. Let $(u_{10}, \delta, v_{10}, v_{20})$ and $(u_{1\tau}, u_{2\tau}, v_{1\tau}, \delta)$ be two points on Π^s and Π^u, respectively. By virtue of the relation $\{H = 0\}$, where H is as in (2.6), we have

$$v_{20} = \delta^{-1} u_{10} v_{10} \quad \text{and} \quad u_{2\tau} = \delta^{-1} u_{1\tau} v_{1\tau}. \tag{2.24}$$

This proves the lemma for the case $\lambda_1 = \lambda_2$.

A straightforward calculation (see [2]) shows that for the cases of systems (2.8) and (2.12), and their corresponding first integrals, we have $H_{v_2} (0, \delta, 0, 0) \neq 0$ and $H_{u_2} (0, 0, 0, \delta) \neq 0$. Then, the proof for the cases $\lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$ follows from the implicit function theorem. □

2.4.2. Local maps: case $\lambda_1 = \lambda_2$

We prove $\mathcal{D} = \emptyset$ by showing that (2.1) never holds. This implies that the Poincaré map along Γ cannot be defined when $\lambda_1 = \lambda_2$. This also proves Theorem A2 for the particular case of $\lambda_1 = \lambda_2$.

Let $\lambda = \lambda_1 = \lambda_2$ and consider the case $u_{20} = v_{2\tau} = \delta$. Evaluating the first two equations of (2.17) at $t = \tau$ and the last two equations at $t = 0$ gives

$$u_{1\tau} = e^{-\lambda\tau} \left[u_{10} + O\left(\delta^2\right) \right], \qquad u_{2\tau} = e^{-\lambda\tau} \delta \left[1 + O\left(\delta\right) \right],$$
$$v_{10} = e^{-\lambda\tau} \left[v_{1\tau} + O\left(\delta^2\right) \right], \qquad v_{20} = e^{-\lambda\tau} \delta \left[1 + O\left(\delta\right) \right]. \tag{2.25}$$

Substituting (2.24) into this relation gives $e^{-\lambda\tau} = \frac{u_{10} v_{10}}{\delta^2} \left[1 + O\left(\delta\right) \right]$. Therefore,

$$v_{1\tau} = e^{\lambda\tau} v_{10} + O\left(\delta^2\right) = \delta^2 u_{10}^{-1} \left[1 + O\left(\delta\right) \right] + O\left(\delta^2\right) = u_{10}^{-1} \delta^2 \left[1 + O\left(\delta\right) \right].$$

For a given sufficiently small δ, we have

$$\lim_{u_{10}\to 0} \|(u_{1\tau}, v_{1\tau})\| \geq \lim_{u_{10}\to 0} |v_{1\tau}| = \lim_{u_{10}\to 0} |u_{10}|^{-1}\delta^2 [1 + O(\delta)] = \infty.$$

This means that (2.1) does not hold when ϵ and ϵ_u are chosen sufficiently small. On the other hand, it is easily seen that the same happens for the points (u_{10}, v_{10}) in $\mathbb{D}$. The same also holds for the case of homoclinic figure-eight. Therefore,

Proposition 2.13. *When $\lambda_1 = \lambda_2$, we have $\mathcal{D} = \mathbb{D} = \mathcal{D}^1 = \mathbb{D}^1 = \mathcal{D}^2 = \mathbb{D}^2 = \emptyset$.*

2.4.3. *Local maps: case $\lambda_1 < \lambda_2 < 2\lambda_1$*
    Let $\lambda < \lambda_2 < 2\lambda_1$ and consider the case $u_{20} = v_{2\tau} = \delta$. Evaluating the first two equations of (2.19) at $t = \tau$ and the last two equations at $t = 0$ gives

$$u_{1\tau} = e^{-\lambda_1\tau} u_{10} [1 + O(\delta)] + e^{-\lambda_2\tau} O(\delta v_{1\tau}), \qquad u_{2\tau} = e^{-\lambda_2\tau}\delta [1 + O(\delta)],$$
$$v_{10} = e^{-\lambda_1\tau} v_{1\tau} [1 + O(\delta)] + e^{-\lambda_2\tau} O(\delta u_{10}), \qquad v_{20} = e^{-\lambda_2\tau}\delta [1 + O(\delta)]. \tag{2.26}$$

This, in particular, implies

$$v_{1\tau} = e^{\lambda_1\tau} v_{10} [1 + O(\delta)] + e^{(\lambda_1-\lambda_2)\tau} O(\delta u_{10}). \tag{2.27}$$

First integral (2.10) vanishes at $(u_{10}, \delta, v_{10}, v_{20}) \in \Pi^s$. Thus $v_{20} = \frac{\gamma}{\delta} \cdot u_{10}v_{10} [1 + o(1)]$. Therefore, (2.26) implies

$$e^{-\lambda_2\tau} = \gamma\delta^{-2} u_{10}v_{10} [1 + O(\delta)], \tag{2.28}$$

and therefore

$$e^{-\lambda_1\tau} = \left(\gamma\delta^{-2} u_{10}v_{10}\right)^\gamma [1 + O(\delta)]. \tag{2.29}$$

By these relations, we rewrite (2.27) as

$$\eta_2(u_{10}, v_{10}) = v_{1\tau} = e^{\lambda_1\tau} v_{10} [1 + O(\delta)]. \tag{2.30}$$

Substituting this into the equation of $u_{1\tau}$ in (2.26) gives

$$\eta_1(u_{10}, v_{10}) = u_{1\tau} = e^{-\lambda_1\tau} u_{10} [1 + O(\delta)] + e^{(\lambda_1-\lambda_2)\tau} O(\delta v_{10}). \tag{2.31}$$

Let us now explore the domain $\mathcal{D}$ of the map T^{loc}. By choosing δ sufficiently small such that $|O(\delta)| \leq 1$, we have

$$\left|\frac{u_{1\tau}}{v_{1\tau}}\right| \leq 2e^{-\lambda_2\tau} \frac{|u_{10}|}{|v_{10}|} + e^{-\lambda_2\tau} \leq \frac{4\gamma}{\delta^2}\left(u_{10}^2 + |u_{10}v_{10}|\right) \leq \frac{8\gamma}{\delta^2}\epsilon^2.$$

Therefore, for any given (fixed) sufficiently small δ, we have $u_{1\tau} = v_{1\tau} O(\epsilon^2)$. Thus,
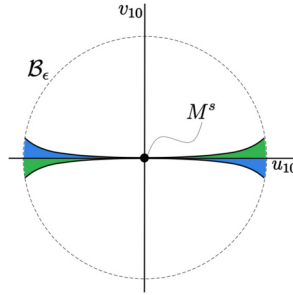
Fig. 8. The regions $\mathcal{D}$ and $\mathbb{D}$ for the case $\lambda_1 < \lambda_2 < 2\lambda_1$ are shown in green and blue, respectively. They are surrounded by horizontal axis, ϵ-ball $\mathcal{B}_\epsilon$ and the curves characterized by (2.32). Since $\gamma = \lambda_1 \lambda_2^{-1} > 0.5$, these curves are tangent to the horizontal axis at $M^s = (0,0)$.

$$\|(u_{1\tau}, v_{1\tau})\| = \sqrt{u_{1\tau}^2 + v_{1\tau}^2} = |v_{1\tau}|\left[1 + O\left(\epsilon^2\right)\right] = \left(\gamma \delta^{-2}\right)^{-\gamma} |u_{10}|^{-\gamma} |v_{10}|^{1-\gamma} \left[1 + O\left(\delta\right)\right].$$

Therefore, $\|(u_{1\tau}, v_{1\tau})\| < \epsilon_u$ if and only if

$$|v_{10}| < \epsilon_u^{\frac{1}{1-\gamma}} \left(\gamma \delta^{-2}\right)^{\frac{\gamma}{1-\gamma}} |u_{10}|^{\frac{\gamma}{1-\gamma}} \left[1 + O\left(\delta\right)\right], \qquad (\gamma = \lambda_1 \lambda_2^{-1} > 0.5). \tag{2.32}$$

By virtue of (2.28), we see that if $(u_{10}, v_{10}) \in \mathcal{D}$, then $u_{10}v_{10} > 0$. It is also easy to see that analogous statements hold for the points in $\mathbb{D}$. This gives:

Proposition 2.14. *Let $\lambda < \lambda_2 < 2\lambda_1$. For a given sufficiently small δ, we can choose ϵ and ϵ_u so that the domain $\mathcal{D}$ (resp. $\mathbb{D}$) becomes the set of all points (u_{10}, v_{10}) in Π^s such that $0 < u_{10}v_{10}$ (resp. $u_{10}v_{10} < 0$), $\|(u_{10}, v_{10})\| < \epsilon$ and (2.32) holds (see Fig. 8).*

Remark 2.15. *The case of homoclinic figure-eight is the same. Relation $u_{1\tau} = v_{1\tau} O\left(\epsilon^2\right)$ holds for any (u_{10}, v_{10}) on Π_i^s, and the domains $\mathcal{D}^i$ and $\mathbb{D}^i$ are given by Proposition 2.14 ($i = 1, 2$).*

2.4.4. Local maps: case $2\lambda_1 < \lambda_2$

Let $2\lambda_1 < \lambda_2$. Evaluating the first two equations of (2.20) at $t = \tau$ and the last two equations at $t = 0$ gives

$$u_{1\tau} = e^{-\lambda_1 \tau} u_{10} [1 + O(\delta)] + e^{-2\lambda_1 \tau} O(\delta v_{1\tau}), \qquad u_{2\tau} = e^{-\lambda_2 \tau} \left[u_{20} + O\left(\delta^2\right)\right],$$
$$v_{10} = e^{-\lambda_1 \tau} v_{1\tau} [1 + O(\delta)] + e^{-2\lambda_1 \tau} O(\delta u_{10}), \qquad v_{20} = e^{-\lambda_2 \tau} \left[v_{2\tau} + O\left(\delta^2\right)\right]. \tag{2.33}$$

For the particular case of $u_{20} = v_{2\tau} = \delta$, we have

$$u_{1\tau} = e^{-\lambda_1 \tau} u_{10} [1 + O(\delta)] + e^{-2\lambda_1 \tau} O(\delta |v_{1\tau}|), \qquad u_{2\tau} = e^{-\lambda_2 \tau} \delta [1 + O(\delta)],$$
$$v_{10} = e^{-\lambda_1 \tau} v_{1\tau} [1 + O(\delta)] + e^{-2\lambda_1 \tau} O(\delta |u_{10}|), \qquad v_{20} = e^{-\lambda_2 \tau} \delta [1 + O(\delta)]. \tag{2.34}$$

This, in particular, implies

$$\eta_2 (u_{10}, v_{10}) = v_{1\tau} = e^{\lambda_1 \tau} v_{10} [1 + O(\delta)] + e^{-\lambda_1 \tau} O(\delta |u_{10}|). \tag{2.35}$$
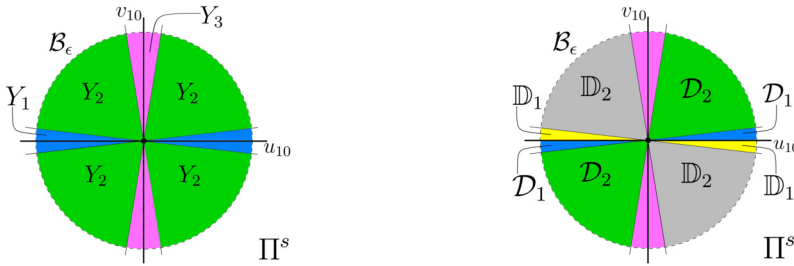
Fig. 9. We divide the ϵ-ball in Π^s centered at M^s into three disjoint regions: Y_1 (blue), Y_2 (green) and Y_3 (pink), shown in the left figure. To investigate the sets $\mathcal{D}$ and $\mathbb{D}$ when $\lambda_2 > 2\lambda_1$, we consider the intersection of each of these sets with the regions Y_1, Y_2 and Y_3. We then define $\mathcal{D}_i = \mathcal{D} \cap Y_i$ and $\mathbb{D}_i = \mathbb{D} \cap Y_i$. The regions $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathbb{D}_1$ and $\mathbb{D}_2$ are shown by blue, green, yellow and gray, respectively in the right figure. The sets $\mathcal{D}_3$ and $\mathbb{D}_3$ are subsets of the pink region.

Substituting this into the equation of $u_{1\tau}$ in (2.34) gives

$$\eta_1 (u_{10}, v_{10}) = u_{1\tau} = e^{-\lambda_1 \tau} u_{10} [1 + O(\delta)] + e^{-\lambda_1 \tau} O(\delta |v_{10}|). \tag{2.36}$$

Then, local map (2.21) maps (u_{10}, v_{10}) to $(u_{1\tau}, v_{1\tau})$, where $u_{1\tau}$ and $v_{1\tau}$ are as in (2.36) and (2.35), respectively, and τ is a function of (u_{10}, v_{10}). It is not as straightforward as the previous two cases to express τ as a function of (u_{10}, v_{10}). This is not straightforward either to find the domain $\mathcal{D}$ of T^{loc}. Below, we divide $\mathcal{D}$ into three regions (it is shown that $\mathcal{D} \neq \emptyset$) and study each region separately.

Let $\mathcal{B}_\epsilon$ be the ϵ-ball in Π^s centered at M^s. For a given $m > 1$ define

$$Y_1^m = \left\{ (u_{10}, v_{10}) \in \mathcal{B}_\epsilon : |v_{10}| < m^{-1} |u_{10}| \right\},$$

$$Y_2^m = \left\{ (u_{10}, v_{10}) \in \mathcal{B}_\epsilon : m^{-1} |u_{10}| \leq |v_{10}| \leq m |u_{10}| \right\} \tag{2.37}$$

$$Y_3^m = \{ (u_{10}, v_{10}) \in \mathcal{B}_\epsilon : m |u_{10}| < |v_{10}| \}$$

(see Fig. 9). Obviously, $\mathcal{B}_\epsilon = Y_1^m \cup Y_2^m \cup Y_3^m$. We define

$$\mathcal{D}_1^\epsilon := \mathcal{D} \cap Y_1^m, \qquad \mathcal{D}_2^\epsilon := \mathcal{D} \cap Y_2^m, \qquad \mathcal{D}_3^\epsilon := \mathcal{D} \cap Y_3^m. \tag{2.38}$$

Analogously, we define $\mathbb{D}_1^\epsilon := \mathbb{D} \cap Y_1^m$, $\mathbb{D}_2^\epsilon := \mathbb{D} \cap Y_2^m$ and $\mathbb{D}_3^\epsilon := \mathbb{D} \cap Y_3^m$. We may drop the subscript ϵ and m, when no confusion arises.

For $(u_{10}, v_{10}) \in Y_1^m \cup Y_2^m$, we have $|v_{10}| \leq m |u_{10}|$ and therefore $v_{10} = O(u_{10})$. By virtue of this relation and taking into account that first integral (2.13) vanishes at $(u_{10}, \delta, v_{10}, v_{20}) \in \Pi^s$, we derive

$$v_{20} = \gamma \delta^{-1} u_{10} v_{10} [1 + O(\delta)]. \tag{2.39}$$

This relation together with (2.34) implies that any point $(u_{10}, v_{10}) \in Y_1^m \cup Y_2^m$ reaches Π^u if $u_{10} v_{10} > 0$, and reaches Σ if $u_{10} v_{10} < 0$. Therefore, to find $\mathcal{D}_1 \cup \mathcal{D}_2$ ($\mathbb{D}_1 \cup \mathbb{D}_2$), it is sufficient to find the points in $Y_1^m \cup Y_2^m$ for which $\|(u_{1\tau}, v_{1\tau})\| < \epsilon_u$.

Like the preceding two cases, relation (2.39) yields (2.28) and (2.29). Let δ be sufficiently small. With (2.35), (2.36) and some straightforward calculation, we derive

$$|u_{1\tau}| \leq (2+\delta) e^{-\lambda_1 \tau} \epsilon, \qquad |v_{1\tau}| \leq \left[4m^{1-\gamma} \left(\delta^2 \gamma^{-1} \right)^\gamma + 1 \right] \epsilon^{1-2\gamma}.$$

This gives the following:

Proposition 2.16. *For given m, sufficiently small δ and sufficiently small ϵ_u, we can choose ϵ sufficiently small such that for $i = 1, 2$ we have*

$$\mathcal{D}_i = \{(u_{10}, v_{10}) \in Y_i^m, \quad u_{10}v_{10} > 0\}, \qquad \mathbb{D}_i = \{(u_{10}, v_{10}) \in Y_i^m, \quad u_{10}v_{10} < 0\}.$$

Now, consider $(u_{10}, v_{10}) \in Y_2^m \cup Y_3^m$. We have $|u_{10}| \leq m|v_{10}|$ and hence $u_{10} = O(v_{10})$. By virtue of this relation and relation (2.33), we obtain

$$v_{10} = e^{-\lambda_1 \tau} v_{1\tau} [1 + O(\delta)] \qquad \text{and} \qquad v_{1\tau} = e^{\lambda_1 \tau} v_{10} [1 + O(\delta)]. \tag{2.40}$$

This relation together with (2.33) gives

$$\frac{u_{1\tau}}{v_{1\tau}} = \frac{e^{-\lambda_1 \tau} u_{10} [1 + O(\delta)] + e^{-2\lambda_1 \tau} O(\delta v_{1\tau})}{v_{1\tau}} = e^{-2\lambda_1 \tau} \left[\frac{u_{10}}{v_{10}} + O(\delta) \right] = o(1), \tag{2.41}$$

which implies $u_{1\tau} = o(v_{1\tau})$. Thus, when $(u_{10}, v_{10}) \in Y_2^m \cup Y_3^m$, we have

$$\left\| (u_{1\tau}, v_{1\tau}) \right\| = \sqrt{u_{1\tau}^2 + v_{1\tau}^2} = |v_{1\tau}| [1 + o(1)].$$

Note that it was relation (2.39) that enabled us to, first, identify the points in Y_i^m for which $\|(u_{1\tau}, v_{1\tau})\| < \epsilon_u$ holds, and second, distinguish $\mathcal{D}_i$ from $\mathbb{D}_i$ for $i = 1, 2$. For the case of $i = 3$, we cannot deduce such a relation from first integral (2.13). However, as we see later, the dynamics on Y_3^m is quite simple and can be analyzed without knowing $\mathcal{D}_3$ and $\mathbb{D}_3$ precisely.

Meanwhile, we have shown the following

Proposition 2.17. *If $(u_{10}, v_{10}) \in Y_2^m$, then (2.39), (2.28), (2.29), (2.40), $u_{10} = O(v_{10})$ and $v_{10} = O(u_{10})$ hold. If $(u_{10}, v_{10}) \in Y_3^m$, then (2.41) and $u_{1\tau} = o(v_{1\tau})$ hold.*

3. Analysis near homoclinics and super-homoclinics

The purpose of this section is to study the dynamics near (single and figure-eight) homoclinic and super-homoclinic orbits. In particular, we prove in this section, all the theorems stated in the Introduction. In the first section below, we introduce some concepts and notations. The second and the third sections are dedicated to study the dynamics near a single homoclinic orbit. We prove Theorems A1, A2 and A3 in these two sections. The ideas and techniques which are used to prove these theorems are also used in the later sections. In the fourth section, we extend the results obtained for a single homoclinic to the case of the homoclinic figure-eight. The proofs of Theorems B1, B2 and B3 are provided in this section. Finally, we study the case of a super-homoclinic and prove Theorems C1 and C2 in the fifth (and the last) section.

3.1. Set-up and notations

Choose a sufficiently small $\delta > 0$ such that all the statements of the previous sections hold. Fix this δ. According to (2.3) and (1.8), for $(u_{10}, v_{10}) \in \mathcal{D} \subset \Pi^s$, we have

$$\begin{pmatrix} \overline{u}_{10} \\ \overline{v}_{10} \end{pmatrix} = T \begin{pmatrix} u_{10} \\ v_{10} \end{pmatrix} = \begin{pmatrix} [a + o\,(1)]\,u_{1\tau} + [b + o\,(1)]\,v_{1\tau} \\ [c + o\,(1)]\,u_{1\tau} + [d + o\,(1)]\,v_{1\tau} \end{pmatrix}, \tag{3.1}$$

where a, b, c and d are real constants (in fact, these coefficients are functions of δ but since δ is assumed to be fixed, we treat these coefficients as constants). Our job is to analyze this map for different values of a, b, c and d, and for each of the cases $\lambda_2 < 2\lambda_1$ and $2\lambda_1 < \lambda_2$. In this strand, we first introduce some notations:

Notation 3.1. Let $\mathcal{N} \subset \mathcal{M}$ be two arbitrary sets and $f : \mathcal{N} \to \mathcal{M}$ be an injective map. We denote the set of the points in $\mathcal{N}$ whose forward orbits lie entirely in $\mathcal{N}$ by $\Lambda^s_{\mathcal{N}, f}$ or $\Lambda^s_{\mathcal{N}}$, when no confusion arises. Indeed,

$$\Lambda^s_{\mathcal{N}, f} = \Lambda^s_{\mathcal{N}} = \{x \in \mathcal{N} : f^n\,(x) \in \mathcal{N}, \quad \forall n \geq 0\}.$$

We denote the set of the points in $\mathcal{N}$ whose backward orbits lie entirely in $\mathcal{N}$ by $\Lambda^u_{\mathcal{N}, f}$ or $\Lambda^u_{\mathcal{N}}$, when no confusion arises. Indeed,

$$\Lambda^u_{\mathcal{N}, f} = \Lambda^u_{\mathcal{N}} = \{x \in \mathcal{N} : \text{for all } n \geq 0, \quad f^{-n}\,(x) \text{ exists and belongs to } \mathcal{N}\}.$$

Remark 3.2. Recall $W^s_{\mathcal{U}}(O)$ and $W^u_{\mathcal{U}}(O)$ from the Introduction. Taking into account that $W^s_{\mathcal{U}}(O) \cap \Lambda^s_{\mathcal{D}, T} = \emptyset$ and $W^u_{\mathcal{U}}(O) \cap \Lambda^u_{\mathcal{D}, T} = \emptyset$, we can reformulate Theorem A1 as follows: The forward (resp. backward) orbit of any point on $\Lambda^s_{\mathcal{D}, T}$ (resp. $\Lambda^u_{\mathcal{D}, T}$) converges to the homoclinic orbit Γ.

Notation 3.3. Given a point (u_{10}, v_{10}) on a given cross-section, we denote the quantity $\frac{v_{10}}{u_{10}}$ (when $u_{10} \neq 0$) by $w(u_{10}, v_{10})$ or w. Consider the case $(u_{10}, v_{10}) \in \mathcal{D}$ and let $(\overline{u}_{10}, \overline{v}_{10}) \in \Pi^s$ be its image under the Poincaré map T. We denote the quantity $\frac{\overline{v}_{10}}{\overline{u}_{10}}$ (when $\overline{u}_{10} \neq 0$) by $\overline{w}\,(u_{10}, v_{10})$ or $\overline{w}$.

Notation 3.4. We denote the straight line $\{v_{10} = \frac{d}{b}u_{10}\}$ in Π^s by ℓ^*.

3.2. Dynamics near the homoclinic orbit Γ: case $\lambda_2 < 2\lambda_1$

Here, we show that when $\lambda_2 < 2\lambda_1$, any point in the domain $\mathcal{D}$ of the Poincaré map T leaves $\mathcal{D}$ by both forward and backward iterations of the Poincaré map. The proof of the case $\lambda_1 = \lambda_2$ directly follows from Proposition 2.13 in which we have shown that the domain $\mathcal{D}$ of the Poincaré map is empty. For the case of $\lambda_1 < \lambda_2 < 2\lambda_1$, we prove that the image of the domain $\mathcal{D}$ under the Poincaré map T has no intersection with $\mathcal{D}$ (see Fig. 10). We formalize this discussion in the following lemma:

Lemma 3.5. *When* $\lambda_2 < 2\lambda_1$, *we have* $\Lambda^s_{\mathcal{D}, T} = \Lambda^u_{\mathcal{D}, T} = \emptyset$.
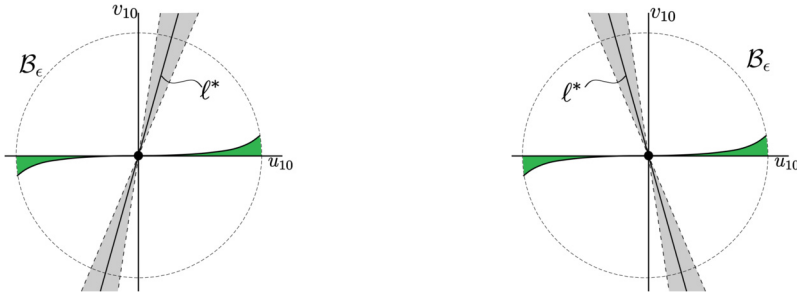
Fig. 10. Case $\lambda_1 < \lambda_2 < 2\lambda_1$: the domain $\mathcal{D}$ of the Poincaré map T is shown in green. The images of the points in $\mathcal{D}$ under the Poincaré map T accumulate near the straight line ℓ^* (the line whose slope is $\frac{d}{b}$) in the gray region. As it is shown, the green and the gray regions have no intersection which means $\mathcal{D} \cap T(\mathcal{D}) = \emptyset$. This implies that the backward and forward orbits of any point of the domain $\mathcal{D}$ leave $\mathcal{D}$. The left and right figures correspond to the cases $bd > 0$ and $bd < 0$, respectively.

Proof. When $\lambda_1 = \lambda_2$, the statement follows from Proposition 2.13.

Suppose $\lambda_1 < \lambda_2 < 2\lambda_1$. By Proposition 2.14, the domain $\mathcal{D}$ of the Poincaré map is

$$\{(u_{10}, v_{10}) \in \Pi^s : u_{10}v_{10} > 0, \quad \|(u_{10}, v_{10})\| < \epsilon, \quad |v_{10}| < K_{\epsilon_u}|u_{10}|^{\frac{\gamma}{1-\gamma}}[1 + O(\delta)]\},$$

where $\frac{1}{2} < \gamma = \frac{\lambda_1}{\lambda_2} < 1$ and K_{ϵ_u} is some constant (see (2.32)). Since $u_{1\tau} = v_{1\tau}O(\epsilon^2)$, Poincaré map (3.1) can be written as

$$(\overline{u}_{10}, \overline{v}_{10}) = \left(\left[b + O(\epsilon^2)\right]v_{1\tau}, \left[d + O(\epsilon^2)\right]v_{1\tau}\right),$$

which implies $\overline{w} = \frac{d}{b} + O(\epsilon^2)$. This means that the images of the points in the domain $\mathcal{D}$ under the Poincaré map T accumulate near ℓ^*. However, for a fixed δ and a sufficiently small ϵ, this line has no intersection with the domain $\mathcal{D}$ (see Fig. 10). This implies $\Lambda^s_{\mathcal{D},T} = \Lambda^u_{\mathcal{D},T} = \emptyset$, as desired. $\square$

We can now prove Theorems A1 (case $\lambda_2 < 2\lambda_1$) and A2. The proof of Theorem A1 for the case $2\lambda_1 < \lambda_2$ is provided in the next section.

Proof of Theorem A1: case $\lambda_2 < 2\lambda_1$. The proof is an immediate consequence of Remark 3.2 and Lemma 3.5. $\square$

Proof of Theorem A2. Any orbit in $W^s_{loc}(\Gamma)$ other than Γ must intersect Π^s at $\Lambda^s_{\mathcal{D},T}$. However, by Lemma 3.5, we have $\Lambda^s_{\mathcal{D},T} = \emptyset$. This implies $W^s_{loc}(\Gamma) = \Gamma$. The proof of $W^u_{loc}(\Gamma) = \Gamma$ is the same. $\square$

3.3. Dynamics near the homoclinic orbit Γ: case $2\lambda_1 < \lambda_2$

In this section, we study the dynamics near the homoclinic orbit Γ for the case $2\lambda_1 < \lambda_2$, and prove Theorems A1 (case $2\lambda_1 < \lambda_2$) and A3.

Recall from Section 2.4.4 that when $2\lambda_1 < \lambda_2$, we divide the domain $\mathcal{D}$ of the Poincaré map T into three subsets $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, i.e. $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ (see Fig. 11). In order to understand the
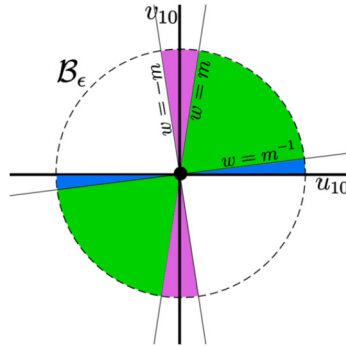
Fig. 11. When $2\lambda_1 < \lambda_2$, we write the domain $\mathcal{D}$ of the Poincaré map T as the disjoint union of three subsets $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$, i.e. $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. The subset $\mathcal{D}_1$ is shown in blue and $\mathcal{D}_2$ is shown in green. The set $\mathcal{D}_3$ is a subset of the purple region.

dynamics near the homoclinic loop Γ, we need to investigate the set of the points on the domain $\mathcal{D}$ whose forward or backward orbits (under the iterations of the Poincaré map T) lie in $\mathcal{D}$, i.e. the sets $\Lambda^s_{\mathcal{D},T}$ and $\Lambda^u_{\mathcal{D},T}$ (see Notation 3.1). To this end, we take the following three steps:

- *Step 1*: Investigating the set of the points in $\mathcal{D}_2 \cup \mathcal{D}_3$ whose forward or backward orbits lie entirely in $\mathcal{D}_2 \cup \mathcal{D}_3$, i.e. the sets $\Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ and $\Lambda^u_{\mathcal{D}_2\cup\mathcal{D}_3,T}$.
- *Step 2*: Investigating the set of the points in $\mathcal{D}_1$ whose forward or backward orbits lie entirely in $\mathcal{D}_1$, i.e. the sets $\Lambda^s_{\mathcal{D}_1,T}$ and $\Lambda^u_{\mathcal{D}_1,T}$.

Obviously, $\Lambda^s_{\mathcal{D}_1,T}$ and $\Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ are subsets of $\Lambda^s_{\mathcal{D},T}$. In addition, $\Lambda^u_{\mathcal{D}_1,T}$ and $\Lambda^u_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ are subsets of $\Lambda^u_{\mathcal{D},T}$. In the third step, we show that the reverse directions also hold: $\Lambda^s_{\mathcal{D},T} \subset \Lambda^s_{\mathcal{D}_1,T} \cup \Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ and $\Lambda^u_{\mathcal{D},T} \subset \Lambda^u_{\mathcal{D}_1,T} \cup \Lambda^u_{\mathcal{D}_2\cup\mathcal{D}_3,T}$. Equivalently,

- *Step 3*: We show $\Lambda^s_{\mathcal{D},T} = \Lambda^s_{\mathcal{D}_1,T} \cup \Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ and $\Lambda^u_{\mathcal{D},T} = \Lambda^u_{\mathcal{D}_1,T} \cup \Lambda^u_{\mathcal{D}_2\cup\mathcal{D}_3,T}$.

Notice that the statement of Step 3 is not trivial. In fact, at the first stage, one can consider the possibility of the existence of a point $x \in \mathcal{D}$ such that its forward orbit lies entirely in $\mathcal{D}$, i.e. $x \in \Lambda^s_{\mathcal{D},T}$, but it does not lie entirely in only one of the sets $\mathcal{D}_1$ or $\mathcal{D}_2 \cup \mathcal{D}_3$, i.e. $x \notin \Lambda^s_{\mathcal{D}_1,T}$ and $x \notin \Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$. In other words, the forward orbit of x stays in $\mathcal{D}$ but switches between $\mathcal{D}_1$ and $\mathcal{D}_2 \cup \mathcal{D}_3$. In Step 3, we indeed show that this scenario does not happen.

We take Step 1 in the following lemma. This lemma helps us to understand the dynamics of the Poincaré map T on the set $\mathcal{D}_2 \cup \mathcal{D}_3$. We explore in this lemma how T behaves on this set, with which rate the orbits of this set grow, and how $\Lambda^s_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ and $\Lambda^u_{\mathcal{D}_2\cup\mathcal{D}_3,T}$ look like. From a technical point of view, part (vii) of this lemma which shows the existence of the unstable manifold of the Poincaré map T is the main result of this section. The techniques which are used in the proof of this part are also used in Section 3.4 for the proof of the existence of the unstable manifold of the homoclinic figure-eight. We prove Lemma 3.6 in Section 3.3.1.

Lemma 3.6. *Let w and ℓ^* be as in Notations 3.3 and 3.4, respectively. Assume $2\lambda_1 < \lambda_2$ and consider $(u_{10}, v_{10}) \in \mathcal{D}_2 \cup \mathcal{D}_3$. Then*
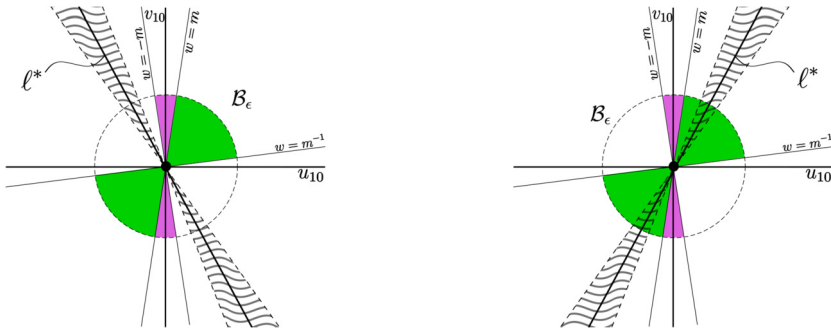
Fig. 12. The straight line whose slope is $\frac{d}{b}$ is denoted by ℓ^*. The left figure corresponds to the case $bd < 0$ and the right one corresponds to the case $bd > 0$. The set $\mathcal{D}_2$ is shown in green. The set $\mathcal{D}_3$ is a subset of the purple region. The image of $\mathcal{D}_2 \cup \mathcal{D}_3$ under the Poincaré map, i.e. $T(\mathcal{D}_2 \cup \mathcal{D}_3)$, is a subset of the wavy region. Informally speaking, the Poincaré map T *preserves* and *expands* the region $\mathcal{D}_2 \cup \mathcal{D}_3$. We show in Lemma 3.6 that when $bd > 0$, there exists an unstable invariant manifold for the Poincaré map T in the wavy region, tangent to ℓ^* at M^s.

   (i) $\overline{w} = w\left(T(u_{10}, v_{10})\right) = \frac{d}{b} + o(1)$, where $o(1)$ stands for a function of (u_{10}, v_{10}) that converges to zero as $(u_{10}, v_{10}) \to (0, 0)$.

  (ii) There exists a constant $C > 0$ such that $\|(u_{10}, v_{10})\|^{1-2\gamma} < C\|T(u_{10}, v_{10})\|$ holds for arbitrary (u_{10}, v_{10}), where $\gamma = \lambda_1 \lambda_2^{-1} < 0.5$.

 (iii) If $bd > 0$, then $T(u_{10}, v_{10})$ lies in $\mathcal{D}_2$ unless it leaves $\mathcal{B}_\epsilon$.

 (iv) $\Lambda^s_{\mathcal{D}_2 \cup \mathcal{D}_3, T} = \emptyset$.

  (v) $\Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T} = \Lambda^u_{\mathcal{D}_2, T}$.

 (vi) When $bd < 0$ we have $\Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T} = \emptyset$.

(vii) When $bd > 0$, the set $\{M^s\} \cup \Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T}$ is a one-dimensional $\mathcal{C}^1$-manifold which is tangent to ℓ^* at M^s.

It follows from this lemma that the image of $\mathcal{D}_2 \cup \mathcal{D}_3$ under the Poincaré map T lies near ℓ^*, and the Poincaré map increases the norm of any point of this set. Informally speaking, for the particular case of $bd > 0$, this means that the Poincaré map T *preserves* and *expands* the region $\mathcal{D}_2 \cup \mathcal{D}_3$. A geometrical picture of this behavior is illustrated in Fig. 12.

We now take the second step in the next lemma. In this lemma, we study the dynamics of T^{-1} on the set $\mathcal{D}_1$. Most of the statements of the following lemma are analogous to the statements of the preceding lemma. This is not a coincidence. In fact, we see later in the proof of Lemma 3.7 that the dynamics of T^{-1} on $\mathcal{D}_1$ can be obtained from the dynamics of T on $\mathcal{D}_2 \cup \mathcal{D}_3$ by a permutation and reversion of time. The proof of this lemma is postponed to Section 3.3.2.

Lemma 3.7. *Let $2\lambda_1 < \lambda_2$ and $(u_{10}, v_{10}) \in \mathcal{D}_1$.*

   (i) If $cd > 0$, then $T(\mathcal{D}) \cap \mathcal{D}_1 = \emptyset$.

  (ii) If $cd < 0$, then $w\left(T^{-1}(u_{10}, v_{10})\right) = o(1)$, where $o(1)$ stands for a function of (u_{10}, v_{10}) that converges to zero as $(u_{10}, v_{10}) \to (0, 0)$. In other words, $T^{-1}(\mathcal{D}_1)$ accumulates near the horizontal axis.

 (iii) If $cd < 0$, then $\|(u_{10}, v_{10})\|^{1-2\gamma} < C\|T^{-1}(u_{10}, v_{10})\|$ for some constant $C > 0$.

 (iv) If $cd < 0$, then $T^{-1}(u_{10}, v_{10})$ remains in $\mathcal{D}_1$ unless it leaves $\mathcal{B}_\epsilon$.

  (v) $\Lambda^s_{\mathcal{D}_1, T^{-1}} = \emptyset$. Equivalently, $\Lambda^u_{\mathcal{D}_1, T} = \emptyset$.

(vi) *If $cd < 0$, then the set $\{M^s\} \cup \Lambda^u_{\mathcal{D}_1, T^{-1}}$ (equivalently, the set $\{M^s\} \cup \Lambda^s_{\mathcal{D}_1, T}$) is a one-dimensional $\mathcal{C}^1$-manifold which is tangent to the horizontal axis at M^s.*

In the preceding two lemmas, we have shown that the sets $\Lambda^s_{\mathcal{D}_2 \cup \mathcal{D}_3, T}$ and $\Lambda^u_{\mathcal{D}_1, T}$ are always empty. It was also shown that $\Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T} = \Lambda^u_{\mathcal{D}_2, T}$. This allows us to reformulate Step 3 as in the following lemma:

Lemma 3.8. *(i) $\Lambda^s_{\mathcal{D}, T} = \Lambda^s_{\mathcal{D}_1, T}$.*   *(ii) $\Lambda^u_{\mathcal{D}, T} = \Lambda^u_{\mathcal{D}_2, T}$.*

Proof. Let $x \in \mathcal{D}_2 \cup \mathcal{D}_3$. It follows from parts (i), (ii) and (iii) of Lemma 3.6 that if $bd < 0$, then $T(x) \notin \mathcal{D}$, and if $bd > 0$, then for some k, $T^k(x) \notin \mathcal{B}_\epsilon$. Thus, any point in $\Lambda^s_{\mathcal{D}, T}$ must belong to $\mathcal{D}_1$. This proves part (i).

To prove part (ii), notice that if $\Lambda^u_{\mathcal{D}, T} = \emptyset$, then $\Lambda^u_{\mathcal{D}_2, T} = \emptyset$ and therefore $\Lambda^u_{\mathcal{D}, T} = \Lambda^u_{\mathcal{D}_2, T}$. So we assume that $\Lambda^u_{\mathcal{D}, T}$ is non-empty. Let $x \in \Lambda^u_{\mathcal{D}, T}$. We need to show $x \in \mathcal{D}_2$. To do this, we first prove $x \notin \mathcal{D}_1$. Assume the contrary, i.e. $x \in \mathcal{D}_1$. It follows from parts (i) and (iii) of Lemma 3.6 that if $T^{-1}(x) \in \mathcal{D}_2 \cup \mathcal{D}_3$, then $x = T\left(T^{-1}(x)\right)$ either belongs to $\mathcal{D}_2$ or lies outside the domain $\mathcal{D}$ which contradicts the assumption $x \in \mathcal{D}_1$. Therefore, $T^{-1}(x) \notin \mathcal{D}_2 \cup \mathcal{D}_3$, and so $T^{-1}(x) \in \mathcal{D}_1$. By virtue of part (i) of Lemma 3.7, this relation implies $cd < 0$. On the other hand, when $cd < 0$, it follows from parts (iii) and (iv) of Lemma 3.7 that there exists a $k > 0$ such that $T^{-k}(x) \notin \mathcal{B}_\epsilon$ and hence $T^{-k}(x) \notin \mathcal{D}$. This contradicts the preliminary assumption $x \in \Lambda^u_{\mathcal{D}, T}$. Therefore, if $x \in \Lambda^u_{\mathcal{D}, T}$, then $x \notin \mathcal{D}_1$, or equivalently, $T^{-n}(x) \notin \mathcal{D}_1$ for all $n \geq 0$.

To finish the proof, it is sufficient to show that $x \notin \mathcal{D}_3$. Assume the contrary, i.e. $x \in \mathcal{D}_3$. Since $x \in \Lambda^u_{\mathcal{D}, T}$ implies $T^{-n}(x) \notin \mathcal{D}_1$ for all $n \geq 0$, we have $T^{-1}(x) \notin \mathcal{D}_1$. On the other hand, parts (i) and (iii) of Lemma 3.6 imply that if $T^{-1}(x) \in \mathcal{D}_2 \cup \mathcal{D}_3$, then $x = T\left(T^{-1}(x)\right)$ either belongs to $\mathcal{D}_2$ or lies outside the domain $\mathcal{D}$ which contradicts the assumption $x \in \mathcal{D}_3$. Therefore, $x \notin \mathcal{D}_3$, as desired. $\square$

Recall that the local stable (unstable) set of the homoclinic loop Γ, denoted by $W^s_{\text{loc}}(\Gamma)$ $(W^u_{\text{loc}}(\Gamma))$, is the union of Γ itself and the set of the points in a sufficiently small neighborhood $\mathcal{U}$ of Γ whose forward (backward) orbits lie in $\mathcal{U}$ and their ω-limit sets (α-limit sets) coincide with $\Gamma \cup \{O\}$. By this definition, the intersection of $W^s_{\text{loc}}(\Gamma)$ and Π^s must belong to $\{M^s\} \cup \Lambda^s_{\mathcal{D}}$, and the intersection of $W^u_{\text{loc}}(\Gamma)$ and Π^s must belong to $\{M^s\} \cup \Lambda^u_{\mathcal{D}}$. On the other hand, we have shown in the above lemmas that when $\Lambda^s_{\mathcal{D}}$ ($\Lambda^u_{\mathcal{D}}$) is non-empty, any point on this set converges to M^s by the forward (backward) iterations of the Poincaré map T. This leads to the following:

Proposition 3.9. *Let ϕ_t be the flow of system (2.12). Then*

$$W^s_{loc}(\Gamma) = \Gamma \cup \phi_t \left(\Lambda^s_{\mathcal{D}, T} \right) \text{ for } t \geq 0, \quad \text{and} \quad W^u_{loc}(\Gamma) = \Gamma \cup \phi_t \left(\Lambda^u_{\mathcal{D}, T} \right) \text{ for } t \leq 0.$$

In system (2.12), the local unstable invariant manifold of the equilibrium O is straightened, i.e. $W^u_{loc}(O) = \{u = 0\}$. Thus, the intersection of this manifold and the cross-section $\Pi^u = \{v_2 = \delta\} \cap \{H = 0\}$ is the straight line $\{u_1 = 0\}$, i.e. v_1-axis. Consider the restriction of this line to a small neighborhood of M^u (in Fig. 13, it is shown by blue color on Π^u). The global map T^{glo} maps this restricted piece to a curve, denote it by γ^u, on Π^s (shown by blue color on Π^s in Fig. 13). This curve is in fact at the intersection of the global unstable invariant manifold of O
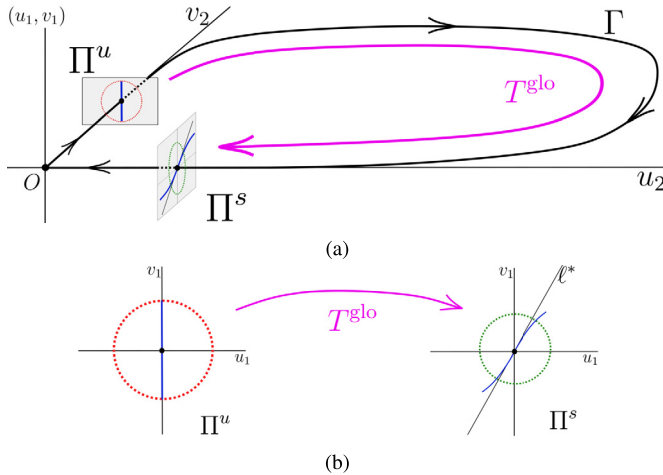
Fig. 13. The local unstable invariant manifold of the equilibrium O intersects Π^u at the v_1-axis. Thus, the blue curve (v_1-axis restricted to a small neighborhood of M^u in Π^u) lies at the intersection of the local unstable invariant manifold of O and the cross-section Π^u. This curve is mapped to the blue curve on Π^s by T^{glo} which means that the blue curve on Π^s lies in $W^u_{\mathrm{glo}}(O) \cap \Pi^s$. Since v_1-axis on Π^u is mapped to ℓ^* on Π^s by dT^{glo}, the straight line ℓ^* is tangent to the blue curve on Π^s at M^s.

and the cross-section Π^s. Since T^{glo} is a diffeomorphism and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is tangent to v_1-axis at M^u, the vector $dT^{\mathrm{glo}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$ is tangent to γ^u at M^s in Π^s, i.e. γ^u is tangent to ℓ^* at M^s (recall that ℓ^* is the line in Π^s whose slope is $\frac{d}{b}$). Therefore, it follows from Lemma 3.8 and part (vii) of Lemma 3.6 that when $bd > 0$, $W^s_{\mathrm{glo}}(O) \cap \Pi^s$ and $\{M^s\} \cup \Lambda^u_{\mathcal{D},T}$ are tangent at M^s. On the other hand, it follows from Lemma 3.8 and part (vi) of Lemma 3.7 that when $cd < 0$ the intersection of the local stable manifold of O and the cross-section Π^s, i.e. the horizontal axis, is tangent to $\{M^s\} \cup \Lambda^s_{\mathcal{D},T}$ at M^s. Moreover, by Assumption 4, the homoclinic orbit Γ is at the transverse intersection of the global stable and unstable invariant manifolds of the equilibrium O. Therefore, the intersection of these two manifolds with the cross-section Π^s, i.e. the horizontal axis and the curve γ^u, intersect transversely at M^s. Since γ^u is tangent to ℓ^* at M^s, we have that the intersection of $W^s_{\mathrm{glo}}(O)$ and $W^u_{\mathrm{glo}}(O)$ at Γ is transverse if and only if the horizontal axis on Π^s and the straight line ℓ^* are distinct. These statements give

Proposition 3.10.

(i) *When $bd > 0$, the 2-dimensional $\mathcal{C}^1$-smooth invariant manifold $W^u_{loc}(\Gamma)$ is tangent to $W^u_{glo}(O)$ at every point of Γ.*

(ii) *When $cd < 0$, the 2-dimensional $\mathcal{C}^1$-smooth invariant manifold $W^s_{loc}(\Gamma)$ is tangent to $W^s_{glo}(O)$ at every point of Γ.*

(iii) *The intersection of $W^s_{glo}(O)$ and $W^u_{glo}(O)$ at Γ is transverse if and only if $d \neq 0$.*

By virtue of the above results, we can prove Theorems A1 (case $2\lambda_1 < \lambda_2$) and A3:

Proof of Theorem A1: case $2\lambda_1 < \lambda_2$. The proof is an immediate consequence of Remark 3.2 and Lemmas 3.6, 3.7 and 3.8.   □

Proof of Theorem A3. By Proposition 3.9 and the preceding Lemmas we have that $W^u_{\text{loc}}(\Gamma) = \Gamma$ when $bd < 0$, and $W^s_{\text{loc}}(\Gamma) = \Gamma$ when $cd > 0$. The rest of the theorem is already proved (see Proposition 3.10).   □

The following remark suggests an alternative formulation of Theorem A3:

Remark 3.11. Consider the global stable and unstable invariant manifolds of the equilibrium O of system (2.12). Let γ^u (resp. γ^s) be a curve at the intersection of the global unstable (resp. stable) invariant manifold of O and the cross-section Π^s (resp. Π^u) which passes through M^s (resp. M^u). Following the discussion above, the slope of the tangent line to the curve γ^u at M^s is $\frac{d}{b}$. Moreover, the slope of the tangent line to the curve γ^s at M^u is $\frac{-c}{d}$. This suggests an alternative way to detect $\text{sgn}(bd)$ and $\text{sgn}(cd)$ which are required in the statement of Theorem A3. Indeed, instead of computing the coefficients a, b, c and d in Theorem A3, one can look at the slopes of the intersection curves of the global stable and unstable invariant manifolds of the equilibrium O with the cross-sections Π^u and Π^s at the points M^u and M^s.

3.3.1. Proof of Lemma 3.6

Proof of part (i). By (2.41) and Proposition 2.17, $(u_{10}, v_{10}) \in \mathcal{D}_2 \cup \mathcal{D}_3$ implies $u_{1\tau} = o(v_{1\tau})$. Thus, Poincaré map (3.1) takes the form

$$(\overline{u}_{10}, \overline{v}_{10}) = \Big([b + o(1)] v_{1\tau}, [d + o(1)] v_{1\tau} \Big), \tag{3.2}$$

which implies $\overline{w} = \frac{d}{b} + o(1)$.   □

Proof of part (ii). For $(u_{10}, v_{10}) \in \mathcal{D}_2$, relations (2.29) and (3.2) imply

$$\| T(u_{10}, v_{10}) \| = \| (\overline{u}_{10}, \overline{v}_{10}) \| = \Big[b^2 + d^2 + o(1) \Big]^{\frac{1}{2}} |v_{1\tau}| = K |u_{10}|^{-\gamma} |v_{10}|^{1-\gamma},$$

where $K = K(u_{10}, v_{10}) = \gamma^{-\gamma} \delta^{2\gamma} \sqrt{b^2 + d^2 + o(1)}$. For $C > K^{-1} m^\gamma (1+m)^{\frac{1}{2}}$, we have

$$\frac{\| (u_{10}, v_{10}) \|}{\| T(u_{10}, v_{10}) \|} = \frac{|v_{10}| \sqrt{1 + |\frac{u_{10}}{v_{10}}|}}{K |u_{10}|^{-\gamma} |v_{10}|^{1-\gamma}} \leq K^{-1} m^\gamma (1+m)^{\frac{1}{2}} |v_{10}|^{2\gamma} < C \| (u_{10}, v_{10}) \|^{2\gamma},$$

as desired.   □

Proof of part (iii). By part (i) of Lemma 3.6, $T(u_{10}, v_{10})$ is somewhere close to the line ℓ^* and since, for $bd > 0$, the restriction of $\ell^* \setminus \{M^s\}$ to $\mathcal{B}_\epsilon$ lies in $\mathcal{D}_2$ we have that if $T(u_{10}, v_{10})$ lies in $\mathcal{B}_\epsilon$, then it must belong to $\mathcal{D}_2$.   □

Proofs of parts (iv), (v) and (vi). All are easy consequences of the previous parts.   □

Proof of part (vii). The proof is based on a theory of invariant manifolds for cross-maps (see [23] and [13]). An introduction to this theory is provided in Appendix D.
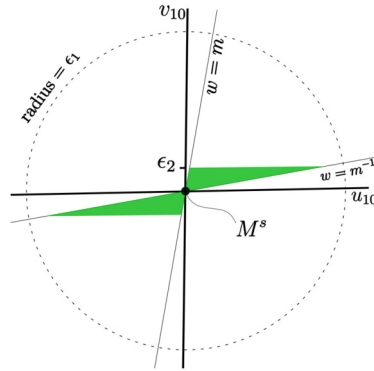
Fig. 14. The set $\mathcal{X} \subset \mathcal{D}_2^{\epsilon_1}$ in (u_{10}, v_{10})-plane is shown by green color.

Fig. 15. The set $\mathcal{Y}$ (the set $\mathcal{X}$ equipped with (w, z)-coordinates) is shown by green color. It contains two connected components (below and above the horizontal axis).

Consider $\mathcal{D}_2^{\epsilon_1}$ for a sufficiently small $\epsilon_1 > 0$ (see (2.38)). Choose $\epsilon_2 < \epsilon_1$ such that $\mathcal{X} \subset \mathcal{D}_2^{\epsilon_1}$, where $\mathcal{X} = \{(u_{10}, v_{10}) \in \Pi^s : m^{-1} \leq \frac{v_{10}}{u_{10}} \leq m, \ u_{10} \neq 0, \ |v_{10}| \leq \epsilon_2\}$ and m is as in (2.37) (see Fig. 14). Recall w in Notation 3.3 and define the new variable z by

$$z = z(u_{10}, v_{10}) = \operatorname{sgn}(v_{10}) |v_{10}|^{\alpha}, \qquad (0 < \alpha \text{ will be specified later}). \tag{3.3}$$

Let $\mathcal{Y}$ be the set $\mathcal{X}$ equipped with (w, z)-coordinates. Thus, $\mathcal{Y} = [m^{-1}, m] \times ([-\epsilon_2^{\alpha}, \epsilon_2^{\alpha}] \setminus \{0\})$ (see Fig. 15). Consider the restriction of the Poincaré map T to the set $\mathcal{X}$, i.e. $T|_{\mathcal{X}}$, and denote the representation of this map in (w, z)-coordinates by $\mathcal{T}$. We write

$$\mathcal{T} : (w, z) \mapsto (\overline{w}, \overline{z}) = (f(w, z), g(w, z)), \tag{3.4}$$

for some smooth functions f and g defined on $\mathcal{Y}$. Note that by (2.29) and the relation $\overline{z} = g(w, z) = \operatorname{sgn}(\overline{v}_{10}) |\overline{v}_{10}|^{\alpha}$, we can derive

$$\overline{z} = g(w, z) = \operatorname{sgn}(dz) |d|^{\alpha} \left(\frac{\gamma}{\delta^2}\right)^{-\gamma\alpha} w^{\gamma\alpha} |z|^{1-2\gamma} [1 + O(\delta)] = O\left(|z|^{1-2\gamma}\right) \tag{3.5}$$

and

$$z = O\left(|\overline{z}|^{\frac{1}{1-2\gamma}}\right). \tag{3.6}$$

We now make a statement which is proved in Appendix C:

Lemma 3.12. $g_z(w, z)$ is non-zero for any $(w, z) \in \mathcal{Y}$.

According to this lemma and the implicit function theorem, the variable z is a $\mathcal{C}^q$-smooth (q is as in Lemma 2.5) function of $(w, \overline{z})$ for $w \in \left[m^{-1}, m\right]$ and $\overline{z} \in g\left(\mathcal{Y}\right)$. Denote this function by G. Regarding the domain of this function, note that not every $(w, \overline{z})$ necessarily belongs to the domain of G. In other words, for an arbitrary $(w, \overline{z})$, there might not exist $z \in [-\epsilon_2^\alpha, \epsilon_2^\alpha] \setminus \{0\}$ such that $z = G\left(w, \overline{z}\right)$. However, by (3.5), this relation holds if $\overline{z}$ is chosen sufficiently small, i.e. for a sufficiently small $\theta > 0$ we have

$$\left[m^{-1}, m\right] \times ([-\theta, \theta] \setminus \{0\}) \subset \text{domain}\,(G).$$

Denote this set by $\mathcal{R}$, i.e. $\mathcal{R} = \left[m^{-1}, m\right] \times ([-\theta, \theta] \setminus \{0\})$. Without loss of generality, assume $\theta < \epsilon_2^\alpha$. Having the function G in hand means that we can write the Poincaré map $\mathcal{T}$ in cross-form: we define the cross-map $\mathcal{T}^\times : (w, \overline{z}) \mapsto (\overline{w}, z)$ by

$$(\overline{w}, z) = (F\left(w, \overline{z}\right), G\left(w, \overline{z}\right)), \qquad \text{where} \quad F\left(w, \overline{z}\right) = f\left(w, G\left(w, \overline{z}\right)\right), \qquad (3.7)$$

and $(w, \overline{z}) \in \text{domain}\,(G)$. It follows from part (i) of Lemma 3.6 (proved earlier), relation (3.6) and the fact that $z = 0$ if and only if $\overline{z} = 0$ (follows from (3.5)) that $\mathcal{T}^\times\left(\mathcal{R}\right) \subset \mathcal{R}$. Hereafter, we focus on the restriction of $\mathcal{T}^\times$ on $\mathcal{R}$. Our approach to prove the existence of the desired invariant manifold for the Poincaré map $\mathcal{T}$ is to apply Theorem 8 (see Appendix D) on the cross-map $\mathcal{T}^\times$. However, to do this, there are two issues that we need to take care of. The first is that the domain $\mathcal{R}$ does not satisfy the assumption of Theorem 8 (in that proposition, the domain must be written as a Cartesian product of two convex closed sets but $\mathcal{R}$ is not of this form since it does not contain the line $\overline{z} = 0$). Second, we need to compute the partial derivatives of the cross-map $\mathcal{T}^\times$. The second issue is resolved by the following lemma:

Lemma 3.13. *Let $\beta = \alpha^{-1} \min\{4\gamma, 1 - 2\gamma\}$. We have*

$$F_w\left(w, \overline{z}\right) = O\left(|\overline{z}|^{\frac{\beta}{1-2\gamma}}\right), \qquad F_{\overline{z}}\left(w, \overline{z}\right) = O\left(|\overline{z}|^{\frac{\beta - 1 + 2\gamma}{1-2\gamma}}\right),$$

$$G_w\left(w, \overline{z}\right) = O\left(|\overline{z}|^{\frac{1}{1-2\gamma}}\right), \qquad G_{\overline{z}}\left(w, \overline{z}\right) = O\left(|\overline{z}|^{\frac{2\gamma}{1-2\gamma}}\right).$$

This lemma is proved in Appendix C. We now extend the domain $\mathcal{R}$ to $\widetilde{\mathcal{R}}$, where $\widetilde{\mathcal{R}} = \left[m^{-1}, m\right] \times [-\theta, \theta]$. We also extend the map $\mathcal{T}^\times$ to the map $\widetilde{\mathcal{T}}^\times$ defined on $\widetilde{\mathcal{R}}$ by

$$\widetilde{\mathcal{T}}^\times\left(w, \overline{z}\right) := \begin{cases} \mathcal{T}^\times\left(w, \overline{z}\right) = (F\left(w, \overline{z}\right), G\left(w, \overline{z}\right)) & \overline{z} \neq 0, \\ \left(\frac{d}{b}, 0\right) & \overline{z} = 0, \end{cases}$$

Lemma 3.13 implies that for a fixed sufficiently small α, the map $\widetilde{\mathcal{T}}^\times : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}$ is a $\mathcal{C}^1$-smooth extension of $\mathcal{T}^\times$ to $\widetilde{\mathcal{R}}$.

Now, let us come back to the Poincaré map $\mathcal{T}$ defined on $\mathcal{Y}$. We extend this map to

$$\widetilde{\mathcal{T}}\left(w, z\right) := \begin{cases} \mathcal{T}\left(w, z\right) & (w, z) \in \mathcal{Y}, \\ \left(\frac{d}{b}, 0\right) & z = 0. \end{cases}$$

It is clear that the map $\widetilde{\mathcal{T}}^{\times}$ is in fact the cross-map of $\widetilde{\mathcal{T}}$ on $\widetilde{\mathcal{R}}$. Note that since $\theta < \epsilon_2^{\alpha}$, we have $\widetilde{\mathcal{R}} \subset \mathcal{Y}$. Thus, both of the maps $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{T}}^{\times}$ are defined on $\widetilde{\mathcal{R}}$. Therefore, for a sufficiently small θ, the map $\widetilde{\mathcal{T}}^{\times}$ satisfies the assumptions of Theorem 8 and Proposition D.2. This implies that the map $\widetilde{\mathcal{T}}$ possesses a $\mathcal{C}^1$-smooth invariant manifold

$$M^* = \big\{ (w, z) : w = h^*(z) \big\} \subset \widetilde{\mathcal{R}},$$

where h^* is some $\mathcal{C}^1$-smooth function defined on $[-\theta, \theta]$. Moreover, by Proposition D.2, if the backward orbit of a point in $\widetilde{\mathcal{R}}$ remains in $\widetilde{\mathcal{R}}$, then it must belong to M^*. Therefore, $\Lambda^u_{\widetilde{\mathcal{R}}, \widetilde{\mathcal{T}}} \subset M^*$. Removing the point $(\frac{d}{b}, 0)$ from M^*, we obtain a set which is invariant under the map $\mathcal{T}$. Moreover, we have $\Lambda^u_{\mathcal{R}, \mathcal{T}} \subset M^* \setminus \{(\frac{d}{b}, 0)\}$.

Let us now come back to (u_{10}, v_{10})-coordinates and the Poincaré map T. Equip $\mathcal{R}$ with (u_{10}, v_{10})-coordinates and choose $0 < \epsilon < \theta$. Thus, $\mathcal{D}_2^{\epsilon} \subset \mathcal{R}$. Consider the manifold M^* in (u_{10}, v_{10})-coordinates and restrict it to $\mathcal{D}_2^{\epsilon}$. Denote this restriction by $\mathcal{M}^*$. We have that $\mathcal{M}^* \setminus \{M^s\}$ is invariant under T, and $\Lambda^u_{\mathcal{D}_2^{\epsilon}, T} \subset \mathcal{M}^* \setminus \{M^s\}$. Choosing a sufficiently small ϵ also guarantees that $\mathcal{M}^*$ is a connected piece of M^* and hence is a $\mathcal{C}^1$-manifold.

The manifold $\mathcal{M}^*$ is our desired manifold if we show $\Lambda^u_{\mathcal{D}_2^{\epsilon}, T} = \mathcal{M}^* \setminus \{M^s\}$. So far, we have shown that $\Lambda^u_{\mathcal{D}_2^{\epsilon}, T} \subset \mathcal{M}^* \setminus \{M^s\}$ and so it is sufficient to show $\mathcal{M}^* \setminus \{M^s\} \subset \Lambda^u_{\mathcal{D}_2^{\epsilon}, T}$. However, this is just a direct consequence of part (ii) of Lemma 3.6 (proved earlier). The fact that $\mathcal{M}^*$ is tangent to ℓ^* at M^s is also a direct consequence of part (i) of this lemma. This ends the proof of part (vii). $\quad\square$

Remark 3.14. Lemma 3.6 states that when $bd > 0$, the set $\{M^s\} \cup \Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T}$ is a $\mathcal{C}^1$-smooth curve which is tangent to ℓ^* at M^s, and any point on this curve converges to M^s by the backward iterations of the Poincaré map T. It follows from part (iii) of Theorem 8 and the proof of Lemma 3.6 that, when $bd > 0$, if we take a curve ζ in $\mathcal{D}_2$, then $\{T^n(\zeta)|_{\mathcal{D}_2}\}_{n=1}^{\infty}$ converges uniformly to the curve $\{M^s\} \cup \Lambda^u_{\mathcal{D}_2 \cup \mathcal{D}_3, T}$.

3.3.2. Proof of Lemma 3.7

Reverse the time direction in system (2.12) (i.e. $t \to -t$) and exchange the stable and unstable components, i.e. apply the linear change of coordinates

$$(\widetilde{u}_1, \widetilde{u}_2, \widetilde{v}_1, \widetilde{v}_2) = (v_1, v_2, u_1, u_2) \tag{3.8}$$

This gives a system which is of the form of system (2.12), where all the assumptions of Lemma 2.5 are satisfied. The global map along Γ for this system is $J \left(T^{\mathrm{glo}}\right)^{-1} J^{-1}$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and T^{glo} is the global map of system (2.12). Thus, the differential of this map at M^s is

$$J \left(dT^{\mathrm{glo}}(M^s)\right)^{-1} J^{-1} = J \cdot \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot J^{-1} = \frac{1}{ad - bc} \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}.$$

This implies that if we replace conditions $bd > 0$ and $bd < 0$ in Lemma 3.6 by $cd < 0$ and $cd > 0$, respectively, and the line ℓ^* by the straight line whose slope is $\frac{-d}{c}$, then all the statements of Lemma 3.6 also hold for this system and the region $\mathcal{D}_2 \cup \mathcal{D}_3 \subset \Pi^s$. Consequently, by applying the inverse of change of coordinates (3.8), all the statements of Lemma 3.6 also hold for the

system which is derived from system (2.12) by a reversion of time and the region $\{(u_1, v_1) \in \mathcal{B}_{\epsilon_u} \subset \Pi^u : 0 < \frac{v_1}{u_1} \le m,\ u_1 \ne 0\} \subset \Pi^u$. In this case, the line ℓ^* is replaced by the straight line in Π^u whose slope is $\frac{-c}{d}$. The homoclinic loop Γ in this system leaves and enters O along the positive sides of u_2 and v_2, respectively, and the corresponding Poincaré map, call it $\widetilde{T}$, is defined on Π^u. Therefore, the statements of Lemma 3.6 also hold for the map

$$T^{\mathrm{glo}} \circ \widetilde{T} \circ \left(T^{\mathrm{glo}}\right)^{-1} \tag{3.9}$$

and the set

$$K = T^{\mathrm{glo}}\left(\{(u_1, v_1) \in \mathcal{B}_{\epsilon_u} \subset \Pi^u : 0 < \frac{v_1}{u_1} \le m,\ u_1 \ne 0\}\right), \tag{3.10}$$

where the line ℓ^* is replaced by the horizontal axis in Π^s. The later one is simply because

$$dT^{\mathrm{glo}}\left(M^u\right)\begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} ad - bc \\ 0 \end{pmatrix}.$$

Notice that map (3.9) is conjugate to the inverse of the Poincaré map T^{-1}.

The map (3.9) coincides with T^{-1} on $T(\mathcal{D})$. Note that, for sufficiently large m, the set $T^{\mathrm{glo}}\left(\{(u_1, v_1) \in \mathcal{B}_{\epsilon_u} \subset \Pi^u : u_1 \ne 0,\ m < \frac{v_1}{u_1}\}\right)$ has no intersection with $\mathcal{D}_1$. Therefore, Lemma 3.7 will be proved once we show that $\mathcal{D}_1 \subset K$ if $cd < 0$, and $K \cap \mathcal{D}_1 = \emptyset$ if $cd > 0$. However, this is an immediate consequence of the discussion above. In fact, it follows from the above discussion that the line ℓ^* passes through $\mathcal{D}_2 \cup \mathcal{D}_3$ if and only if the horizontal axis passes through K. The later case, for sufficiently large m, is equivalent to the condition $\mathcal{D}_1 \subset K$ and happens if and only if $cd < 0$. This ends the proof of Lemma 3.7.

Remark 3.15. Lemma 3.7 states that if $cd < 0$, then the set $\{M^s\} \cup \Lambda^u_{\mathcal{D}_1, T^{-1}}$ is a $\mathcal{C}^1$-smooth curve which is tangent to the horizontal axis at M^s. Moreover, any point on this curve converges to M^s by the forward iterations of the Poincaré map T. It follows from Remark 3.14 and the proof of Lemma 3.7 that if we take a curve ζ in $K \cap \mathcal{B}_\epsilon$, then $\{T^{-n}(\zeta)|_{K \cap \mathcal{B}_\epsilon}\}_{n=1}^\infty$ converges uniformly to the curve $\{M^s\} \cup \Lambda^u_{\mathcal{D}_1, T^{-1}}$.

3.4. Dynamics near the homoclinic figure-eight

In this section, we study the dynamics near the homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$. In particular, we prove Theorems B1, B2 and B3 in this section. We start with recalling some definitions and notations from Section 2.1.

For $i = 1, 2$, we denote by $\mathcal{D}^i$ the set of the points (u_{10}, v_{10}) on Π^s_i whose forward orbits go along the homoclinic orbit Γ_i and intersect Π^u_i at $(u_{1\tau}, v_{1\tau})$ such that

$$\|(u_{10}, v_{10})\| < \epsilon \quad \text{and} \quad \|(u_{1\tau}, v_{1\tau})\| < \epsilon_u, \tag{3.11}$$

for some sufficiently small constants $0 < \epsilon \le \epsilon_u < \delta$. We denote by $\mathbb{D}^1$ ($\mathbb{D}^2$) the set of the points (u_{10}, v_{10}) on Π^s_1 (Π^s_2) whose forward orbits go along the negative (positive) side of v_2-axis and intersect Π^u_2 (Π^u_1) at $(u_{1\tau}, v_{1\tau})$ such that (3.11) holds (see Fig. 16). We also denote
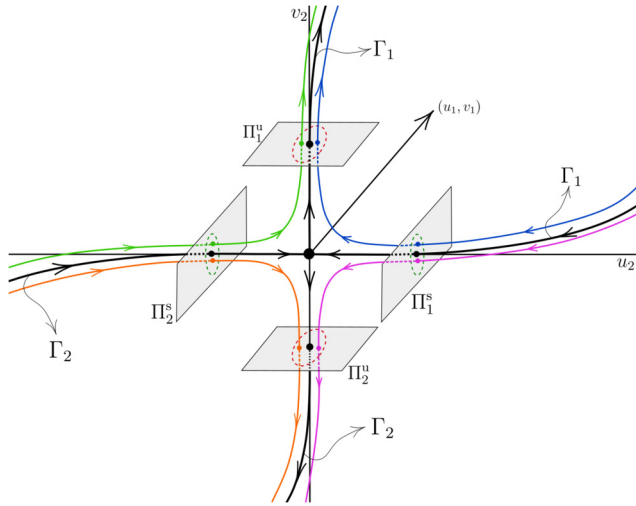
Fig. 16. The homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$ and the cross-sections Π_1^s, Π_1^u, Π_2^s and Π_2^u are shown. We consider ϵ-neighborhoods of M_1^s and M_2^s (green dashed circles) in Π_1^s and Π_2^s, respectively, as well as ϵ_u-neighborhoods of M_1^u and M_2^u (red dashed circles) in Π_1^u and Π_2^u, respectively. The set $\mathcal{D}^1$ (resp. $\mathcal{D}^2$) is the set of the points in the ϵ-neighborhood in Π_1^s (resp. Π_2^s) whose forward orbits go along Γ_1 (resp. Γ_2) and intersect the ϵ_u-neighborhood in Π_1^u (resp. Π_2^u). The blue point on Π_1^s and the brown point on Π_2^s belong to $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively. We denote by $\mathbb{D}^1$ (resp. $\mathbb{D}^2$) the set of the points in the ϵ-neighborhood in Π_1^s (resp. Π_2^s) whose forward orbits go along the negative (resp. positive) side of v_2-axis and intersect the ϵ_u-neighborhood in Π_2^u (resp. Π_1^u).

by T_i, T_i^{loc} and T_i^{glo} the Poincaré, local and global maps along Γ_i ($i = 1, 2$), respectively (see Fig. 7). The maps T_1^{glo} and T_2^{glo} are defined on the open ϵ_u-balls around M_1^s and M_2^s, respectively. Regarding the other maps, we have domain (T_1^{loc}) = domain $(T_1) = \mathcal{D}^1$ and domain (T_2^{loc}) = domain $(T_2) = \mathcal{D}^2$. We also define the map $T_{12}^{\text{loc}} : \mathbb{D}^1 \subset \Pi_1^s \to \Pi_2^u$ (resp. $T_{21}^{\text{loc}} : \mathbb{D}^2 \subset \Pi_2^s \to \Pi_1^u$) by $(u_{10}, v_{10}) \mapsto (u_{1\tau}, v_{1\tau})$, where $(u_{10}, v_{10}) \in \mathbb{D}^1$ (resp. $\in \mathbb{D}^2$) and $(u_{1\tau}, v_{1\tau}) \in \Pi_2^u$ (resp. $\in \Pi_1^u$) (see Fig. 7).

Let $\mathcal{V}$ be a sufficiently small neighborhood of $\Gamma_1 \cup \Gamma_2$ and define $\Xi = \mathcal{D}^1 \cup \mathbb{D}^1 \cup \mathcal{D}^2 \cup \mathbb{D}^2$. For any $x \in \Xi$, we correspond a (finite or infinite) sequence $\{x_k\}$ to x in the following way: *(i)* $x_0 = x$, *(ii)* if $x_k \in \Xi$ ($k \geq 0$), we define x_{k+1} to be the first intersection point of the forward orbit of x_k and $\Pi_1^s \cup \Pi_2^s$. Similarly, if $x_k \in \Xi$ ($k \leq 0$), we define x_{k-1} to be the first intersection point of the backward orbit of x_k and $\Pi_1^s \cup \Pi_2^s$. In order to understand the dynamics in $\mathcal{V}$, we need to find the set of the points whose forward or backward orbits lie entirely in $\mathcal{V}$, i.e. the set of the points $x \in \Xi$ for which the sequence $\{x_k\}$ is well-defined for all $k \geq 0$ or $k \leq 0$.

When $\lambda_2 < 2\lambda_1$, the dynamics near the homoclinic figure-eight is quite similar to the case of a single homoclinic loop: the forward and backward orbit of any arbitrary point in $\mathcal{V}$ leaves $\mathcal{V}$. When $\lambda_1 = \lambda_2$, it follows from Proposition 2.13 that $\Xi = \emptyset$ and so there is no dynamics near the homoclinic figure-eight. For the case of $\lambda_1 < \lambda_2 < 2\lambda_1$, we show in the next proof that for any $x \in \Xi$ whose corresponding x_1 is defined, the point x_1 lies close to the straight lines with slope $\frac{d_1}{b_1}$ (if x_1 lies in $\mathcal{D}^1 \cup \mathbb{D}^1$) or $\frac{d_2}{b_2}$ (if x_1 lies in $\mathcal{D}^2 \cup \mathbb{D}^2$), and hence, it lies outside of the set Ξ (see Fig. 17).

Proof of Theorem B2. The proof for the case $\lambda_1 = \lambda_2$ is an immediate consequence of Proposition 2.13.
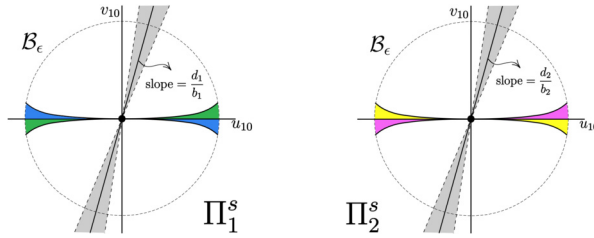
Fig. 17. This figure corresponds to the case $\lambda_1 < \lambda_2 < 2\lambda_1$, $b_1 d_1 > 0$ and $b_2 d_2 > 0$. The regions $\mathcal{D}^1$, $\mathbb{D}^1$, $\mathcal{D}^2$ and $\mathbb{D}^2$ are shown in green, blue, pink and yellow, respectively. We define Ξ as the union of these four regions. Let x_1 be the first intersection point of the forward orbit of $\mathrm{x} \in \Xi$ and $\Pi_1^s \cup \Pi_2^s$. It is shown in the proof of Theorem B2 that for any $\mathrm{x} \in \Xi$ the point x_1 lies in one of the gray regions on Π_1^s or Π_2^s.

Suppose $\lambda_1 < \lambda_2 < 2\lambda_1$. By Proposition 2.14, we have

$$\Xi = \{(u_{10}, v_{10}) : \quad \| (u_{10}, v_{10}) \| < \epsilon, \text{ and } 0 < |v_{10}| < K_{\epsilon_u} |u_{10}|^{\frac{\gamma}{1-\gamma}} [1 + O(\delta)]\},$$

where $K_{\epsilon_u} > 0$ is some constant and $\gamma = \lambda_1 \lambda_2^{-1} > 0.5$ (see Fig. 17). Consider $(u_{10}, v_{10}) \in \Xi$. Since $u_{1\tau} = v_{1\tau} O(\epsilon^2)$ (see Remark 2.15), the forward orbit of this point intersects one of the cross-sections Π_1^u or Π_2^u at a point close to the vertical axis and then it ends up either in the cross-section Π_1^s close to the straight line with the slope $\frac{d_1}{b_1}$ or in the cross-section Π_2^s close to the straight line with the slope $\frac{d_2}{b_2}$. In both cases, this point is outside of the set Ξ (see Fig. 17). This proves Theorem B2. $\quad\square$

A point in $\Pi_1^s \cup \Pi_2^s$ whose forward orbit lies entirely in $\mathcal{V}$ and does not lie on the stable manifold of O must belong to Ξ. We denote the set of these points by Λ^s. The same holds for backward orbits. We also define the set Λ^u analogously. In order to understand the dynamics in $\mathcal{V}$, we need to investigate these two sets. For the case of $\lambda_2 < 2\lambda_1$, Theorem B2 states that both of these sets are empty. Our approach to investigate Λ^s and Λ^u for the case $2\lambda_1 < \lambda_2$ is similar to what we have done in the previous section for the case of a single homoclinic loop.

Recall from Section 2.4.4 that when $2\lambda_1 < \lambda_2$, we divide each of the sets $\mathcal{D}^1$, $\mathbb{D}^1$, $\mathcal{D}^2$ and $\mathbb{D}^2$ into three regions, i.e. for $i = 1, 2$, we write $\mathcal{D}^i = \mathcal{D}_1^i \cup \mathcal{D}_2^i \cup \mathcal{D}_3^i$ and $\mathbb{D}^i = \mathbb{D}_1^i \cup \mathbb{D}_2^i \cup \mathbb{D}_3^i$ (see Fig. 18). Write $\Xi = \mathcal{I} \cup \mathcal{J} \subset \Pi_1^s \cup \Pi_2^s$, where

$$\mathcal{I} := \bigcup_{i=1,2} \left(\mathcal{D}_1^i \cup \mathbb{D}_1^i \right) \quad \text{and} \quad \mathcal{J} := \bigcup_{\substack{i=1,2 \\ j=2,3}} \left(\mathcal{D}_j^i \cup \mathbb{D}_j^i \right).$$

Definition 4. We define $\Lambda_{\mathcal{I}}^s (\Lambda_{\mathcal{I}}^u)$ as the set of the points in $\mathcal{I}$ whose forward (backward) orbits intersect Ξ infinitely many times and all the intersection points belong to $\mathcal{I}$. More precisely,

$$\Lambda_{\mathcal{I}}^s = \{\mathrm{x} = \mathrm{x}_0 : \quad \mathrm{x}_k \in \mathcal{I} \text{ for all } k \geq 0\} \quad \text{and} \quad \Lambda_{\mathcal{I}}^u = \{\mathrm{x} = \mathrm{x}_0 : \quad \mathrm{x}_k \in \mathcal{I} \text{ for all } k \leq 0\}$$

The sets $\Lambda_{\mathcal{J}}^s$ and $\Lambda_{\mathcal{J}}^u$ are defined analogously.

Similar to the case of a single homoclinic, we take three steps to investigate the sets Λ^s and Λ^u. In the first step, we investigate the sets $\Lambda_{\mathcal{J}}^s$ and $\Lambda_{\mathcal{J}}^u$. This is done in Lemma 3.16.
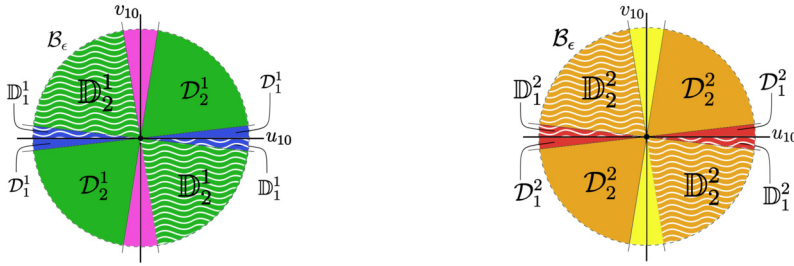
Fig. 18. The left and right figures show Π_1^s and Π_2^s, respectively. When $2\lambda_1 < \lambda_2$, we divide $\mathcal{D}^i$ into three subsets $\mathcal{D}_1^i$, $\mathcal{D}_2^i$ and $\mathcal{D}_3^i$ ($i = 1, 2$). Similarly, we divide $\mathbb{D}^i$ into three subsets $\mathbb{D}_1^i$, $\mathbb{D}_2^i$ and $\mathbb{D}_3^i$ ($i = 1, 2$). The sets $\mathcal{D}_3^1$ and $\mathbb{D}_3^1$ are subsets of the purple region, and the sets $\mathcal{D}_3^2$ and $\mathbb{D}_3^2$ are subsets of the yellow region.

From technical point of view, part (viii) of this lemma which proves the existence of an unstable invariant manifold of the homoclinic figure-eight is the main result of this section. The techniques which are used in the proof of this part rely on the proof of part (vii) of Lemma 3.6. In the second step, we investigate the sets $\Lambda_{\mathcal{I}}^s$ and $\Lambda_{\mathcal{I}}^u$. This is also done in Lemma 3.17. Finally, in Lemma 3.18, we clarify the relations between the sets $\Lambda_{\mathcal{J}}^s$, $\Lambda_{\mathcal{J}}^u$, $\Lambda_{\mathcal{I}}^s$ and $\Lambda_{\mathcal{I}}^u$, and the sets Λ^s and Λ^u. This enables us to prove Theorem B2. We start with the following:

Lemma 3.16. *Assume $2\lambda_1 < \lambda_2$ and let w be as in Notation 3.3. For $\mathrm{x} \in \mathcal{J}$, we have*

(i) *If $\mathrm{x} \in \mathcal{D}_2^1 \cup \mathcal{D}_3^1 \cup \mathbb{D}_2^2 \cup \mathbb{D}_3^2$ (i.e. $\mathrm{x} \in \mathcal{J} \cap \Pi_1^s$), then $w(\mathrm{x}_1) = \frac{d_1}{b_1} + o(1)$. If $\mathrm{x} \in \mathcal{D}_2^2 \cup \mathcal{D}_3^2 \cup \mathbb{D}_2^1 \cup \mathbb{D}_3^1$ (i.e. $\mathrm{x} \in \mathcal{J} \cap \Pi_2^s$), then $w(\mathrm{x}_1) = \frac{d_2}{b_2} + o(1)$. Here, $o(1)$ stands for a function of x that converges to zero as $\mathrm{x} \to M_{1,2}^s$.*

(ii) *There exists a constant $C > 0$ such that $\|\mathrm{x}\|^{1-2\gamma} < C\|\mathrm{x}_1\|$ holds for arbitrary x ($0 < \gamma = \frac{\lambda_1}{\lambda_2} < 0.5$).*

(iii) *$\mathrm{x}_1 \in \mathcal{B}_\epsilon$ implies $\mathrm{x}_1 \in \mathcal{J}$.*

(iv) *$\Lambda_{\mathcal{J}}^s = \emptyset$.*

(v) *If $b_1 d_1 > 0$ and $b_2 d_2 < 0$, then $\Lambda_{\mathcal{J}}^u = W_{loc}^u(\Gamma_1) \cap \mathcal{D}_2^1$.*

(vi) *If $b_1 d_1 < 0$ and $b_2 d_2 > 0$, then $\Lambda_{\mathcal{J}}^u = W_{loc}^u(\Gamma_2) \cap \mathcal{D}_2^2$.*

(vii) *If $b_1 d_1 > 0$ and $b_2 d_2 > 0$, then $\Lambda_{\mathcal{J}}^u = \left[W_{loc}^u(\Gamma_1) \cap \mathcal{D}_2^1 \right] \bigcup \left[W_{loc}^u(\Gamma_2) \cap \mathcal{D}_2^2 \right]$.*

(viii) *If $b_1 d_1 < 0$ and $b_2 d_2 < 0$, then $\Lambda_{\mathcal{J}}^u \subset \mathbb{D}_2^1 \cup \mathbb{D}_2^2$. More precisely, for each $i = 1, 2$, the union of M_i^s and $\Lambda_{\mathcal{J}}^u \cap \mathbb{D}^i$ is a one-dimensional $\mathcal{C}^1$-manifold in Π_i^s which at M_i^s is tangent to the straight line with slope $\frac{d_i}{b_i}$. Moreover, the backward orbit of any point in $\Lambda_{\mathcal{J}}^u$ intersects these two manifolds alternately, i.e. for any $\mathrm{x} \in \Lambda_{\mathcal{J}}^u$, all the points x_k for even and negative ks belong to only one of the manifolds and all the other x_k (odd and negative ks) belong to the other manifold.*

Proof. The same techniques that were used in the proof of Lemma 3.6 also prove parts (i), (ii) and (iii).

Part (iv) is an immediate consequence of (ii) and (iii).

In the rest of the proof, we assume $\mathrm{x} \in \Lambda_{\mathcal{J}}^u$. Notice that $\mathrm{x} = \mathrm{x}_0 \in \Lambda_{\mathcal{J}}^u$ implies that x_k is defined for all $k \leq 0$ and $\mathrm{x}_k \in \Lambda_{\mathcal{J}}^u$. Since $\Lambda_{\mathcal{J}}^u \subset \mathcal{J}$, we have two possibilities for x_k:

$$\mathrm{x}_k \in \mathbb{D}_2^2 \cup \mathbb{D}_3^2 \cup \mathcal{D}_2^1 \cup \mathcal{D}_3^1 \qquad \text{or} \qquad \mathrm{x}_k \in \mathcal{D}_2^2 \cup \mathcal{D}_3^2 \cup \mathbb{D}_2^1 \cup \mathbb{D}_3^1.$$

Our strategy for proving the rest of this lemma is to consider both of these possibilities and keep track of the sequence $\mathrm{x}_k, \mathrm{x}_{k+1}, \cdots, \mathrm{x}_{-1}, \mathrm{x}_0$. We analyze the behaviors and patterns of this sequence for arbitrary $\mathrm{x} \in \Lambda_{\mathcal{J}}^u$.

To prove part (v), suppose $b_1 d_1 > 0$ and $b_2 d_2 < 0$. By part (i), for x_{-2}, we observe

(1) $\mathrm{x}_{-2} \in \mathbb{D}_2^2 \cup \mathbb{D}_3^2 \cup \mathcal{D}_2^1 \cup \mathcal{D}_3^1 \implies \mathrm{x}_{-1} \in \mathcal{D}_2^1 \implies \mathrm{x} \in \mathcal{D}_2^1$, and
(2) $\mathrm{x}_{-2} \in \mathcal{D}_2^2 \cup \mathcal{D}_3^2 \cup \mathbb{D}_2^1 \cup \mathbb{D}_3^1 \implies \mathrm{x}_{-1} \in \mathbb{D}_2^2 \implies \mathrm{x} \in \mathcal{D}_2^1$.

According to this observation, $\mathrm{x} \in \Lambda_{\mathcal{J}}^u$ implies $\mathrm{x} \in \mathcal{D}_2^1$. In other words, $\Lambda_{\mathcal{J}}^u$ is in fact the set of all $\mathrm{x} \in \mathcal{D}_2^1$ whose backward orbits only intersect Π_1^s (and not Π_2^s), and all the intersection points belong to $\mathcal{D}_2^1$. It follows from Theorem A3 that this set is nothing but $W_{\mathrm{loc}}^u(\Gamma_1) \cap \mathcal{D}_2^1$. This proves part (v).

The proof of part (vi) is analogous to the proof of part (v).

To prove part (vii), let $b_1 d_1 > 0$ and $b_2 d_2 > 0$. By (i), for x_{k-2} ($k \leq 0$) we observe

(1) $\mathrm{x}_{k-2} \in \mathbb{D}_2^2 \cup \mathbb{D}_3^2 \cup \mathcal{D}_2^1 \cup \mathcal{D}_3^1 \implies \mathrm{x}_{k-1} \in \mathcal{D}_2^1 \implies \mathrm{x}_k \in \mathcal{D}_2^1 \implies \cdots \implies \mathrm{x} \in \mathcal{D}_2^1$, and
(2) $\mathrm{x}_{k-2} \in \mathcal{D}_2^2 \cup \mathcal{D}_3^2 \cup \mathbb{D}_2^1 \cup \mathbb{D}_3^1 \implies \mathrm{x}_{k-1} \in \mathcal{D}_2^2 \implies \mathrm{x}_k \in \mathcal{D}_2^2 \implies \cdots \implies \mathrm{x} \in \mathcal{D}_2^2$.

This observation holds for any arbitrary $k \leq 0$ which means that the set $\Lambda_{\mathcal{J}}^u$ consists of two disjoint sets: the first is the set of all $\mathrm{x} \in \mathcal{D}_2^1$ whose backward orbits intersect Ξ infinitely many times and every time at $\mathcal{D}_2^1$, and the second is the set of all $\mathrm{x} \in \mathcal{D}_2^2$ whose backward orbits intersect Ξ infinitely many times and every time at $\mathcal{D}_2^2$. According to Theorem A3, the first set is in fact $W_{\mathrm{loc}}^u(\Gamma_1) \cap \mathcal{D}_2^1$ and the second one is $W_{\mathrm{loc}}^u(\Gamma_2) \cap \mathcal{D}_2^2$. This proves part (vii).

To prove part (viii), let $b_1 d_1 < 0$ and $b_2 d_2 < 0$. By (i), for x_{k-1} ($k \leq -4$), we observe

(1) $\mathrm{x}_{k-1} \in \mathbb{D}_2^2 \cup \mathbb{D}_3^2 \cup \mathcal{D}_2^1 \cup \mathcal{D}_3^1 \implies \mathrm{x}_k \in \mathbb{D}_2^1 \implies \mathrm{x}_{k+1} \in \mathbb{D}_2^2 \implies \mathrm{x}_{k+2} \in \mathbb{D}_2^1$
$\implies \mathrm{x}_{k+3} \in \mathbb{D}_2^2 \implies \cdots \implies \mathrm{x} \in \mathbb{D}_2^1$ (if $\mathrm{x}_{-1} \in \mathbb{D}_2^2$) or $\mathrm{x} \in \mathbb{D}_2^2$ (if $\mathrm{x}_{-1} \in \mathbb{D}_2^1$), and
(2) $\mathrm{x}_{k-1} \in \mathcal{D}_2^2 \cup \mathcal{D}_3^2 \cup \mathbb{D}_2^1 \cup \mathbb{D}_3^1 \implies \mathrm{x}_k \in \mathbb{D}_2^2 \implies \mathrm{x}_{k+1} \in \mathbb{D}_2^1 \implies \mathrm{x}_{k+2} \in \mathbb{D}_2^2$
$\implies \mathrm{x}_{k+3} \in \mathbb{D}_2^1 \implies \cdots \implies \mathrm{x} \in \mathbb{D}_2^1$ (if $\mathrm{x}_{-1} \in \mathbb{D}_2^2$) or $\mathrm{x} \in \mathbb{D}_2^2$ (if $\mathrm{x}_{-1} \in \mathbb{D}_2^1$).

This observation holds for any arbitrary $k \leq -4$ and means that the backward orbit of x intersects $\mathcal{J}$ at $\mathbb{D}_2^1$ and $\mathbb{D}_2^2$ alternately.

Define the maps $\mathbb{T}_{12} : \mathbb{D}^1 \to \Pi_2^s$ and $\mathbb{T}_{21} : \mathbb{D}^2 \to \Pi_1^s$ by $\mathbb{T}_{12} := T_2 \circ T_{12}$ and $\mathbb{T}_{21} := T_1 \circ T_{21}$. We then define $\mathbb{T} : \mathbb{D}^1 \to \Pi_1^s$ by $\mathbb{T} := \mathbb{T}_{21} \circ \mathbb{T}_{12}$. According to the above observation, the set $\Lambda_{\mathcal{J}}^u$ is in fact the set of the points $x \in \mathbb{D}_2^1$ such that $\mathbb{T}^{-n}(x) \in \mathbb{D}_2^1$ for all integers $n > 0$.

Recall (w, z) coordinate system and the map $\widetilde{T}$ introduced in the proof of Lemma 3.6. Similar to that proof, we equip $\mathbb{D}_2^1$ and $\mathbb{D}_2^2$ with (w, z) coordinates and define the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{\mathbb{T}}_{21}$ by

$$\widetilde{\mathbb{T}}_{12}(w, z) := \begin{cases} (\overline{w}, \overline{z}) & z \neq 0, \\ \left(\frac{d_2}{b_2}, 0\right) & z = 0, \end{cases} \qquad \text{for } (w, z) \in \widetilde{\mathcal{R}}_1,$$

and

$$
\widetilde{\mathbb{T}}_{21}\left(w,z\right) := \begin{cases} (\overline{w},\overline{z}) & z \neq 0, \\ \left(\frac{d_1}{b_1},0\right) & z = 0, \end{cases} \quad \text{for } (w,z) \in \widetilde{\mathcal{R}}_2,
$$

where $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ are some appropriate rectangles defined analogous to the proof of Lemma 3.6. According to Remark C.2, the estimates given by Lemma C.1 also hold for the local maps T_{12} and T_{21}. Therefore, with exactly the same proof as the proof of Lemma 3.6, we see that both of the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{\mathbb{T}}_{21}$ can be written in cross-form and the partial derivatives of the cross-map satisfies the estimates given by Lemma 3.13. Moreover, as it can be seen from the proof of Lemma 3.6, we can make the estimates in Lemma 3.13 sufficiently small by choosing θ small enough. This means that the maps $\widetilde{\mathbb{T}}_{12}$ and $\widetilde{\mathbb{T}}_{21}$ satisfy the assumptions of Lemma D.1 for sufficiently small K_1 and K_2. Thus, Lemma D.1 implies that by choosing an appropriate norm, the map $\widetilde{\mathbb{T}} := \widetilde{\mathbb{T}}_{21} \circ \widetilde{\mathbb{T}}_{12}$ (which is in fact the representation of $\mathbb{T}$ in (w,z) coordinates) can be written in cross-form and the cross-map has sufficiently small partial derivatives. Therefore, this cross-map satisfies the assumptions of Theorem 8. The rest of the proof follows from the proof of Lemma 3.6. $\square$

The following lemma is analogous to Lemma 3.16. The proof of this lemma is a simple modification of the proof of Lemma 3.7 for the case of homoclinic figure-eight.

Lemma 3.17. *Assume $2\lambda_1 < \lambda_2$ and let w be as in Notation 3.3. For $\mathrm{x} \in \mathcal{I}$, we have*

(i) *$w\left(\mathrm{x}_{-1}\right) = o(1)$, where $o\left(1\right)$ stands for a function of x that converges to zero as $\mathrm{x} \to M_{1,2}^s$.*
(ii) *There exists a constant $C > 0$ such that $\|\mathrm{x}\|^{1-2\gamma} < C\|\mathrm{x}_{-1}\|$ holds for any x ($0 < \gamma = \frac{\lambda_1}{\lambda_2} < 0.5$).*
(iii) *$\mathrm{x}_{-1} \in \mathcal{B}_\epsilon$ implies $\mathrm{x}_{-1} \in \mathcal{I}$.*
(iv) *$\Lambda_\mathcal{I}^u = \emptyset$.*
(v) *If $c_1 d_1 < 0$ and $c_2 d_2 > 0$, then $\Lambda_\mathcal{I}^s = W_{loc}^s\left(\Gamma_1\right) \cap \mathcal{D}_1^1$.*
(vi) *If $c_1 d_1 > 0$ and $c_2 d_2 < 0$, then $\Lambda_\mathcal{I}^s = W_{loc}^s\left(\Gamma_2\right) \cap \mathcal{D}_1^2$.*
(vii) *If $c_1 d_1 < 0$ and $c_2 d_2 < 0$, then $\Lambda_\mathcal{I}^s = \left[W_{loc}^u\left(\Gamma_1\right) \cap \mathcal{D}_1^1\right] \bigcup \left[W_{loc}^s\left(\Gamma_2\right) \cap \mathcal{D}_1^2\right]$.*
(viii) *If $c_1 d_1 > 0$ and $c_2 d_2 > 0$, then $\Lambda_\mathcal{I}^s \subset \mathbb{D}_1^1 \cup \mathbb{D}_1^2$. More precisely, for each $i = 1,2$, the union of M_i^s and $\Lambda_\mathcal{I}^s \cap \mathbb{D}_1^i$ is a one-dimensional $\mathcal{C}^1$-manifold in Π_i^s which at M_i^s is tangent to the horizontal axis. Moreover, the forward orbit of any point in $\Lambda_\mathcal{I}^s$ intersects these two manifolds alternately, i.e. for any $\mathrm{x} \in \Lambda_\mathcal{I}^s$, all the points x_k for even and negative ks belong to only one of the manifolds and all the other x_k (odd and negative ks) belong to the other manifold.*

The following Lemma states that the forward (resp. backward) orbit of a point in $\mathcal{V}$ lies in $\mathcal{V}$ if and only if it intersects the cross-sections Π_1^s and Π_2^s only at $\mathcal{I}$ (resp. $\mathcal{J}$).

Lemma 3.18. *We have $\Lambda^u = \Lambda_\mathcal{J}^u$ and $\Lambda^s = \Lambda_\mathcal{I}^s$.*

Proof. It follows from parts (ii) and (iii) of Lemma 3.17 that if $\mathrm{x} \in \mathcal{I}$, then the sequence $\{\mathrm{x}_k\}$ is not defined for all $k \leq 0$. Indeed, for some $k_0 \leq 0$, we have $\{\mathrm{x}_{k_0}, \cdots, \mathrm{x}_{-1}\} \subset \mathcal{I}$ such that x_{k_0-1} lies outside the ϵ-balls around M_1^s or M_2^s. This means that if x belongs to Λ^u, then it must belong to $\mathcal{J}$. Therefore, $\mathrm{x} \in \Lambda^u$ implies $\mathrm{x} \in \Lambda_\mathcal{J}^u$. On the other hand, we know $\Lambda_\mathcal{J}^u \subset \Lambda^u$. This proves the first part of the lemma. The proof of the other part is the same. $\square$

By virtue of the preceding lemmas, we are now in a position to prove Theorem B3.

Proof of Theorem B3. The local stable (resp. unstable) set of the homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$, denoted by $W^s_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$ (resp. $W^u_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$), is the union of $\Gamma_1 \cup \Gamma_2$ itself and the set of the points in a sufficiently small neighborhood $\mathcal{V}$ of $\Gamma_1 \cup \Gamma_2$ whose forward (resp. backward) orbits lie in $\mathcal{V}$ and their ω-limit sets (resp. α-limit sets) coincide with $\Gamma_1 \cup \Gamma_2 \cup \{O\}$. By this definition, the intersection of $W^s_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$ and any of the cross-sections Π^s_1 and Π^s_2 must belong to $\{M^s_1, M^s_2\} \cup \Lambda^s$. Similarly, the intersection of $W^u_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$ and the cross-sections Π^s_1 and Π^s_2 must belong to $\{M^s_1, M^s_2\} \cup \Lambda^u$.

It follows from Lemma 3.17 that in any cases except the case $c_1 d_1 > 0$ and $c_2 d_2 > 0$, the ω-limit set of any orbit in Λ^s coincides with either $\Gamma_1 \cup \{O\}$ or $\Gamma_2 \cup \{O\}$. Therefore, in all of these cases, we have $W^s_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2$.

Denote the flow of system (2.12) by ϕ_t. When $c_1 d_1 > 0$ and $c_2 d_2 > 0$, it follows from parts (ii) and (viii) of Lemma 3.17 that the set $\Gamma_1 \cup \Gamma_2 \cup \phi_t(\Lambda^s)$ for $t \geq 0$ is a 2-dimensional $\mathcal{C}^1$ manifold, and the forward orbit of any point on this manifold converges to $\Gamma_1 \cup \{O\} \cup \Gamma_2$ as $t \to \infty$. This means that this manifold is in fact the local stable set of the homoclinic figure-eight $\Gamma_1 \cup \Gamma_2$. The fact that this manifold is tangent to $W^s_{\mathrm{glo}}(O)$ at every point of $\Gamma_1 \cup \Gamma_2$ is an straightforward consequence of the discussion before Proposition 3.10.

The proof for the case of $W^u_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$ is the same. This ends the proof.   □

Corollary 3.19. *Let ϕ_t be the flow of system (2.12). Then*

$$W^s_{loc}(\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2 \cup \phi_t(\Lambda^s), \qquad for\ t \geq 0,$$

$$W^u_{loc}(\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2 \cup \phi_t(\Lambda^u), \qquad for\ t \leq 0.$$

Finally, we prove

Proof of Theorem B1. Denote the set $W^s_{\mathrm{loc}}(\Gamma_1) \cup W^s_{\mathrm{loc}}(\Gamma_2) \cup W^s_{\mathrm{loc}}(\Gamma_1 \cup \Gamma_2)$ by $\mathcal{W}^s$. By definition, the forward orbit of any point on $W^s_{\mathcal{V}}(O) \cup \mathcal{W}^s$ lies in $\mathcal{V}$. Consider a point in $\mathcal{V} \setminus W^s_{\mathcal{V}}(O)$ whose forward orbit lies entirely in $\mathcal{V}$. The forward orbit of this point must intersect $\Pi^s_1 \cup \Pi^s_2$ at Λ^s. Therefore, it follows from the proof of Theorem B2 (for the case $\lambda_2 < 2\lambda_1$) and Corollary 3.19 (for the case $2\lambda_1 < \lambda_2$) that this point lies on $\mathcal{W}^s$. This finishes the proof for the case of forward orbits.

The proof of the case of backward orbits is the same. This finishes the proof.   □

3.5. Dynamics near super-homoclinic orbits

In this section, we prove Theorem C1. The idea of the proof is to show that there exist sequences of curves $\{l^u_k\}^\infty_{k=1} \subset W^u_{\mathrm{glo}}(O) \cap \Pi^s$ and $\{l^s_k\}^\infty_{k=1} \subset W^s_{\mathrm{glo}}(O) \cap \Pi^s$ that accumulate to $W^u_{\mathrm{loc}}(\Gamma) \cap \Pi^s$ and $W^s_{\mathrm{loc}}(\Gamma) \cap \Pi^s$, respectively (see Fig. 19). Then, the flow near the super-homoclinic orbit defines a map which maps the first sequence to a sequence of curves, denoted by $\{m^u_k\}$ in Fig. 19, such that each of the curves $\{m^u_k\}$ intersects each of the curves $\{l^s_k\}$ at a single point. Each of these intersection points correspond to a homoclinic orbit. The proof of Theorem C2 is exactly the same.

Proof of Theorem C1. Let $\mathcal{W}^s = W^s_{\mathrm{loc}}(\Gamma) \cap \mathcal{D}_1$ and $\mathcal{W}^u = W^u_{\mathrm{loc}}(\Gamma) \cap \mathcal{D}_2$. We have shown in Section 3.3 (after Proposition 3.9) that $T^{\mathrm{glo}}(W^u_{\mathrm{loc}}(O) \cap \Pi^u)$ intersects Π^s at a curve which is
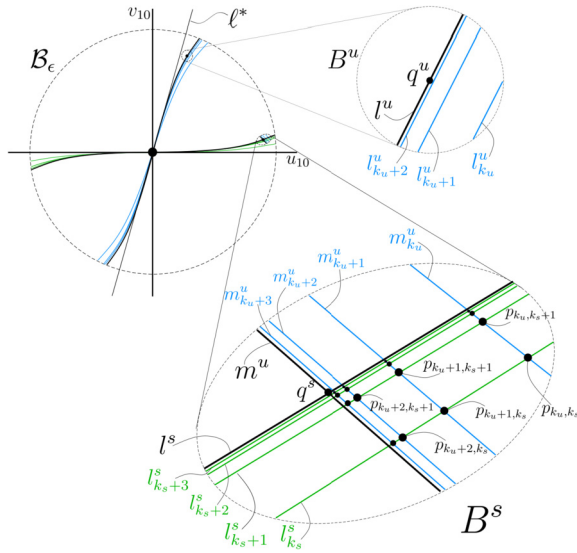
Fig. 19. The blue and green curves belong to the intersection of Π^s and the global unstable and stable invariant manifolds of the equilibrium O, respectively. The blue curves accumulate to $\mathcal{W}^u$ and the green curves accumulate to $\mathcal{W}^s$, where $\mathcal{W}^u = W^u_{\text{loc}}(\Gamma) \cap \Pi^s$ and $\mathcal{W}^s = W^s_{\text{loc}}(\Gamma) \cap \Pi^s$. Let $q^u \in \mathcal{W}^u$ and $q^s \in \mathcal{W}^s$ be at the intersection of the super-homoclinic orbit and the cross-section Π^s. The flow near the super-homoclinic orbit defines a map on a small neighborhood B^u of q^u onto a small neighborhood B^s of q^s. This map maps the blue curves restricted to B^u to the blue curves in B^s. The blue and green curves in B^s intersect transversely. Any point of these intersections belongs to both stable and unstable invariant manifolds of O. Thus, the orbits passing through these points are homoclinic to O.

tangent to ℓ^* at M^s. For a sufficiently small ϵ, the restriction of this curve to $\mathcal{B}_\epsilon \setminus \{M^s\}$ lies in $\mathcal{D}_2$. Denote this restricted curve by L^u_0, and let L^u_k ($k \geq 1$) be the restriction of $T\left(L^u_{k-1}\right)$ to $\mathcal{B}_\epsilon \setminus \{M^s\}$. By Remark 3.14, the sequence $\{L^u_k\}_{k=1}^\infty$ converges to $\mathcal{W}^u$ uniformly.

Now, consider the restriction of $W^s_{\text{loc}}(O) \cap \Pi^s$ to $\mathcal{B}_\epsilon \setminus \{M^s\}$ and denote it by L^s_0. We have $L^s_0 \subset K$, where K is as in (3.10). Let L^s_k ($k \geq 1$) be the restriction of $T^{-1}\left(L^s_{k-1}\right)$ to $\mathcal{B}_\epsilon \setminus \{M^s\}$. By Remark 3.15, the sequence $\{L^s_k\}_{k=1}^\infty$ converges to $\mathcal{W}^s$ uniformly.

The super-homoclinic orbit $\mathcal{S}$ intersects Π^s at $\mathcal{W}^u$ and $\mathcal{W}^s$ infinitely many times. Denote the furthest points of $\mathcal{S} \cap \mathcal{W}^u$ and $\mathcal{S} \cap \mathcal{W}^s$ from M^s by q^u and q^s, respectively. Let B^u be a sufficiently small open ball in $\mathcal{D}_2$ centered at q^u. The orbits starting from B^u leave the small neighborhood $\mathcal{U}$ of Γ and go along the super-homoclinic orbit $\mathcal{S}$, and after a finite time, they come back and intersect Π^s at some points close to q^s. These orbits induce a global map

$$T_\mathcal{S} : B^u \subset \Pi^s \to B^s \subset \Pi^s$$

along the super-homoclinic orbit $\mathcal{S}$, where $B^s = T_\mathcal{S}(B^u)$ and $T_\mathcal{S}(q^u) = q^s$. Since B^u is sufficiently small and the map $T_\mathcal{S}$ is a diffeomorphism, the neighborhood B^s is small, connected and convex.

Define $l^u = \mathcal{W}^u \cap B^u$ and $l^s = \mathcal{W}^s \cap B^s$. Since the sequence $\{L^s_k\}_{k=1}^\infty$ converges to $\mathcal{W}^s$ uniformly, there exists a sufficiently large k_s such that for all $k \geq k_s$, the curve L^s_k intersects B^s. Let $l^s_k = L^s_k \cap B^s$ for $k \geq k_s$. This implies that $l^s_k \xrightarrow{\text{unif}} l^s$. Similarly, for some sufficiently large k_u, all the curves L^u_k for $k \geq k_u$ intersect B^u. Let $l^u_k = L^u_k \cap B^u$ for $k \geq k_u$. Therefore, $l^u_k \xrightarrow{\text{unif}} l^u$.

The map T_S maps B^u to B^s. Thus, the curves l^u and l^u_k in B^u are mapped to some curves in B^s by T_S. Let $m^u = T_S(l^u)$ and $m^u_k = T_S(l^u_k)$ for $k \geq k_u$. Since the super-homoclinic orbit S is at the transverse intersection of the stable and unstable invariant manifolds of the homoclinic orbit Γ, the curves m^u and l^s intersect each other transversely. On the other hand, the sequences of the curves m^u_k and l^s_k converge to m^u and l^s, respectively. This implies that the curves m^u_k intersect the curves l^s_k transversely. Moreover, without loss of generality, we can assume that the integers k_u and k_s are large enough such that the curves m^u_i and l^s_j intersect each other at a unique point $p_{i,j}$ for any $i \geq k_u$ and $j \geq k_s$. The orbits passing through the points $p_{i,j}$ are the desired multi-pulse homoclinic orbits. This proves Theorem C1. $\quad\square$

Appendix A. Proofs of Lemmas 2.3, 2.4 and 2.5

In this appendix, we prove normal form lemmas. We start with a brief discussion on some materials needed for the proofs of the lemmas, and then proceed to the proofs.

A.1. Preliminaries

Consider the system

$$
\begin{aligned}
\dot{x} &= f(x, y), \\
\dot{y} &= g(x, y),
\end{aligned}
\tag{A.1}
$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and $f(0,0) = g(0,0) = 0$. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth mapping such that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Let the manifold $\mathcal{M} = \{(x, y) : y = \varphi(x)\}$ be invariant with respect to the flow of this system.

Definition 5. By straightening the invariant manifold $\mathcal{M}$, we mean applying a change of coordinates of the form $(\tilde{x}, \tilde{y}) = (x, y - \varphi(x))$.

Making this change of coordinates transforms the manifold $\mathcal{M}$ to the linear subspace $\{(x, y) : y = 0\}$. Straightening an invariant manifold of the type $\{(x, y) : x = \varphi(y)\}$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth mapping such that $\varphi(0) = 0$ and $\varphi'(0) = 0$, is defined analogously.

Again, consider system (A.1) and let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{R}^m$ be some smooth maps such that $\varphi(0) = 0$ and $\psi(0) = 0$. According to [2], we have

Proposition A.1. *The manifold $\mathcal{M} = \{(x, y) : y = \varphi(x)\}$ is invariant with respect to the flow of system (A.1) if and only if*

$$
g(x, \varphi(x)) = \varphi'(x) \cdot f(x, \varphi(x)).
\tag{A.2}
$$

Analogously, the manifold $\mathcal{N} = \{(x, y) : x = \psi(y)\}$ is invariant with respect to the flow of system (A.1) if and only if

$$
f(\psi(y), y) = \psi'(y) \cdot g(\psi(y), y).
\tag{A.3}
$$

Definition 6. We refer to (A.2) (resp. (A.3)) as the condition of the invariance of the manifold $\mathcal{M}$ (resp. $\mathcal{N}$) with respect to the flow of system (A.1).

Remark A.2. System (A.1) has an equilibrium state at the origin. This equilibrium may possess strong stable $W^{ss}(O)$, strong unstable $W^{uu}(O)$, extended stable $W^{sE}(O)$ and extended unstable invariant manifolds $W^{uE}(O)$ (see [23]). Let system (A.1) be invariant with respect to some linear symmetry. Then, changes of coordinates that straighten the manifolds $W^{ss}(O)$, $W^{uu}(O)$, $W^{sE}(O)$ and $W^{uE}(O)$ commute with that symmetry.

A.2. Proofs of Lemmas 2.3, 2.4 and 2.5

Proof of Lemma 2.3. To reduce system (1.3) to the form (2.4), we straighten the local stable and local unstable invariant manifolds of the equilibrium state O, i.e. we apply a change of coordinates

$$
\begin{aligned}
\tilde{u}_1 &= u_1 - \varphi_{1s}(v_1, v_2), & \tilde{u}_2 &= u_2 - \varphi_{2s}(v_1, v_2), \\
\tilde{v}_1 &= v_1 - \psi_{1u}(u_1, u_2), & \tilde{v}_2 &= v_2 - \psi_{2u}(u_1, u_2),
\end{aligned}
\tag{A.4}
$$

where $\{u_1 = \varphi_{1s}(v_1, v_2), u_2 = \varphi_{2s}(v_1, v_2)\}$ and $\{v_1 = \psi_{1u}(u_1, u_2), v_2 = \psi_{2u}(u_1, u_2)\}$ are the equations of the local stable and the local unstable invariant manifolds of O, respectively. Thus, after applying (A.4), the equations of the local stable and the local unstable manifolds of O become $\{v_1 = v_2 = 0\}$ and $\{u_1 = u_2 = 0\}$, respectively. This implies that system (1.3) can be written in the form (2.4) such that (2.5) is satisfied. Notice that change of coordinates (A.4) does not affect the quadratic part of (1.4). Therefore, the updated first integral H keeps the form (1.4).

Since H vanishes at every point of the local unstable invariant manifold $\{u_1 = u_2 = 0\}$, it can be written as

$$
H(u_1, u_2, v_1, v_2) = \lambda_1 u_1 [v_1 + H_1(u_1, u_2, v_1, v_2)] - \lambda_2 u_2 [v_2 + H_2(u_1, u_2, v_1, v_2)], \quad (A.5)
$$

for some $\mathcal{C}^\infty$-smooth $H_1, H_2 : \mathbb{R}^4 \to \mathbb{R}$ such that H_1 and H_2 and their first derivatives vanish at O. On the other hand, H vanishes at every point of the local stable invariant manifold $\{v_1 = v_2 = 0\}$. This implies

$$
0 = H(u_1, u_2, 0, 0) = \lambda_1 u_1 H_1(u_1, u_2, 0, 0) - \lambda_2 u_2 H_2(u_1, u_2, 0, 0).
$$

Therefore

$$
\begin{aligned}
H(u_1, u_2, v_1, v_2) = &\lambda_1 u_1 [v_1 + H_1(u_1, u_2, v_1, v_2) - H_1(u_1, u_2, 0, 0)] \\
&- \lambda_2 u_2 [v_2 + H_2(u_1, u_2, v_1, v_2) - H_2(u_1, u_2, 0, 0)].
\end{aligned}
$$

This suggests that, without loss of generality, we can assume that H_1 and H_2 vanish at $\{v_1 = v_2 = 0\}$. Now, consider the change of coordinates

$$
\begin{aligned}
\tilde{u}_1 &= u_1, & \tilde{u}_2 &= u_2, \\
\tilde{v}_1 &= v_1 + H_1(u_1, u_2, v_1, v_2), & \tilde{v}_2 &= v_2 + H_2(u_1, u_2, v_1, v_2).
\end{aligned}
\tag{A.6}
$$

Since $H_1(u_1, u_2, 0, 0) = H_2(u_1, u_2, 0, 0) = 0$, applying this change of coordinates on system (2.4) keeps the local stable and local unstable invariant manifolds straightened and therefore

keeps the form (2.4) of the system such that (2.5) still holds. However, this change of coordinates reduces the first integral H to the form (2.6).

It is a direct consequence of Remark A.2 that change of coordinates (A.4) preserves the symmetric structure of the system and the first integral. Concerning the change of coordinates (A.6), note that since H in (A.5) satisfies (1.6), we have

$$H_1(-u_1, u_2, -v_1, v_2) = -H_1(u_1, u_2, v_1, v_2),$$

$$H_2(-u_1, u_2, -v_1, v_2) = H_2(u_1, u_2, v_1, v_2).$$

This implies that the change of coordinates (A.6) commutes with symmetry (1.5), and therefore, preserves the invariance of the system with respect to symmetry (1.5). This ends the proof of Lemma 2.3. □

Our proof of Lemma 2.4 is based on a theorem in [23] (Theorem A.1). A special case of this theorem that we need for the proof of that lemma is stated below:

Lemma A.3. *([23], Theorem A.1) Consider system (2.4) and assume $\lambda_1 < \lambda_2$. There exists a $\mathcal{C}^\infty$-smooth change of coordinates which brings system (2.4) to the form*

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + f_{11}(u_1, u_2, v_1, v_2)u_1 + f_{12}(u_1, u_2, v_1, v_2)u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + f_{21}(u_1, u_2, v_1, v_2)u_1 + f_{22}(u_1, u_2, v_1, v_2)u_2, \\
\dot{v}_1 &= +\lambda_1 v_1 + g_{11}(u_1, u_2, v_1, v_2)v_1 + g_{12}(u_1, u_2, v_1, v_2)v_2, \\
\dot{v}_2 &= +\lambda_2 v_2 + g_{21}(u_1, u_2, v_1, v_2)v_1 + g_{22}(u_1, u_2, v_1, v_2)v_2,
\end{aligned}
\tag{A.7}
$$

where the functions f_{ij}, g_{ij} are $\mathcal{C}^\infty$-smooth and

$$
\begin{aligned}
f_{ij}(0,0,0,0) &= 0, \quad f_{1i}(u_1, u_2, 0, 0) \equiv 0, \quad f_{j1}(0,0,v_1,v_2) \equiv 0, \\
g_{ij}(0,0,0,0) &= 0, \quad g_{1i}(0,0,v_1,v_2) \equiv 0, \quad g_{j1}(u_1,u_2,0,0) \equiv 0, \quad (i,j=1,2).
\end{aligned}
\tag{A.8}
$$

Proof. See [23]. □

As a matter of comparison between this lemma and Lemma 2.4, the functions f_{i1} and g_{i1} $(i=1,2)$ in (2.8) do not depend on u_2 and v_2, respectively, and (2.9) includes all conditions (A.8) as well as two extra constraints

$$f_{22}(0,v) \equiv 0, \tag{A.9}$$

$$g_{22}(u,0) \equiv 0. \tag{A.10}$$

Remark A.4. The desired change of coordinates in Lemma A.3, denote it by Φ, is in fact a composition of several changes of coordinates, each describing some invariant manifolds. One can observe that each of these changes of coordinates commutes with symmetry (1.5) (see [2]). Moreover, due to this symmetric property, each of these changes of coordinates can be written in the form

$$\tilde{u}_1 = u_1 [1 + o(1)], \qquad \tilde{u}_2 = u_2 [1 + o(1)] + u_1 v_1 O(1),$$
$$\tilde{v}_1 = v_1 [1 + o(1)], \qquad \tilde{v}_2 = v_2 [1 + o(1)] + u_1 v_1 O(1), \tag{A.11}$$

where $o(1)$ and $O(1)$ stand for $\mathcal{C}^\infty$-smooth functions of (u, v) which converge to zero and are bounded above by a constant, respectively, as $(u, v) \to O$. A straightforward calculation shows that making changes of coordinates of this form preserves the form (2.6) of the first integral H. On the other hand, first integral (2.6) is already of the form (2.10). This implies that the change of coordinates Φ transforms first integral (2.6) to the form (2.10).

Proof of Lemma 2.4. According to Lemmas 2.3 and A.3, there exists a change of coordinates which brings system (1.3) to system (A.7) where the functions f_{ij}, g_{ij} are $\mathcal{C}^\infty$-smooth and satisfy (A.8). We show that there exists a change of coordinates which brings system (A.7) into the form (2.8), where f_{ij}, g_{ij} are $\mathcal{C}^\infty$-smooth and satisfy (2.9).

Consider system (A.7) and, for $i = 1, 2$, let

$$f_{i1}^{\text{new}}(u_1, v) = f_{i1}(u_1, 0, v_1, v_2), \qquad g_{i1}^{\text{new}}(u, v_1) = g_{i1}(u_1, u_2, v_1, 0),$$

$$f_{i2}^{\text{new}}(u_1, u_2, v) = \left[\frac{f_{i1}(u_1, u_2, v_1, v_2) - f_{i1}(u_1, 0, v_1, v_2)}{u_2} \right] u_1 + f_{i2}(u_1, u_2, v_1, v_2),$$

$$g_{i2}^{\text{new}}(u, v_1, v_2) = \left[\frac{g_{i1}(u_1, u_2, v_1, v_2) - g_{i1}(u_1, u_2, v_1, 0)}{v_2} \right] v_1 + g_{i2}(u_1, u_2, v_1, v_2).$$

It is easily seen that $\{f_{ij}^{\text{new}}\}$ and $\{g_{ij}^{\text{new}}\}$ satisfy (A.8). Thus, by rewriting system (A.7) with $\{f_{ij}^{\text{new}}\}$ and $\{g_{ij}^{\text{new}}\}$, this system takes the form (2.8) such that (A.8) holds.

Hereafter, we assume that (A.8) is satisfied for system (2.8). Write this system as

$$\dot{u}_1 = -\lambda_1 u_1 + f_{11}(u_1, v)u_1 + f_{12}(u_1, u_2, v)u_2,$$
$$\dot{u}_2 = -\lambda_2 u_2 + f_{21}(u_1, v)u_1 + J_1(u, v)u_2 + \underline{J_2(v)u_2},$$
$$\dot{v}_1 = +\lambda_1 v_1 + g_{11}(u, v_1)v_1 + g_{12}(u, v_1, v_2)v_2, \tag{A.12}$$
$$\dot{v}_2 = +\lambda_2 v_2 + g_{21}(u, v_1)v_1 + J_3(u, v)v_2 + \underline{J_4(u)v_2},$$

where

$$J_1(u, v) = f_{22}(u, v) - f_{22}(0, v), \qquad J_2(v) = f_{22}(0, v),$$
$$J_3(u, v) = g_{22}(u, v) - g_{22}(u, 0), \qquad J_4(u) = g_{22}(u, 0).$$

In order to obtain conditions (A.9) and (A.10), we need to find a change of coordinates which eliminates the underlined terms in (A.12). We claim that this is possible by applying two consecutive $\mathcal{C}^\infty$-smooth changes of coordinates of the forms

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2 + q_1(v_1, v_2)u_2, \qquad \tilde{v}_1 = v_1, \qquad \tilde{v}_2 = v_2, \tag{A.13}$$

and

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2, \qquad \tilde{v}_1 = v_1, \qquad \tilde{v}_2 = v_2 + q_2(u_1, u_2)v_2, \tag{A.14}$$

where q_1 and q_2 are some functions such that $q_1(0) = q_2(0) = 0$. We show that the underlined terms $J_2(v)u_2$ and $J_4(u)v_2$ can be eliminated by applying a change of coordinates of the forms (A.13) and (A.14), respectively.

Let $z = \frac{u_2}{1+q_1(v)}$. Applying change of coordinates (A.13) brings system (A.12) to

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + [f_{11}(u_1, v)]u_1 + \left[\frac{f_{12}(u_1, z, v)}{1+q_1(v)}\right]u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + [(1+q_1(v))\,f_{21}(u_1, v)]u_1 + Q_1(u, v)u_2 + Q_2(v)u_2 \\
\dot{v}_1 &= \lambda_1 v_1 + [g_{11}(u_1, z, v_1)]v_1 + [g_{12}(u_1, z, v_1, v_2)]v_2, \\
\dot{v}_2 &= \lambda_2 v_2 + [g_{21}(u_1, z, v_1)]v_1 + [g_{22}(u_1, z, v)]v_2,
\end{aligned}
\tag{A.15}
$$

where

$$
\begin{aligned}
Q_1(u, v) =\,& J_1(u_1, z, v) + \frac{q_{1_{v_1}}(v)}{1+q_1(v)} \cdot \Big[g_{11}(u_1, z, v_1)v_1 + g_{12}(u_1, z, v)v_2\Big] \\
&+ \frac{q_{1_{v_2}}(v)}{1+q_1(v)} \cdot \Big[g_{21}(u_1, z, v_1)v_1 + g_{22}(u_1, z, v)v_2 - g_{21}(0, v_1)v_1 - g_{22}(0, v)v_2\Big], \\
Q_2(v) =\,& J_2(v) + \frac{\lambda_1 q_{1_{v_1}}(v)v_1 + q_{1_{v_2}}(v)\big(\lambda_2 v_2 + g_{21}(0, v_1)v_1 + g_{22}(0, v)v_2\big)}{1+q_1(v)}.
\end{aligned}
$$

It is easy to see that Q_1 vanishes at $u = 0$ and also the updated f_{ij} and g_{ij} in system (A.15) satisfy all the conditions (2.9) except for (A.9) and (A.10). In order to get (A.9), it is sufficient to find $q_1(v)$ such that $Q_2(v) \equiv 0$, i.e. $q_1(v)$ satisfies the relation

$$
-(1+q_1(v))\,J_2(v) = q_{1_{v_1}}(v) \cdot \big[\lambda_1 v_1\big] + q_{1_{v_2}}(v) \cdot \big[\lambda_2 v_2 + g_{21}(0, v_1)v_1 + g_{22}(0, v)v_2\big]. \tag{A.16}
$$

Consider the $\mathcal{C}^\infty$-smooth system

$$
\begin{aligned}
\dot{U} &= -(1+U)\,J_2(v), \\
\dot{v}_1 &= \lambda_1 v_1, \\
\dot{v}_2 &= \lambda_2 v_2 + g_{21}(0, v_1)v_1 + g_{22}(0, v)v_2,
\end{aligned}
\tag{A.17}
$$

where $(U, v_1, v_2) \in \mathbb{R}^3$. The linear part of this system at the origin is

$$
\begin{pmatrix}
0 & \frac{\partial J_2}{\partial v_1}(0) & \frac{\partial J_2}{\partial v_2}(0) \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_2
\end{pmatrix},
$$

with the spectrum $\{0, \lambda_1, \lambda_2\}$. Therefore, this system has a $\mathcal{C}^\infty$-smooth 2-dimensional local unstable invariant manifold defined by the equation $\{U = q_1(v_1, v_2)\}$ for some function q_1. Moreover, this function satisfies (A.16) because this relation is nothing but the condition of the invariance of the local unstable invariant manifold with respect to the flow of system (A.17) (see Definition 6). Thereby, as we required, a $\mathcal{C}^\infty$-smooth function $q_1(v_1, v_2)$ that fulfills (A.16) exists.

We have now shown our first claim: the underlined term $J_2(v)u_2$ in system (A.12) can be eliminated by performing a change of coordinates of the form (A.13). The proof for the second claim that the term $J_4(u)v_2$ can be eliminated by applying a change of coordinates of the form (A.14) can be accomplished analogously (see [2]).

Note that, by Remark A.4, since the desired change of coordinates in Lemma A.3 commutes with symmetry (1.5), we have that system (A.12) is invariant with respect to this symmetry. Therefore, system (A.17) is invariant with respect to the symmetries $v_1 \leftrightarrow -v_1$. Thus, $q_1(v_1, v_2) = q_1(-v_1, v_2)$. Analogously, $q_2(u_1, u_2) = q_2(-u_1, u_2)$. This means that changes of coordinates (A.13) and (A.14) commute with symmetry (1.5) too.

To finish the proof, note that changes of coordinates (A.13) and (A.14) are of the form (A.11). Therefore, by Remark A.4, applying changes of coordinates (A.13) and (A.14) together with the change of coordinates used in Lemma A.3 transforms first integral (2.6) to the form (2.10). This ends the proof of Lemma 2.4. □

Proof of Lemma 2.5. By Lemma 2.4, there exists a change of coordinates which commutes with symmetry (1.5) and brings system (1.3) and first integral (1.4) to (2.8) and (2.10), respectively. System (2.8) possesses a C^q-smooth three dimensional extended unstable invariant manifold W^{uE} defined by $\{(u, v) : u_2 = \phi^{uE}(u_1, v_1, v_2)\}$, and a C^q-smooth extended stable invariant manifold W^{sE} defined by $\{(u, v) : v_2 = \phi^{sE}(u_1, u_2, v_1)\}$ (see [23]). We claim that straightening W^{uE}, i.e. applying the C^q-smooth change of coordinates

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2 - \phi^{uE}(u_1, v_1, v_2), \qquad \tilde{v}_1 = v_1, \qquad \tilde{v}_2 = v_2, \qquad \text{(A.18)}$$

and straightening W^{sE}, i.e. applying the C^q-smooth change of coordinates

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2, \qquad \tilde{v}_1 = v_1, \qquad \tilde{v}_2 = v_2 - \phi^{sE}(u_1, u_2, v_1), \qquad \text{(A.19)}$$

reduce system (2.8) to system (2.12), where (2.9) is satisfied, and transforms first integral (2.10) to (2.13). On the other hand, by Remark A.2, straightening these manifolds keeps the invariance of system (2.8) and first integral (2.10) with respect to symmetry (1.5). Thus, we are done once we prove this claim. To this end, we use the following lemma

Lemma A.5. *The following hold for the C^q-smooth functions ϕ^{uE} and ϕ^{sE}:*
(i) $\phi^{uE}(0, v_1, v_2) \equiv \phi_{u_1}^{uE}(0, v_1, v_2) \equiv 0$,   (ii) $\phi^{sE}(u_1, u_2, 0) \equiv \phi_{v_1}^{sE}(u_1, u_2, 0) \equiv 0$.

The following are immediate consequences of this lemma:

Corollary A.6. *We can write ϕ^{uE} and ϕ^{sE} as*

$$\phi^{uE}(u_1, v_1, v_2) = u_1 p_1^{uE}(u_1, v_1, v_2) = u_1^2 p_2^{uE}(u_1, v_1, v_2),$$
$$\phi^{sE}(u_1, u_2, v_1) = v_1 p_1^{sE}(u_1, u_2, v_1) = v_1^2 p_2^{sE}(u_1, u_2, v_1),$$

where p_1^{sE} and p_1^{uE} are some C^{q-1}-smooth functions and p_2^{sE} and p_2^{uE} are some C^{q-2}-smooth functions such that $p_1^{uE} = u_1 p_2^{uE}$ and $p_1^{sE} = v_1 p_2^{sE}$.

Corollary A.7. *We have*
(i) $\phi_{v_1}^{uE}(0, v_1, v_2) \equiv \phi_{v_2}^{uE}(0, v_1, v_2) \equiv 0$,   (ii) $\phi_{u_1}^{sE}(u_1, u_2, 0) \equiv \phi_{u_2}^{sE}(u_1, u_2, 0) \equiv 0$.

We prove Lemma A.5 later. Taking into account that as a result of applying change of co-ordinates (A.18), the equation of $\dot{u}_2$ vanishes at $\{u_2 = 0\}$, one can easily see that change of coordinates (A.18) reduces system (2.8) to

$$
\begin{aligned}
\dot{u}_1 &= -\lambda_1 u_1 + \tilde{f}_{11}\,(u_1, v)\,u_1 + \tilde{f}_{12}\,(u_1, u_2, v)\,u_2, \\
\dot{u}_2 &= -\lambda_2 u_2 + \tilde{f}_{22}\,(u_1, u_2, v)\,u_2, \\
\dot{v}_1 &= +\lambda_1 v_1 + \tilde{g}_{11}\,(u, v_1)\,v_1 + \tilde{g}_{12}\,(u, v_1, v_2)\,v_2, \\
\dot{v}_2 &= +\lambda_2 v_2 + \tilde{g}_{21}\,(u, v_1)\,v_1 + \tilde{g}_{22}\,(u, v_1, v_2)\,v_2,
\end{aligned}
\tag{A.20}
$$

where

$$
\begin{aligned}
\tilde{f}_{11}\,(u_1, v) &= f_{11}\,(u_1, v) + f_{12}\,(u_1, x, v)\,p_1^{uE}\,(u_1, v), \\
\tilde{f}_{12}\,(u, v) &= f_{12}\,(u_1, u_2 + x, v) + P_1\,(u, v)\,x, \\
\tilde{f}_{22}\,(u, v) &= f_{22}\,(u_1, u_2 + x, v) + P_2\,(u, v)\,x - \phi_{u_1}^{uE}\,(u_1, v)\,[f_{12}\,(u_1, u_2 + x, v) + P_3\,(u, v)\,x] \\
&\quad - \phi_{v_1}^{uE}\,(u_1, v)\,[P_4\,(u, v)\,v_1 + P_5\,(u, v)\,v_2] - \phi_{v_2}^{uE}\,(u_1, v)\,[P_6\,(u, v)\,v_1 + P_7\,(u, v)\,v_2], \\
\tilde{g}_{i1}\,(u, v_1) &= g_{i1}\,(u_1, u_2 + x, v_1), \qquad \tilde{g}_{i2}\,(u, v) = g_{i2}\,(u_1, u_2 + x, v), \qquad (i = 1, 2).
\end{aligned}
$$

Here, $x := \phi^{uE}\,(u_1, v)$, the functions f_{ij} and g_{ij} are as in (2.8), and $P_j\,(u, v)$ ($j = 1, .., 7$) are some functions such that P_1 vanishes at $\{v = 0\}$ (see [2] for more details). Moreover, $\tilde{f}_{ij}$ are $\mathcal{C}^{q-1}$-smooth and $\tilde{g}_{ij}$ are $\mathcal{C}^q$-smooth. Using Lemma A.5 and Corollaries A.6 and A.7 and taking into account that the expression $u_2 + \phi^{uE}\,(u_1, v)$ vanish at $u = 0$, and also the functions f_{ij} and g_{ij} satisfy (2.9), one can easily show that $\tilde{f}_{ij}$ and $\tilde{g}_{ij}$ satisfy (2.9) as well.

System (A.20) is of the form (2.8) where $f_{21}\,(u_1, v) \equiv 0$. Similar to the case of straightening the extended unstable manifold, one can use Lemma A.5 and Corollaries A.6 and A.7 and show that making change of coordinates (A.19) reduces system (A.20) to system (2.12) where the corresponding f_{ij} and g_{ij} are $\mathcal{C}^{q-1}$-smooth and satisfy (2.9). This ends the proof of the first part of Lemma 2.5.

Denote the H_1 and H_2 in (2.10) by H_1° and H_2°, respectively, and let $x := (u_1, u_2, v_1)$, $y := \left(u_1, v_1, v_2 + \phi^{sE}\,(x)\right)$ and $z := \left(u_1, u_2 + \phi^{uE}\,(y), v_1, v_2 + \phi^{sE}\,(x)\right)$. Applying changes of coordinates (A.18) and (A.19) brings (2.10) to

$$
H = \lambda_1 u_1 v_1\,[1 + H_1^\circ\,(z)] - \lambda_2\left(u_2 + \phi^{uE}\,(y)\right)\left(v_2 + \phi^{sE}\,(x)\right)[1 + H_2^\circ\,(z)],
$$

which by Corollary A.6, can be written in the form (2.13), for

$$
\begin{aligned}
H_1 &= H_1^\circ\,(z) + \lambda_2 \lambda_1^{-1} p_1^{uE}\,(y)\,p_1^{sE}\,(x)\,[1 + H_2^\circ\,(z)], & H_2 &= H_2^\circ, \\
H_3 &= p_2^{sE}\,(x)\,[1 + H_2^\circ\,(z)], & H_4 &= p_2^{uE}\,(y)\,[1 + H_2^\circ\,(z)].
\end{aligned}
$$

This proves the second part of Lemma 2.5.

All that remains to finish the proof of Lemma 2.5 is proving Lemma A.5. We only prove part (i) of this lemma; the proof of part (ii) is the same.

The first identity

$$\phi^{uE}(0, v_1, v_2) \equiv 0, \tag{A.21}$$

is an immediate consequence of the fact that the extended unstable invariant manifold W^{uE} contains the unstable invariant manifold $\{u_1 = u_2 = 0\}$ (see Section [23]). Indeed, for any (v_1, v_2), we have that $(0, 0, v_1, v_2)$ belongs to $\{(u, v) : u_2 = \phi^{uE}(u_1, v_1, v_2)\}$. This implies $\phi^{uE}(0, v_1, v_2) = 0$, for any (v_1, v_2), which yields (A.21).

It is important to notice that relation (A.21) is sufficient to obtain the statement of part I of Corollary A.7. In other words, (A.21) implies part I of Corollary A.7.

To prove the identity

$$\phi^{uE}_{u_1}(0, v_1, v_2) \equiv 0, \tag{A.22}$$

we consider the condition of the invariance of the manifold W^{uE} with respect to the flow of system (2.8) (see Definition 6), i.e.

$$
\begin{aligned}
-\lambda_2 x + f_{21}(u_1, v) u_1 + f_{22}(u_1, x, v) x &= \phi^{uE}_{u_1}(u_1, v)\big[-\lambda_1 u_1 + f_{11}(u_1, v) u_1 \\
&+ f_{12}(u_1, x, v) x \big] + \phi^{uE}_{v_1}(u_1, v)\big[\lambda_1 v_1 + g_{11}(u_1, x, v_1) v_1 + g_{12}(u_1, x, v) v_2\big] \\
&+ \phi^{uE}_{v_2}(u_1, v)\big[\lambda_2 v_2 + g_{21}(u_1, x, v_1) v_1 + g_{22}(u_1, x, v) v_2\big],
\end{aligned}
$$

where $x := \phi^{uE}(u_1, v)$. Both sides of this relation are $\mathcal{C}^{q-1}$-smooth ($q \geq 2$ because $2\lambda_1 < \lambda_2$) functions of u_1, v_1 and v_2. Taking (A.21) as well as conditions (2.9) and Corollary A.7 into account, we can differentiate this relation with respect to u_1 at $u_1 = 0$ and obtain

$$
\begin{aligned}
0 = \Big[(\lambda_2 - \lambda_1)\phi^{uE}_{u_1}(0, v) &+ f_{12}(0, v)\big(\phi^{uE}_{u_1}(0, v)\big)^2\Big] + \big[\lambda_1 v_1\big]\phi^{uE}_{u_1 v_1}(0, v) \\
&+ \big[\lambda_2 v_2 + g_{21}(0, v_1)v_1 + g_{22}(0, v)v_2\big]\phi^{uE}_{u_1 v_2}(0, v).
\end{aligned} \tag{A.23}
$$

Define $z = z(v) = \phi^{uE}_{u_1}(0, v)$. Then, (A.23) can be written as

$$
\begin{aligned}
0 = \Big[(\lambda_2 - \lambda_1) z + f_{12}(0, v)z^2\Big] &+ \big[\lambda_1 v_1\big] \cdot \frac{\partial z(v)}{\partial v_1} \\
&+ \big[\lambda_2 v_2 + g_{21}(0, v_1)v_1 + g_{22}(0, v)v_2\big] \cdot \frac{\partial z(v)}{\partial v_2},
\end{aligned} \tag{A.24}
$$

where $z(0) = 0$ (note that $\phi^{uE}_{u_1}(0, 0, 0) = 0$).

To get (A.22), we need to show $z(v) \equiv 0$. First, note that $z(v) \equiv 0$ satisfies (A.24). Thus, (A.22) holds if we show that $z \equiv 0$ is the unique solution of (A.24). Note that, by Proposition A.1, $z(v)$ satisfies (A.24) if and only if the 2-dimensional manifold

$$\{(v, z) : z = z(v) \text{ and } z(0) = 0\} \tag{A.25}$$

be invariant with respect to the flow of the $\mathcal{C}^{q-1}$-smooth system

$$\dot{v}_1 = -\lambda_1 v_1,$$
$$\dot{v}_2 = -\lambda_2 v_2 - g_{21}(0, v_1)v_1 - g_{22}(0, v)v_2, \tag{A.26}$$
$$\dot{z} = (\lambda_2 - \lambda_1)\, z + f_{12}(0, v)z^2,$$

which is defined on a small neighborhood of the origin in $\mathbb{R}^3$. (Indeed, relation (A.24) is the condition of the invariance of (A.25) with respect to the flow of system (A.26).) Therefore, the uniqueness of the solution of (A.24) can be proved by showing that system (A.26) has a unique invariant manifold of the form (A.25). To do this, first, notice that this system possesses a unique two dimensional stable invariant manifold of form (A.25). Second, we observe that any orbit on manifold (A.25) converges to the origin of system (A.26): the first two equations in (A.26) are independent of z and have $(v_1, v_2) = (0, 0)$ as an asymptotically stable equilibrium. Therefore, as $t \to \infty$, an orbit $(v(t), z(v(t)))$ of system (A.26) which belongs to invariant manifold (A.25) converges to $(0, z(0))$. Since $z(0) = 0$, this means that any invariant manifold of the form (A.26) must be a subset of the stable manifold of system (A.26). However, since both manifolds are 2-dimensional, they must be the same. Therefore, system (A.26) has a unique invariant manifold of the form (A.25) which is in fact its stable invariant manifold. This ends the proof of Lemma A.5 and hence the proof of Lemma 2.5.   □

Appendix B. Proofs of Lemmas 2.7, 2.8 and 2.9

We only prove Lemma 2.7. The proofs of Lemmas 2.8 and 2.9 are similar. We refer the reader to [2] for the proofs of these lemmas. We start with a discussion on the method of the proof and then proceed to the proof of Lemma 2.7.

B.1. The method of the proof

Here, we present the main procedure which is used in the proofs of Lemmas 2.7, 2.8, 2.9 and also Lemma C.1 (see Appendix C). Consider system (2.14) and denote its unique solution that satisfies boundary condition (2.15) by (u^*, v^*), where $u^* = (u_1^*, u_2^*)$ and $v^* = (v_1^*, v_2^*)$. We may also write this as

$$\left(u^*, v^*\right) = \left(u^*(t), v^*(t)\right) = \left(u^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau}), v^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})\right), \quad \text{(B.1)}$$

to emphasize that in addition to time variable t, this solution explicitly depends on τ and the boundary conditions $u_{10}, u_{20}, v_{1\tau}$ and $v_{2\tau}$ as well. It is easy to see that $(u^*(t), v^*(t))$ is a solution of this system with boundary conditions (2.15) if and only if

$$u_1^*(t) = e^{-\lambda_1 t} u_{10} + \int_0^t e^{\lambda_1(s-t)} F_1\left(u^*(s), v^*(s)\right) ds,$$

$$u_2^*(t) = e^{-\lambda_2 t} u_{20} + \int_0^t e^{\lambda_2(s-t)} F_2\left(u^*(s), v^*(s)\right) ds,$$

$$v_1^*(t) = e^{-\lambda_1(\tau-t)} v_{1\tau} - \int_t^\tau e^{-\lambda_1(s-t)} G_1\left(u^*(s), v^*(s)\right) ds, \tag{B.2}$$

$$v_2^*(t) = e^{-\lambda_2(\tau-t)} v_{2\tau} - \int_t^\tau e^{-\lambda_2(s-t)} G_2\left(u^*(s), v^*(s)\right) ds.$$

For a given τ, denote by $\mathcal{I}$ the set of all vector valued functions $(u_1(t), u_2(t), v_1(t), v_2(t))$ defined for $t \in [0, \tau]$ on some small neighborhood of the origin in $\mathbb{R}^4$. Then, the right-hand side of (B.2) defines an integral operator on $\mathcal{I}$, denote it by $\mathfrak{T}$, as follows:

$$\mathfrak{T} : (u_1(t), u_2(t), v_1(t), v_2(t)) \mapsto (\overline{u}_1(t), \overline{u}_2(t), \overline{v}_1(t), \overline{v}_2(t)),$$

where

$$\overline{u}_1(t) = e^{-\lambda_1 t} u_{10} + \int_0^t e^{\lambda_1(s-t)} F_1(u(s), v(s)) ds,$$

$$\overline{u}_2(t) = e^{-\lambda_2 t} u_{20} + \int_0^t e^{\lambda_2(s-t)} F_2(u(s), v(s)) ds,$$

$$\overline{v}_1(t) = e^{-\lambda_1(\tau-t)} v_{1\tau} - \int_t^\tau e^{-\lambda_1(s-t)} G_1(u(s), v(s)) ds,$$

$$\overline{v}_2(t) = e^{-\lambda_2(\tau-t)} v_{2\tau} - \int_t^\tau e^{-\lambda_2(s-t)} G_2(u(s), v(s)) ds.$$

The solution $(u^*(t), v^*(t))$ is in fact the fixed point of this integral operator. According to [23] (Theorems 2.9 and 5.11), this integral operator is a contraction and its fixed point is the limit of the sequence of successive approximations

$$\left\{ \left(u^{(n)}(t), v^{(n)}(t)\right) = \left(u_1^{(n)}(t), u_2^{(n)}(t), v_1^{(n)}(t), v_2^{(n)}(t)\right) \right\}_{n=0}^{n=\infty},$$

where $\left(u^{(0)}, v^{(0)}\right) \equiv (0, 0)$ and

$$\left(u^{(n+1)}(t), v^{(n+1)}(t)\right) = \mathfrak{T}\left(u^{(n)}(t), v^{(n)}(t)\right), \quad \forall n \geq 0.$$

Let $\mathcal{A}$ be a closed subset of $\mathcal{I}$ such that $(u(t), v(t)) \equiv (0, 0) \in \mathcal{A}$ and $\mathfrak{T}(\mathcal{A}) \subset \mathcal{A}$. Since $\left(u^{(0)}, v^{(0)}\right) \equiv (0, 0) \in \mathcal{A}$, the invariance of $\mathcal{A}$ implies that $\left(u^{(n)}(t), v^{(n)}(t)\right)$ belongs to $\mathcal{A}$ for all $n > 0$, and so does the solution $(u^*(t), v^*(t))$.

Remark B.1. Assume that there exists a *'certain estimate'* which for any arbitrary $(u(t), v(t)) \in \mathcal{A}$, its image $\mathfrak{T}(u(t), v(t))$ satisfies. Therefore, since $\mathfrak{T}(u^*, v^*) = (u^*, v^*)$, the solution (u^*, v^*) itself satisfies that certain estimate as well.

Our approach for proving Lemmas 2.7, 2.8 and 2.9 (and Lemma C.1) is based on this remark. We construct the integral operator, introduce the invariant set $\mathcal{A}$ and find an estimate for the image of the elements of this set under $\mathfrak{T}$. Then, this estimate holds for the solution (u^*, v^*) too.

B.2. Proof of Lemma 2.7

Throughout, we use the following notation: $x = (u, v)$, $x(t) = (u(t), v(t))$, and $x(s) = (u(s), v(s))$. Recast system (2.4) into the form (2.14), where

$$
\begin{aligned}
F_i(x) &= \mathtt{f}_{i1}(x)\, u_1^2 + \mathtt{f}_{i2}(x)\, u_1 u_2 + \mathtt{f}_{i3}(x)\, u_1 v_1 + \mathtt{f}_{i4}(x)\, u_1 v_2 + \mathtt{f}_{i5}(x)\, u_2^2 \\
&\quad + \mathtt{f}_{i6}(x)\, u_2 v_1 + \mathtt{f}_{i7}(x)\, u_2 v_2, \\
G_i(x) &= \mathtt{g}_{i1}(x)\, v_1^2 + \mathtt{g}_{i2}(x)\, v_1 v_2 + \mathtt{g}_{i3}(x)\, v_1 u_1 + \mathtt{g}_{i4}(x)\, v_1 u_2 + \mathtt{g}_{i5}(x)\, v_2^2 \\
&\quad + \mathtt{g}_{i6}(x)\, v_2 u_1 + \mathtt{g}_{i7}(x)\, v_2 u_2,
\end{aligned}
\tag{B.3}
$$

for $i = 1, 2$, and some continuous functions $\mathtt{f}_{ij}$ and $\mathtt{g}_{ij}$. Let Ω be a small compact neighborhood of O and define

$$
M^* := \sup_{(u,v)\in\Omega} \left\{ |\mathtt{f}_{ij}(u,v)|, |\mathtt{g}_{ij}(u,v)| \right\}.
\tag{B.4}
$$

Let $\delta > 0$ be small and consider the set

$$
\mathcal{A} = \left\{ x(t): \quad |u_1(t)|, |u_2(t)| \le 2e^{-\lambda t}\delta, \quad |v_1(t)|, |v_2(t)| \le 2e^{-\lambda(\tau-t)}\delta \right\},
\tag{B.5}
$$

where $x(t)$ is any continuous function defined on Ω for $t \in [0, \tau]$.

We first show that $\mathcal{A}$ is invariant with respect to the integral operator $\mathfrak{T}$, i.e. $\mathfrak{T}(\mathcal{A}) \subseteq \mathcal{A}$. By (B.3), (B.4) and (B.5), for any $(u_1(t), u_2(t), v_1(t), v_2(t))$ in $\mathcal{A}$, we have

$$
\max\{|F_1(x(t))|, |F_2(x(t))|\} \le M^* \left(12 e^{-2\lambda t}\delta^2 + 16 e^{-\lambda\tau}\delta^2 \right),
$$

$$
\max\{|G_1(x(t))|, |G_2(x(t))|\} \le M^* \left(12 e^{-2\lambda(\tau-t)}\delta^2 + 16 e^{-\lambda\tau}\delta^2 \right).
$$

Let $M = 32 M^* \lambda^{-1}$. For $i = 1, 2$, we have

$$
\left| \overline{u}_i(t) - e^{-\lambda t} u_{i0} \right| \le \int_0^t e^{\lambda(s-t)} |F_i(x(s))|\, ds
$$

$$
\le 16 M^* \delta^2 \int_0^t e^{\lambda(s-t)} \left(e^{-2\lambda s} + e^{-\lambda\tau} \right) ds \le M e^{-\lambda t}\delta^2,
$$

$$
|\overline{v}_i(t) - e^{-\lambda(\tau-t)} v_{i\tau}| \le \int_t^\tau e^{\lambda(t-s)} |G_i(x(s))|\, ds
$$

$$\leq 16 M^* \delta^2 \int_t^\tau e^{\lambda(t-s)} \left(e^{-2\lambda(\tau-s)} + e^{-\lambda\tau} \right) \leq M e^{-\lambda(\tau-t)} \delta^2.$$

Choose δ sufficiently small such that $M\delta < 1$. Taking into account that $|u_{10}|$, $|u_{20}|$, $|v_{1\tau}|$, $|v_{2\tau}|$ are all bounded by δ, we have $\max\{|\overline{u}_1(t)|, |\overline{u}_2(t)|\} \leq 2e^{-\lambda t}\delta$ and $\max\{|\overline{v}_1(t)|, |\overline{v}_2(t)|\} \leq 2e^{-\lambda(\tau-t)}\delta$. Thus, $(\overline{u}_1(t), \overline{u}_2(t), \overline{v}_1(t), \overline{v}_2(t)) \in \mathcal{A}$, as desired.

Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ can be written in the form (2.16) such that the corresponding ξ_1, ξ_2, ζ_1 and ζ_2 satisfy the estimates given in the statement of the lemma. However, since $(u^{(0)}, v^{(0)}) = (0,0) \in \mathcal{A}$, it follows from Remark B.1 that the same holds for the solution $(u(t), v(t))$ that satisfies boundary condition (2.15). This ends the proof of Lemma 2.7.

Appendix C. Proofs of Lemmas 3.12 and 3.13

The main part of this appendix is the proof of the following lemma:

Lemma C.1. *Let (2.21) be the local map of system (2.12) and suppose $(u_{10}, v_{10}) \in \mathcal{D}_2$. Write $x := (u_{10}, v_{10})$. We have*

$$\frac{\partial\eta_1}{\partial u_1}(x) = (1+\gamma)e^{-\lambda_1\tau}[1+O(\delta)], \qquad \frac{\partial\eta_1}{\partial v_1}(x) = \gamma\frac{u_{10}}{v_{10}}\cdot e^{-\lambda_1\tau}[1+O(\delta)],$$

$$\frac{\partial\eta_2}{\partial u_1}(x) = -\gamma\frac{v_{10}}{u_{10}}\cdot e^{\lambda_1\tau}[1+O(\delta)], \qquad \frac{\partial\eta_2}{\partial v_1}(x) = (1-\gamma)e^{\lambda_1\tau}[1+O(\delta)]. \tag{C.1}$$

It is straightforward to derive Lemmas 3.12 and 3.13 from Lemma C.1 (see [2]). Indeed, by this lemma, an estimate for $\frac{\partial(\overline{u}_{10}, \overline{v}_{10})}{\partial(u_{10}, v_{10})}$ can be obtained. One can use this estimate and the implicit relations between (u_{10}, v_{10}), (w, z), $(\overline{u}_{10}, \overline{v}_{10})$ and $(\overline{w}, \overline{z})$ to prove Lemmas 3.12 and 3.13. We refer the reader to [2] for further details.

Proof of Lemma C.1. Let (2.22) be the solution of system (2.12) that satisfies boundary conditions (2.15), where $u_{20} = v_{2\tau} = \delta$. When the point $(u_{10}, \delta, v_{10}, v_{20})$ on Π^s reaches the cross-section Π^u at $(u_{1\tau}, u_{2\tau}, v_{1\tau}, \delta)$, the corresponding flight time τ is uniquely determined by u_{10} and v_{10}, i.e. $\tau = \tau(u_{10}, v_{10})$, for some function τ. Thus, by (2.23), we have

$$v_{10} = v_1^*(0, \tau(u_{10}, v_{10}), u_{10}, \delta, \eta_2(u_{10}, v_{10}), \delta), \tag{C.2}$$

$$v_{20} = v_2^*(0, \tau(u_{10}, v_{10}), u_{10}, \delta, \eta_2(u_{10}, v_{10}), \delta), \tag{C.3}$$

$$\eta_1(u_{10}, v_{10}) = u_1^*(\tau(u_{10}, v_{10}), \tau(u_{10}, v_{10}), u_{10}, \delta, \eta_2(u_{10}, v_{10}), \delta). \tag{C.4}$$

Recall that, by Lemma 2.12, v_{20} is a function of u_{10} and v_{10} which we denote it by $\kappa(u_{10}, v_{10})$. Both sides of (C.2), (C.3) and (C.4) are functions of u_{10} and v_{10}. Differentiating these three relations with respect to u_{10} and v_{10} gives the following identities

$$0 = \frac{\partial v_1^*}{\partial \tau}\bigg|_{t=0} \cdot \frac{\partial \tau}{\partial u_{10}} + \frac{\partial v_1^*}{\partial u_{10}}\bigg|_{t=0} + \frac{\partial v_1^*}{\partial v_{1\tau}}\bigg|_{t=0} \cdot \frac{\partial \eta_2}{\partial u_{10}}, \tag{C.5}$$

$$1 = \frac{\partial v_1^*}{\partial \tau}\bigg|_{t=0} \cdot \frac{\partial \tau}{\partial v_{10}} + \frac{\partial v_1^*}{\partial v_{1\tau}}\bigg|_{t=0} \cdot \frac{\partial \eta_2}{\partial v_{10}}, \tag{C.6}$$

$$\frac{\partial \kappa}{\partial u_{10}} = \frac{\partial v_2^*}{\partial \tau}\bigg|_{t=0} \cdot \frac{\partial \tau}{\partial u_{10}} + \frac{\partial v_2^*}{\partial u_{10}}\bigg|_{t=0} + \frac{\partial v_2^*}{\partial v_{1\tau}}\bigg|_{t=0} \cdot \frac{\partial \eta_2}{\partial u_{10}}, \tag{C.7}$$

$$\frac{\partial \kappa}{\partial v_{10}} = \frac{\partial v_2^*}{\partial \tau}\bigg|_{t=0} \cdot \frac{\partial \tau}{\partial v_{10}} + \frac{\partial v_2^*}{\partial v_{1\tau}}\bigg|_{t=0} \cdot \frac{\partial \eta_2}{\partial v_{10}}, \tag{C.8}$$

$$\frac{\partial \eta_1}{\partial u_{10}} = \frac{\partial u_1^*}{\partial t}\bigg|_{t=\tau} \cdot \frac{\partial \tau}{\partial u_{10}} + \frac{\partial u_1^*}{\partial \tau}\bigg|_{t=\tau} \cdot \frac{\partial \tau}{\partial u_{10}} + \frac{\partial u_1^*}{\partial u_{10}}\bigg|_{t=\tau} + \frac{\partial u_1^*}{\partial v_{1\tau}}\bigg|_{t=\tau} \cdot \frac{\partial \eta_2}{\partial u_{10}}, \tag{C.9}$$

$$\frac{\partial \eta_1}{\partial v_{10}} = \frac{\partial u_1^*}{\partial t}\bigg|_{t=\tau} \cdot \frac{\partial \tau}{\partial v_{10}} + \frac{\partial u_1^*}{\partial \tau}\bigg|_{t=\tau} \cdot \frac{\partial \tau}{\partial v_{10}} + \frac{\partial u_1^*}{\partial v_{1\tau}}\bigg|_{t=\tau} \cdot \frac{\partial \eta_2}{\partial v_{10}}. \tag{C.10}$$

To obtain the estimates in (C.1): we first estimate the following expressions

(i) $\frac{\partial u_1^*}{\partial t}\big|_{t=\tau}$,   (ii) $\frac{\partial \kappa}{\partial u_{10}}, \frac{\partial \kappa}{\partial v_{10}}$,   (iii) $\frac{\partial u_1^*}{\partial u_{10}}\big|_{t=\tau}, \frac{\partial v_1^*}{\partial u_{10}}\big|_{t=0}, \frac{\partial v_2^*}{\partial u_{10}}\big|_{t=0}$,

(iv) $\frac{\partial u_1^*}{\partial v_{1\tau}}\big|_{t=\tau}, \frac{\partial v_1^*}{\partial v_{1\tau}}\big|_{t=0}, \frac{\partial v_2^*}{\partial v_{1\tau}}\big|_{t=0}$,   (v) $\frac{\partial u_1^*}{\partial \tau}\big|_{t=\tau}, \frac{\partial v_1^*}{\partial \tau}\big|_{t=0}, \frac{\partial v_2^*}{\partial \tau}\big|_{t=0}$.

(i) Estimate for $\frac{\partial u_1^*}{\partial t}\big|_{t=\tau}$: By (2.9), (2.7) and the first equation of (2.12), we have

$$\frac{\partial u_1^*}{\partial t}\bigg|_{t=\tau} = -\lambda_1 u_{1\tau} + O\left(u_{1\tau}^2\right) + O\left(u_{1\tau} u_{2\tau}\right) + O\left(v_{1\tau} u_{2\tau}\right),$$

and by virtue of (2.34), (2.35) and (2.36), for $(u_{10}, v_{10}) \in \mathcal{D}_1 \cup \mathcal{D}_2$, we have

$$\frac{\partial u_1^*}{\partial t}\bigg|_{t=\tau} = -\lambda_1 e^{-\lambda_1 \tau} u_{10} [1 + O(\delta)]. \tag{C.11}$$

(ii) Estimates for $\frac{\partial \kappa}{\partial u_{10}}$ *and* $\frac{\partial \kappa}{\partial v_{10}}$: Following Lemma 2.12, κ is a $\mathcal{C}^q$-smooth ($q \geq 2$) function of (u_{10}, v_{10}) which is defined on an open neighborhood of $M^s \in \Pi^s$. Since its restriction to $\mathcal{D}_1 \cup \mathcal{D}_2$ is of the form (2.39), we have

$$\kappa(0,0) = \frac{\partial \kappa}{\partial u_{10}}(0,0) = \frac{\partial \kappa}{\partial v_{10}}(0,0) = \frac{\partial^2 \kappa}{\partial u_{10}^2}(0,0) = \frac{\partial^2 \kappa}{\partial v_{10}^2}(0,0) = 0, \quad \frac{\partial^2 \kappa}{\partial u_{10} v_{10}}(0,0) = \frac{\gamma}{\delta}.$$

Since $v_{10} = O(u_{10})$ and $u_{10} = O(v_{10})$, by Taylor theorem, for $(u_{10}, v_{10}) \in \mathcal{D}_2$, we derive

$$\frac{\partial \kappa}{\partial u_{10}}(u_{10}, v_{10}) = \frac{\gamma}{\delta} v_{10} [1 + o(1)], \qquad \frac{\partial \kappa}{\partial v_{10}}(u_{10}, v_{10}) = \frac{\gamma}{\delta} u_{10} [1 + o(1)].$$

In order to get estimates for $\frac{\partial u_1^*}{\partial \theta}, \frac{\partial v_1^*}{\partial \theta}$ and $\frac{\partial v_2^*}{\partial \theta}$, where $\theta = u_{10}, v_{1\tau}$ and τ, we solve some boundary value problems. Let (2.22) be the solution of system (2.12) which satisfies boundary conditions (2.15). By writing system (2.12) in the form (2.14), i.e.

$$F_1(u,v) = f_{11}(u_1, v) u_1 + f_{12}(u_1, u_2, v) u_2, \qquad F_2(u,v) = f_{22}(u_1, u_2, v) u_2,$$
$$G_1(u,v) = g_{11}(u, v_1) v_1 + g_{12}(u, v_1, v_2) v_2, \qquad G_2(u,v) = g_{22}(u, v_1, v_2) v_2, \tag{C.12}$$

where f_{ij} and g_{ij} satisfy (2.9) and (2.7), we have

$$
\begin{aligned}
\dot{u}_k^* &= -\lambda_k u_k^* + F_k(u_1^*, u_2^*, v_1^*, v_2^*), \\
\dot{v}_k^* &= +\lambda_k v_k^* + G_k(u_1^*, u_2^*, v_1^*, v_2^*), \qquad (k = 1, 2).
\end{aligned}
\tag{C.13}
$$

Differentiating (C.13) with respect to θ, where $\theta = u_{10}, v_{1\tau}$ and τ, gives

$$
\begin{pmatrix} \dot{U} \\ \dot{V} \end{pmatrix} = \mathbf{diagonal}\, (-\lambda_1, -\lambda_2, \lambda_1, \lambda_2) \cdot \begin{pmatrix} U \\ V \end{pmatrix} + \mathbf{M}(t) \cdot \begin{pmatrix} U \\ V \end{pmatrix},
\tag{C.14}
$$

where $U = (U_1, U_2)$, $V = (V_1, V_2)$, $\mathbf{M}(t) = \left. \frac{\partial (F_1, F_2, G_1, G_2)}{\partial (u_1, u_2, v_1, v_2)} \right|_{(u^*, v^*)}$ and, for $(i = 1, 2)$,

$$
U_i(t) = \frac{\partial u_i^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})}{\partial \theta}, \qquad V_i(t) = \frac{\partial v_i^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})}{\partial \theta}.
\tag{C.15}
$$

The solution $(U(t), V(t))$ of system (C.14) that satisfies the boundary conditions

$$
U_1(0) = U_{10}, \quad U_2(0) = U_{20}, \quad V_1(\tau) = V_{1\tau}, \quad V_2(\tau) = V_{2\tau}
\tag{C.16}
$$

is in fact the fixed point of the integral operator

$$
\mathfrak{T} : \left(U_1(t), U_2(t), V_1(t), V_2(t) \right) \mapsto \left(\overline{U}_1(t), \overline{U}_2(t), \overline{V}_1(t), \overline{V}_2(t) \right),
\tag{C.17}
$$

such that

$$
\overline{U}_i(t) = e^{-\lambda_i t} U_{i0} + \int_0^t e^{\lambda_i (s-t)} P_i(s)\, ds, \qquad \overline{V}_i(t) = e^{-\lambda_i (\tau - t)} V_{i\tau} + \int_t^\tau e^{\lambda_i (t-s)} Q_i(s)\, ds,
$$

where

$$
\begin{aligned}
P_i(t) =\ & F_{i u_1}\left(x^*(t)\right) \cdot U_1(t) + F_{i u_2}\left(x^*(t)\right) \cdot U_2(t) + F_{i v_1}\left(x^*(t)\right) \cdot V_1(t) \\
& + F_{i v_2}\left(x^*(t)\right) \cdot V_2(t), \\
Q_i(t) =\ & G_{i u_1}\left(x^*(t)\right) \cdot U_1(t) + G_{i u_2}\left(x^*(t)\right) \cdot U_2(t) + G_{i v_1}\left(x^*(t)\right) \cdot V_1(t) \\
& + G_{i v_2}\left(x^*(t)\right) \cdot V_2(t),
\end{aligned}
$$

for $i = 1, 2$ and $x^*(t) = (u^*(t), v^*(t))$ (see [23]). Moreover, this integral operator is a contraction and its fixed point $(U(t), V(t))$ is the limit of the sequence

$$
\left\{ \left(U^{(n)}(t), V^{(n)}(t) \right) = \left(U_1^{(n)}(t), U_2^{(n)}(t), V_1^{(n)}(t), V_2^{(n)}(t) \right) \right\}_{n=0}^{n=\infty},
$$

where $\left(U^{(0)}, V^{(0)} \right) = (0, 0)$ and $\left(U^{(n+1)}(t), V^{(n+1)}(t) \right) = \mathfrak{T}\left(U^{(n)}(t), V^{(n)}(t) \right), \forall n \geq 0.$

   (iii) Estimates for $\left. \frac{\partial u_1^*}{\partial u_{10}} \right|_{t=\tau}, \left. \frac{\partial v_1^*}{\partial u_{10}} \right|_{t=0}$ *and* $\left. \frac{\partial v_2^*}{\partial u_{10}} \right|_{t=0}$: Let (U_1, U_2, V_1, V_2) be the solution of system (C.14), i.e. the fixed point of (C.17), where

$$U_i(t) = \frac{\partial u_i^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})}{\partial u_{10}}, \qquad V_i(t) = \frac{\partial v_i^*(t, \tau, u_{10}, u_{20}, v_{1\tau}, v_{2\tau})}{\partial u_{10}}, \qquad (i = 1, 2).$$

Taking into account that (B.2) holds for the solution (u^*, v^*) of system (2.12), we have

$$U_1(0) = U_{10} = 1, \quad U_2(0) = U_{20} = 0, \quad V_1(\tau) = V_{1\tau} = 0, \quad V_2(\tau) = V_{2\tau} = 0. \qquad (C.18)$$

We claim that the solution (U, V) that satisfies (C.18) is of the form

$$
\begin{array}{ll}
U_1(t) = e^{-\lambda_1 t}[1 + O(\delta)], & U_2(t) = e^{-\lambda_2 t} O(\delta), \\
V_1(t) = e^{-\lambda_1(\tau - t)} O(\delta), & V_2(t) = e^{-\lambda_2(\tau - t)} O(\delta).
\end{array}
\qquad (C.19)
$$

To prove the claim, let us first show that the set

$$\mathcal{A} = \Big\{ (U_1(t), U_2(t), V_1(t), V_2(t)) : \quad |U_1(t)| \le 2e^{-\lambda_1 t}, \quad |U_2(t)| \le e^{-\lambda_2 t},$$
$$|V_1(t)| \le e^{-\lambda_1(\tau - t)}, \quad |V_2(t)| \le e^{-\lambda_2(\tau - t)} \Big\},$$

where $(U_1(t), U_2(t), V_1(t), V_2(t))$ is any continuous function defined on $t \in [0, \tau]$, is invariant with respect to integral operator (C.17). Note that since f_{ij} and g_{ij} in (2.12) are $\mathcal{C}^{q-1}$ $(q \ge 2)$ and fulfill (2.9) and (2.7), the first derivatives of F_i and G_i can be written as

$$F_{1u_1}(x) = f_{11}^1(x) u_1 + f_{12}^1(x) u_2, \qquad F_{1u_2}(x) = f_{11}^2(x) v_1 + f_{12}^2(x) v_2,$$

$$F_{1v_1}(x) = f_{11}^3(x) u_1 + f_{12}^3(x) u_2, \qquad F_{1v_2}(x) = f_{11}^4(x) u_1 + f_{12}^4(x) u_2,$$

$$F_{2u_1}(x) = f_{21}^1(x) u_2, \qquad F_{2u_2}(x) = f_{21}^2(x) u_1 + f_{22}^2(x) u_2,$$

$$F_{2v_1}(x) = f_{21}^3(x) u_2 \qquad F_{2v_2}(x) = f_{21}^4(x) u_2, \qquad G_{1u_1}(x) = g_{11}^1(x) v_1 + g_{12}^1(x) v_2,$$

$$G_{1u_2}(x) = g_{11}^2(x) v_1 + g_{12}^2(x) v_2, \qquad G_{1v_1}(x) = g_{11}^3(x) v_1 + g_{12}^3(x) v_2,$$

$$G_{1v_2}(x) = g_{11}^4(x) u_1 + g_{12}^4(x) u_2, \qquad G_{2u_1}(x) = g_{21}^1(x) v_2,$$

$$G_{2u_2}(x) = g_{21}^2(x) v_2, \qquad G_{2v_1}(x) = g_{21}^3(x) v_2, \qquad G_{2v_2}(x) = g_{21}^4(x) v_1 + g_{22}^4(x) v_2,$$

where $x = (u, v)$, and f_{ij}^k and g_{ij}^k are some continuous functions. Consider the constant M given by Lemma 2.9. Recall that δ is sufficiently small such that $M\delta < 1$. Let $M^\dagger = \max\{3, 3M\}$. Then, for the solution $(u^*(t), v^*(t))$ of system (2.12), we have

$$|u_i^*(t)| \le M^\dagger e^{-\lambda_i t} \delta, \qquad |v_i^*(t)| \le M^\dagger e^{-\lambda_i(\tau - t)} \delta, \qquad (i = 1, 2).$$

Let Ω be a small compact neighborhood of the equilibrium O of system (2.12). Define $M^* := \sup_{(u,v) \in \Omega} \{|f_{ij}^k(u, v)|, |g_{ij}^k(u, v)|\}$, and $M^\ddagger := M^* M^\dagger$. We have

$$\big|F_{1u_1}(u^*, v^*)\big|, \big|F_{1v_1}(u^*, v^*)\big|, \big|F_{1v_2}(u^*, v^*)\big|, \big|F_{2u_2}(u^*, v^*)\big|, \big|G_{1v_2}(u^*, v^*)\big|$$
$$\le M^\ddagger e^{-\lambda_1 t} \delta \big|F_{2u_1}(u^*, v^*)\big|, \big|F_{2v_1}(u^*, v^*)\big|, \big|F_{2v_2}(u^*, v^*)\big| \le M^\ddagger e^{-\lambda_2 t} \delta$$

$$\big|F_{1u_2}(u^*, v^*)\big|, \big|G_{1u_1}(u^*, v^*)\big|, \big|G_{1u_2}(u^*, v^*)\big|, \big|G_{1v_1}(u^*, v^*)\big|, \big|G_{2v_2}(u^*, v^*)\big|$$
$$\le M^\ddagger e^{-\lambda_1(\tau - t)} \delta \big|G_{2u_1}(u^*, v^*)\big|, \big|G_{2u_2}(u^*, v^*)\big|, \big|G_{2v_1}(u^*, v^*)\big| \le M^\ddagger e^{-\lambda_2(\tau - t)} \delta.$$

This implies

$$|P_1(t)| \le M^{\ddagger}\delta \left[3e^{-2\lambda_1 t} + 2e^{-\lambda_1 \tau}\right], \qquad |P_2(t)| \le M^{\ddagger}\delta \left[3e^{-(\lambda_1+\lambda_2)t} + 2e^{-\lambda_2 t - \lambda_1(\tau-t)}\right],$$

$$|Q_1(t)| \le M^{\ddagger}\delta \left[3e^{-\lambda_1 \tau} + 2e^{-2\lambda_1(\tau-t)}\right], \quad |Q_2(t)| \le M^{\ddagger}\delta \left[3e^{-\lambda_1 t - \lambda_2(\tau-t)} + 2e^{-(\lambda_1+\lambda_2)(\tau-t)}\right].$$

Define $M = 6M^{\ddagger}\lambda_1^{-1}$. Using the above relations, we have

$$\left|\overline{U}_1(t) - e^{-\lambda_1 t}\right| \le \int_0^t \left|e^{\lambda_1(s-t)} P_1(s)\right| ds \le 3M^{\ddagger}\delta \int_0^t e^{\lambda_1(s-t)} \left[e^{-2\lambda_1 s} + e^{-\lambda_1 \tau}\right] ds \le Me^{-\lambda_1 t}\delta,$$

$$\left|\overline{U}_2(t)\right| \le \int_0^t \left|e^{\lambda_2(s-t)} P_2(s)\right| ds \le 3M^{\ddagger}\delta \int_0^t e^{\lambda_2(s-t)} \left[e^{-(\lambda_1+\lambda_2)s} + e^{-\lambda_2 s - \lambda_1(\tau-s)}\right] ds \le Me^{-\lambda_2 t}\delta,$$

$$\left|\overline{V}_1(t)\right| \le \int_t^\tau \left|e^{\lambda_1(t-s)} Q_1(s)\right| ds \le 3M^{\ddagger}\delta \int_t^\tau e^{\lambda_1(t-s)} \left[e^{-\lambda_1 \tau} + e^{-2\lambda_1(\tau-s)}\right] ds$$

$$\le Me^{-\lambda_1(\tau-t)}\delta,$$

$$\left|\overline{V}_2(t)\right| \le \int_t^\tau \left|e^{\lambda_2(t-s)} Q_2(s)\right| ds \le 3M^{\ddagger}\delta \int_t^\tau e^{\lambda_2(t-s)} \left[e^{-\lambda_1 s - \lambda_2(\tau-s)} + e^{-(\lambda_1+\lambda_2)(\tau-s)}\right] ds$$

$$\le Me^{-\lambda_2(\tau-t)}\delta.$$

Choosing δ sufficiently small such that $M\delta < 1$, the above relation immediately implies $\left(\overline{U}_1(t), \overline{U}_2(t), \overline{V}_1(t), \overline{V}_2(t)\right) \in \mathcal{A}$, as desired.

Meanwhile, we have shown that the image of any element of $\mathcal{A}$ under $\mathfrak{T}$ is of the form (C.19). However, since $\left(U^{(0)}, V^{(0)}\right) \equiv (0,0) \in \mathcal{A}$, it follows from Remark B.1 that the same holds for the solution $(U(t), V(t))$ that satisfies boundary condition (C.16). This gives (C.19) and therefore,

$$\frac{\partial u_1^*}{\partial u_{10}}\bigg|_{t=\tau} = e^{-\lambda_1 \tau}[1 + O(\delta)], \qquad \frac{\partial v_1^*}{\partial u_{10}}\bigg|_{t=0} = e^{-\lambda_1 \tau} O(\delta), \qquad \frac{\partial v_2^*}{\partial u_{10}}\bigg|_{t=0} = e^{-\lambda_2 \tau} O(\delta).$$

(iv) Estimates for $\frac{\partial u_1^*}{\partial v_{1\tau}}\Big|_{t=\tau}, \frac{\partial v_1^*}{\partial v_{1\tau}}\Big|_{t=0}$ *and* $\frac{\partial v_2^*}{\partial v_{1\tau}}\Big|_{t=0}$: Let (U_1, U_2, V_1, V_2) be the solution of system (C.14), where U_i and V_i are as in (C.15) for $\theta = v_{1\tau}$. With the same method that we derived (C.19), one can prove that when $(u_{10}, v_{10}) \in \mathcal{D}_2$, the solution (U, V) is of the form

$$U_1(t) = e^{-\lambda_1(\tau+t)} O(\delta), \qquad U_2(t) = e^{-\lambda_2 t} O(\delta),$$
$$V_1(t) = e^{-\lambda_1(\tau-t)}[1 + O(\delta)], \qquad V_2(t) = e^{-\lambda_2(\tau-t)} O(\delta),$$

(see [2]). Therefore, when $(u_{10}, v_{10}) \in \mathcal{D}_2$, we have

$$\frac{\partial u_1^*}{\partial v_{1\tau}}\bigg|_{t=\tau} = e^{-2\lambda_1 \tau} O(\delta), \qquad \frac{\partial v_1^*}{\partial v_{1\tau}}\bigg|_{t=0} = e^{-\lambda_1 \tau}[1 + O(\delta)], \qquad \frac{\partial v_2^*}{\partial v_{1\tau}}\bigg|_{t=0} = e^{-\lambda_2 \tau} O(\delta).$$

(v) Estimates for $\frac{\partial u_1^*}{\partial \tau}\Big|_{t=\tau}$, $\frac{\partial v_1^*}{\partial \tau}\Big|_{t=0}$ *and* $\frac{\partial v_2^*}{\partial \tau}\Big|_{t=0}$: Let (U_1, U_2, V_1, V_2) be the solution of system (C.14), where U_i and V_i are as in (C.15) for $\theta = \tau$. With the same method that we derived (C.19), one can prove that when $(u_{10}, v_{10}) \in \mathcal{D}_2$, the solution (U, V) is of the form

$$
\begin{aligned}
U_1(t) &= e^{-\lambda_1 t} O\left(\delta |u_{10}|\right), & U_2(t) &= e^{-\lambda_2 t} O\left(\delta^2\right), \\
V_1(t) &= e^{-\lambda_1(\tau-t)} v_{1\tau} \left[-\lambda_1 + O\left(\delta\right)\right], & V_2(t) &= e^{-\lambda_2(\tau-t)} \delta \left[-\lambda_2 + O\left(\delta\right)\right],
\end{aligned}
$$

(see [2]). Therefore, when $(u_{10}, v_{10}) \in \mathcal{D}_2$, we have

$$
\frac{\partial u_1^*}{\partial \tau}\Big|_{t=\tau} = e^{-\lambda_1 \tau} O\left(\delta |u_{10}|\right), \qquad \frac{\partial v_1^*}{\partial \tau}\Big|_{t=0} = -\lambda_1 e^{-\lambda_1 \tau} v_{1\tau} \left[1 + O\left(\delta\right)\right],
$$

$$
\frac{\partial v_2^*}{\partial \tau}\Big|_{t=0} = -\lambda_2 e^{-\lambda_2 \tau} \delta \left[1 + O\left(\delta\right)\right].
$$

So far, we have obtained all the estimates that we required. Substituting these estimates into (C.6) and (C.8) gives

$$
1 = -\lambda_1 e^{-\lambda_1 \tau} v_{1\tau} \left[1 + O\left(\delta\right)\right] \cdot \frac{\partial \tau}{\partial v_{10}} + e^{-\lambda_1 \tau} \left[1 + O\left(\delta\right)\right] \cdot \frac{\partial \eta_2}{\partial v_{10}} \tag{C.20}
$$

$$
\frac{\gamma u_{10}}{\delta} \left[1 + O\left(\delta\right)\right] = -\lambda_2 e^{-\lambda_2 \tau} \delta \left[1 + O\left(\delta\right)\right] \cdot \frac{\partial \tau}{\partial v_{10}} + e^{-\lambda_2 \tau} O\left(\delta\right) \cdot \frac{\partial \eta_2}{\partial v_{10}}. \tag{C.21}
$$

Relation (C.21) implies

$$
\frac{\partial \tau}{\partial v_{10}} = \frac{-1}{\lambda_2 \delta} \left(e^{\lambda_2 \tau} \frac{\gamma u_{10}}{\delta} + O\left(\delta\right) \cdot \frac{\partial \eta_2}{\partial v_{10}}\right) \left[1 + O\left(\delta\right)\right]. \tag{C.22}
$$

By substituting this into (C.20), we have

$$
\frac{\partial \eta_2}{\partial v_{10}} = \left(e^{\lambda_1 \tau} - \frac{\gamma^2}{\delta^2} \cdot e^{\lambda_2 \tau} u_{10} v_{1\tau}\right) \left[1 + O\left(\delta\right)\right] \xrightarrow{\text{Proposition 2.17}} \frac{\partial \eta_2}{\partial v_{10}} = (1-\gamma) e^{\lambda_1 \tau} \left[1 + O\left(\delta\right)\right],
$$

as desired in (C.1). By Proposition 2.17, substituting this estimate into (C.22) yields

$$
\frac{\partial \tau}{\partial v_{10}} = -\frac{1}{\lambda_2} \cdot \frac{1}{v_{10}} \left[1 + O(v_{1\tau})\right] = -\frac{1}{\lambda_2} \cdot \frac{1}{v_{10}} \left[1 + O(\delta)\right].
$$

Similarly, we can estimate derivatives of τ and η_2 with respect to u_{10}. By substituting the obtained estimates into (C.7), we have

$$
\frac{\partial \tau}{\partial u_{10}} = \frac{-1}{\lambda_2 \delta} \left(\frac{\gamma}{\delta} e^{\lambda_2 \tau} v_{10} + O\left(\delta\right) + O\left(\delta\right) \cdot \frac{\partial \eta_2}{\partial u_{10}}\right) \left[1 + O\left(\delta\right)\right]. \tag{C.23}
$$

Substituting these estimates into (C.5) and simplifying the result using Proposition 2.17 give the desired estimate for $\frac{\partial \eta_2}{\partial u_{10}}$. In addition, we have

$$
\frac{\partial \tau}{\partial u_{10}} = -\frac{1}{\lambda_2} \cdot \frac{1}{u_{10}} \left[1 + O(\delta)\right].
$$

It is easily seen that substituting the estimates that we have derived so far into (C.9) and (C.10) gives the desired estimates for $\frac{\partial \eta_1}{\partial u_{10}}$ and $\frac{\partial \eta_1}{\partial v_{10}}$. This ends the proof. $\quad\square$

Remark C.2. In the case of homoclinic figure-eight, the estimates given by Lemma C.1 also hold for the local maps T_1^{loc} (on $\mathcal{D}_2^1$), T_{12}^{loc} (on $\mathbb{D}_2^1$), T_{21}^{loc} (on $\mathbb{D}_2^2$) and T_2^{loc} (on $\mathcal{D}_2^2$). For instance, applying Lemma C.1 on the local map T^{loc} on $\mathcal{D}_2$ of the system which is derived from system (2.12) by applying the linear change of coordinates $(\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2) = (u_1, u_2, -v_1, -v_2)$ gives the estimates in Lemma C.1 for T_{12}^{loc} on $\mathbb{D}_2^1$.

Appendix D. Invariant manifolds theory for cross-maps

In this appendix, we briefly discuss the method of cross-maps which is used in this paper to prove the existence of the invariant manifolds of the Poincaré map along the homoclinic orbits. We start with the formal definition of cross-maps:

Definition 7. Let $(X^*, \|\cdot\|_{X^*})$ and $(Y^*, \|\cdot\|_{Y^*})$ be two Banach spaces, and $\mathcal{U}$ be a subset of $X^* \times Y^*$. Let

$$T : \mathcal{U} \to T(\mathcal{U})$$
$$(x, y) \mapsto (\overline{x}, \overline{y})\,,$$

be a map. We say T can be written in cross-form if and only if

$$\overline{x} = F(x, \overline{y})\,,$$
$$y = G(x, \overline{y})\,, \tag{D.1}$$

holds for some functions F and G. The map defined by (D.1) (which maps $(x, \overline{y})$ to $(\overline{x}, y)$), is called the cross-map of T and denoted by $T^\times$.

In general, the composition of two maps which each can be written in cross-form cannot necessarily be written in cross-form. Here we provide a specific setting in which the property of 'being written in cross-form' can transfer to the composition map: let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, and X_1, X_2, Y_1 and Y_2 be convex subsets of $\mathcal{B}$. Consider the maps $T_1 : X_1 \times Y_1 \to X_2 \times Y_2$ and $T_2 : X_2 \times Y_2 \to X_1 \times Y_1$ and suppose that both of them can be written in cross-form in the following way:

$$(\overline{x}, \overline{y}) = T_1(x, y) \quad \text{if and only if} \quad \overline{x} = p_1(x, \overline{y}) \text{ and } y = q_1(x, \overline{y})\,,$$

and

$$(\hat{x}, \hat{y}) = T_2(\overline{x}, \overline{y}) \quad \text{if and only if} \quad \hat{x} = p_2(\overline{x}, \hat{y}) \text{ and } \overline{y} = q_2(\overline{x}, \hat{y})\,,$$

where $p_1 : X_1 \times Y_2 \to X_2$, $q_1 : X_1 \times Y_2 \to Y_1$, $p_2 : X_2 \times Y_1 \to X_1$ and $q_2 : X_2 \times Y_1 \to Y_2$ are some smooth functions. Let

$$\max\left\{\left\|\frac{\partial p_1}{\partial x}\right\|, \left\|\frac{\partial p_1}{\partial \overline{y}}\right\|, \left\|\frac{\partial q_1}{\partial x}\right\|, \left\|\frac{\partial q_1}{\partial \overline{y}}\right\|\right\} \le K_1,$$

$$\max\left\{\left\|\frac{\partial p_2}{\partial \overline{x}}\right\|, \left\|\frac{\partial p_2}{\partial \hat{y}}\right\|, \left\|\frac{\partial q_2}{\partial \overline{x}}\right\|, \left\|\frac{\partial q_2}{\partial \hat{y}}\right\|\right\} \le K_2$$

for some constants K_1 and K_2.

Lemma D.1. *([29], Lemma 4) Define $T := T_2 \circ T_1 : X_1 \times Y_1 \to X_1 \times Y_1$. If $K_1 K_2 < 1$, then*

(i) *the map T can be written in cross-form, i.e. there exist functions p and q such that*

$$(\hat{x}, \hat{y}) = T(x, y) \quad \text{if and only if} \quad \hat{x} = p(x, \hat{y}) \text{ and } y = q(x, \hat{y}).$$

Moreover, the functions p and q are smooth and defined everywhere on $X_1 \times Y_1$.

(ii) *Equip $X_1 \times Y_1$ with the norm $\|(x, y)\|_* = \max\{\sqrt{K_1}\|x\|, \sqrt{K_2}\|y\|\}$. We have*

$$\left\|\frac{\partial (p, q)}{\partial (x, \hat{y})}\right\|_* \le \frac{\sqrt{K_1 K_2}}{1 - \sqrt{K_1 K_2}}.$$

The next theorem provides a setting in which if a map T possesses a cross-map $T^\times$ which satisfies certain properties, then it has an invariant manifold that contains ω-limit points of every forward orbit of the domain. This theorem becomes powerful when one is looking for the invariant manifolds of a non-smooth map whose cross-map is smooth. This result was first obtained by Afraimovich and Shilnikov [1] for maps defined on an annulus. The following formulation of this result which holds for arbitrary Banach spaces is stated in [23].

Theorem 8. *([23], Theorem 4.3) With the setting in Definition 7, let X and Y be two convex closed subsets of X^* and Y^*, respectively, such that $\mathcal{R} = X \times Y \subset \mathcal{U}$, $T^\times$ is defined on $\mathcal{R}$ and $T^\times(\mathcal{R}) \subset \mathcal{R}$. Let F and G in (D.1) be $\mathcal{C}^1$-smooth and satisfy*

$$\sqrt{\sup_{(x,\overline{y}) \in X \times Y} \left\{\left\|\frac{\partial F}{\partial x}\right\| \cdot \left\|\frac{\partial G}{\partial \overline{y}}\right\|\right\}} + \sqrt{\left\|\frac{\partial F}{\partial \overline{y}}\right\|_\circ \cdot \left\|\frac{\partial G}{\partial x}\right\|_\circ} < 1$$

and

$$\left\|\frac{\partial F}{\partial x}\right\|_\circ + \sqrt{\left\|\frac{\partial F}{\partial \overline{y}}\right\|_\circ \cdot \left\|\frac{\partial G}{\partial x}\right\|_\circ} < 1,$$

where $\|\varphi(x, \overline{y})\|_\circ = \sup_{(x,\overline{y}) \in X \times Y} \|\varphi(x, \overline{y})\|$ for any vector-valued or matrix-valued function φ. Then

(i) *the map T has a $\mathcal{C}^1$-smooth invariant manifold $M^* = \{(x, y) \in \mathcal{R} : x = h^*(y)\}$, where $h^* : Y \to X$ is a Lipschitz function with the Lipschitz constant*

$$\mathcal{L} = \sqrt{\left\|\frac{\partial F}{\partial \overline{y}}\right\|_\circ \left(\left\|\frac{\partial G}{\partial x}\right\|_\circ\right)^{-1}}.$$

(ii) *for any* $x = (x, y) \in \mathcal{R}$ *and any arbitrary* $\epsilon > 0$, *there exists an integer* $\mathcal{N}_\epsilon^x \in \mathbb{N}$ *such that for any* $n > \mathcal{N}_\epsilon^x$ *if* $\{T^i(x)\}_{i=0}^{i=n} \subset \mathcal{R}$, *then* $\text{dist}(T^n(x), M^*) < \epsilon$. *In particular,* M^* *contains the* ω-*limit set of any point of* $\mathcal{R}$ *whose forward orbit lies entirely in* $\mathcal{R}$.

(iii) *if* $\mathcal{R}$ *is bounded, then the integer* $\mathcal{N}_x^\epsilon$ *given above can be chosen independent of* x, *i.e. for any arbitrary* $\epsilon > 0$, *there exists an integer* $\mathcal{N}_\epsilon \in \mathbb{N}$ *such that for any* $n > \mathcal{N}_\epsilon$ *and any* $x \in \mathcal{R}$ *if* $\{T^i(x)\}_{i=0}^{i=n} \subset \mathcal{R}$, *then* $\text{dist}(T^n(x), M^*) < \epsilon$.

(iv) *let* M *be a* $\mathcal{L}$-*surface (i.e.* M *is the graph of some* $\mathcal{L}$-*Lipschitz function* $h : Y \to X$). *Then* $T(M)|_{X \times Y}$ *is a* $\mathcal{L}$-*surface as well. Moreover, the sequence* $\{T^n(M)|_{X \times Y}\}$ *converges to* M^*.

Proof. See [23], Theorem 4.3 as well as Theorem 4.2 and its proof. $\square$

Proposition D.2. *With the setting of Theorem* 8, *if* $\mathcal{R}$ *is bounded,* T^{-1} *exists and the backward orbit of a point* $x \in \mathcal{R}$ *lies entirely in* $\mathcal{R}$ *then* $x \in M^*$.

Proof. The proof is by contradiction. Assume $x \notin M^*$. This implies $\text{dist}(x, M^*) > 0$. Choose an $0 < \epsilon < \text{dist}(x, M^*)$ and consider $\mathcal{N}_\epsilon$ given by Theorem 8. We have $\text{dist}(M^*, T^{\mathcal{N}_\epsilon}(T^{-\mathcal{N}_\epsilon}(x))) < \epsilon$ and thereby $\text{dist}(M^*, x) < \epsilon$, which is a contradiction. $\square$

References

[1] V.S. Afraimovich, L.P. Shilnikov, The ring principle in problems of interaction between two self-oscillating systems, J. Appl. Math. Mech. 41 (1977) 632–641.
[2] S. Bakrani, Invariant manifolds of homoclinic orbits: super-homoclinics and multi-pulse homoclinic loops, PhD thesis, Imperial College London, 2020, https://doi.org/10.25560/81572.
[3] P.G. Barrientos, S. Ibáñez, J.A. Rodríguez, Robust cycles unfolding from conservative bifocal homoclinic orbits, Dyn. Syst. Int. J. 31 (4) (2016) 546–579.
[4] I.U. Bronstein, A.Ya. Kopanskii, Finitely smooth normal forms of vector fields in the vicinity of a rest point, in: Global Analysis—Studies and Applications, V, in: Lecture Notes in Math., vol. 1520, Springer, Berlin, 1992, pp. 157–172, MR1167014 (93j:58120).
[5] P.G. Barrientos, A. Raibekas, A.A.P. Rodrigues, Chaos near a reversible homoclinic bifocus, Dyn. Syst. Int. J. 34 (3) (2019) 504–516.
[6] L.A. Belyakov, L.P. Shilnikov, Homoclinic curves and complex solitary waves, Sel. Math. Sov. 9 (3) (1990) 219–228, Selected translations.
[7] B. Buffoni, E. Séré, A global condition for quasi-random behavior in a class of conservative systems, Commun. Pure Appl. Math. 49 (3) (1996) 285–305.
[8] T. Chawanya, P. Ashwin, A minimal system with a depth-two heteroclinic network, Dyn. Syst. Int. J. 25 (3) (2010) 397–412.
[9] R.L. Devaney, Homoclinic orbits in Hamiltonian systems, J. Differ. Equ. 21 (2) (1976) 431–438.
[10] V.M. Eleonsky, V.G. Korolev, On coding of series of homoclinic loops, Int. J. Bifurc. Chaos 6 (6) (1996) 1029–1057.
[11] V.M. Eleonsky, V.G. Korolev, N.E. Kulagin, L.P. Shilnikov, Homoclinic orbits and their bifurcations in dynamical systems with two degrees of freedom: a method of qualitative and numerical analysis, Int. J. Bifurc. Chaos 3 (2) (1993) 385–397.
[12] V.M. Eleonsky, N.E. Kulagin, D.V. Turaev, L.P. Shilnikov, Classification of self-localized states of an electromagnetic field in a nonlinear medium, Dokl. Akad. Nauk SSSR 309 (4) (1989) 848–852.
[13] S. Gonchenko, M.-Ch. Li, Shilnikov's cross-map method and hyperbolic dynamics of three-dimensional Hénon-like maps, Regul. Chaotic Dyn. 15 (2–3) (2010) 165–184.
[14] A.J. Homburg, Global Aspects of Homoclinic Bifurcations of Vector Fields, Memoirs of the American Mathematical Society, vol. 121(578), 1996, viii+128.
[15] Y. Ilyashenko, W. Li, Nonlocal Bifurcations, Mathematical Surveys and Monographs, vol. 66, American Mathematical Society, Providence, RI, 1999.
[16] P. Kirrmann, G. Schneider, A. Mielke, The validity of modulation equations for extended systems with cubic nonlinearities, Proc. R. Soc. Edinb., Sect. A, Math. 122 (1–2) (1992) 85–91.

[17] L.M. Lerman, Complex dynamics and bifurcations in a Hamiltonian system having a transversal homoclinic orbit to a saddle focus, Chaos Interdis. J. Nonlinear Sci. 1 (2) (1991) 174–180.

[18] L.M. Lerman, Homo- and heteroclinic orbits, hyperbolic subsets in a one-parameter unfolding of a Hamiltonian system with heteroclinic contour with two saddle-foci, Regul. Chaotic Dyn. 2 (3–4) (1997) 139–155, V. I. Arnold (on the occasion of his 60th birthday) (in Russian).

[19] L.M. Lerman, Dynamical phenomena near a saddle-focus homoclinic connection in a Hamiltonian system, J. Stat. Phys. 101 (1–2) (2000) 357–372, Dedicated to Grégoire Nicolis on the occasion of his sixtieth birthday (Brussels, 1999).

[20] H. Najafi Alishah, P. Duarte, T. Peixe, Asymptotic Poincaré maps along the edges of polytopes, Nonlinearity 33 (1) (2020) 469–510.

[21] I.M. Ovsyannikov, L.P. Shilnikov, Systems with a saddle-focus homoclinic curve, Mat. Sb. (N.S.) 130(172) (4) (1986) 552–570.

[22] B. Sandstede, Center manifolds for homoclinic solutions, J. Dyn. Differ. Equ. 12 (3) (2000) 449–510.

[23] L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, L.O. Chua, Methods of Qualitative Theory in Nonlinear Dynamics. Part I, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises., vol. 4, World Scientific Publishing Co., Inc., River Edge, NJ, 1998, with the collaboration of Sergey Gonchenko (Sections 3.7 and 3.8), Oleg Stenkin (Section 3.9 and Appendix A) and Mikhail Shashkov (Sections 6.1 and 6.2).

[24] L.P. Shilnikov, D.V. Turaev, Super-homoclinic orbits and multi-pulse homoclinic loops in Hamiltonian systems with discrete symmetries, Regul. Chaotic Dyn. 2 (3–4) (1997) 126–138, V. I. Arnold (on the occasion of his 60th birthday) (in Russian).

[25] M.V. Shashkov, D.V. Turaev, An existence theorem of smooth nonlocal center manifolds for systems close to a system with a homoclinic loop, J. Nonlinear Sci. 9 (5) (1999) 525–573.

[26] D.V. Turaev, L.P. Shilnikov, Hamiltonian systems with homoclinic saddle curves, Dokl. Akad. Nauk SSSR 304 (4) (1989) 811–814.

[27] D.V. Turaev, On a case of bifurcations of a contour composed by two homoclinic curves of a saddle, in: Methods of Qualitative Theory of Differential Equations, 1984, pp. 162–175 (in Russian).

[28] D.V. Turaev, Multi-pulse homoclinic loops in systems with a smooth first integral, in: Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems, Springer, Berlin, 2001, pp. 691–716.

[29] D.V. Turaev, Hyperbolic sets near homoclinic loops to a saddle for systems with a first integral, Regul. Chaotic Dyn. 19 (6) (2014) 681–693.

[30] J. Yang, Classification of the solitary waves in coupled nonlinear Schrödinger equations, Physica D Nonlinear Phen. 108 (1–2) (1997) 92–112.

[31] J. Yang, Multiple permanent-wave trains in nonlinear systems, Stud. Appl. Math. 100 (2) (1998) 127–152.