# Spin-s Spin-Glass Phases in the d=3 Ising Model

E. Can Artun<sup>1</sup> and A. Nihat Berker<sup>1,2</sup>

<sup>1</sup>Faculty of Engineering and Natural Sciences, Kadir Has University, Cibali, Istanbul 34083, Turkey <sup>2</sup>Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

All higher-spin ( $s \ge 1/2$ ) Ising spin glasses are studied by renormalization-group theory in spatial dimension d=3, exactly on a d=3 hierarchical model and, simultaneously, by the Migdal-Kadanoff approximation on the cubic lattice. The s-sequence of global phase diagrams, the chaos Lyapunov exponent, and the spin-glass runaway exponent are calculated. It is found that, in d=3, a finite-temperature spin-glass phase occurs for all spin values, including the continuum limit of  $s\to\infty$ . The phase diagrams, with increasing spin s, saturate to a limit value. The spin-glass phase, for all s, exhibits chaotic behavior under rescalings, with the calculated Lyapunov exponent of  $\lambda=1.93$  and runaway exponent of  $y_R=0.24$ , showing simultaneous strong-chaos and strong-coupling behaviors. The ferromagnetic-spinglass and spinglass-antiferromagnetic phase transitions occurring, along their whole length, respectively at  $p_t=0.37$  and 0.63 are unaffected by s, confirming the percolative nature of this phase transition.

# I. INTRODUCTION: SPIN-S ISING SPIN-GLASS SYSTEMS

Since the blossoming days of modern phase transitions and critical phenomena, theoretical insight has been obtained from model spin systems studied systematically as a function of the number of local states, namely for all s.[1-15]. Indeed some of the authors in these References are the founders of the field. However, the same systematic studies have not been done on the complex ordering systems of current interest, such as in the presence of frozen disorder, presumably due to calculational difficulties, now surmountable by grobal renormalization group, as shown below.

Frozen disorder of the interactions introduces many qualitatively and quantitatively new effects to statistical mechanical systems, such as the immediate (i.e., with infinitesimal disorder) conversion of first-order phase transitions into second-order phase transitions [16–19] or the creation of an entirely new phase such as the spin-glass phase [20]. The latter occurs under frozen (quenched) competing interactions causing local minimum-energy degeneracies dubbed frustration [21]. The signature of the spin-glass phase is the appearance of a chaotic sequence of interactions [22–32] under the successive scale changes of a renormalization-group transformation. This translates to a chaotic spin-spin correlation function, as function of distance, at a given scale.[33] This chaotic behavior signifies that a small change at the microscopic level in the coupling constant (J) or temperature (1/J), or randomness (p) in the system causes major changes in the correlation function between two distant spins.[33]

The spin-glass phase and its rescaling chaos appears with the introduction, by rewiring, of infinitesimal frustration to the Mattis phase [34] obtained by random local spin redefinitions (gauge transformations) in the usual ferromagnetic or antiferromagnetic phase.[35] On the other hand, strong chaos, signalled by a large Lyapunov exponent, of the spin-glass phase in fully frustated systems continues [40] until the lower-critical dimension  $d_c \simeq 2.5$  of the spin-glass phase [36–42]. Thus both grad-

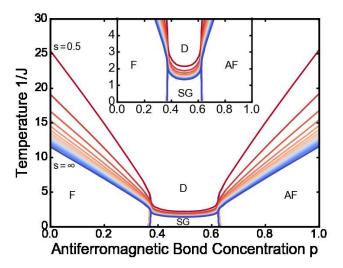


FIG. 1. Calculated phase diagrams of the spin-s Ising spin glasses in d=3. Fom top to bottom, s=1/2,1,3/2,2,5/2,3,... to  $s\to\infty$ . There is an accumulation, from above, of the phase diagrams at the lowermost, but still at finite-temperature, phase diagram of the continuum limit  $s\to\infty$ . The phase diagrams shown here are the results of our numerical calculations and are accurate to the thickness of the lines in this figure.

ual [35] or abrupt [40] onsets of chaos are seen.

Most spin-glass studies have been on the classical spin s=1/2 Ising model, where locally  $s_i=\pm 1.[44]$  Spin-glass studies have also been done on q-state clock models and their continuum limit the XY model [45, 46], chiral (helical [47]) Potts and clock models, in fact leading to a chiral spin-glass Potts [48] and clock [49, 50] phases, and quantum Heisenberg models [51]. The position-space renormalization-group method appears to be a method suited for such studies, where the rescaling behavior of the distribution of the quenched random interactions is followed and analyzed [52]. This is best effected (Fig. 2) by use of the Migdal-Kadanoff approximation [53, 54] or, equivalently, the exact recursion of a hierarchical lattice

[56–59]. In the current work, we quantitatively and globally study, in spatial dimension d=3, the Ising spin glass for all spins s=1/2,1,3/2,2,5/2,... to the limiting value  $s\to\infty$ , obtaining the global s-sequence phase diagram (Fig. 1) and chaotic behaviors.

The spin-s Ising model is defined by the Hamiltonian

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} J_{ij}(s_i/s)(s_j/s), \qquad (1)$$

where  $\beta=1/kT$ , at each site i of the lattice the spin  $s_i=\pm 1/2,\pm 1,\pm 3/2,...,\pm s,$  and  $\langle ij\rangle$  denotes summation over all nearest-neighbor site pairs. The division by s is done to conserve the energy scale across the different spin-s models and thereby make meaningful temperature comparisons between them. Note that for s=1/2, this formalism yields the much studied  $s_i/s=\pm 1$  case. The bond  $J_{ij}$  is ferromagnetic +J>0 or antiferromagnetic -J with respective probabilities 1-p and p. Under renormalization-group tranformation, this "double-delta" distribution of interactions is not conserved. A more complicated distribution of interactions ensues and is kept track of, as explained below.

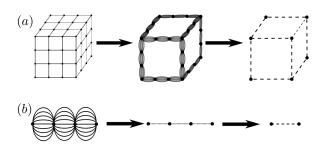


FIG. 2. (a) Migdal-Kadanoff approximate renormalization-group transformation for the d=3 cubic lattice with the length-rescaling factor of b=3. In this intuitive approximation, bond moving is followed by decimation. (b) Exact renormalization-group transformation of the d=3, b=3 hierarchical lattice for which the Migdal-Kadanoff renormalization-group recursion relations are exact. The construction of a hierarchical lattice proceeds in the opposite direction of its renormalization-group solution. From [49, 56].

## II. METHOD: RENORMALIZATION-GROUP FLOWS OF THE QUENCHED PROBABILITY DISTRIBUTION OF THE INTERACTIONS

Under renormalization group, for s > 1/2, the Hamiltonian does not conserve its form in Eq.(1). Thus, for any s, the Hamiltonian is most generally expressed as

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} E_{ij}(s_i, s_j), \qquad (2)$$

where  $E_{ij}(s_i, s_j)$  is the interaction between nearestneighbor spins, starting out as in Eq.(1) as  $J_{ij}(s_i/s)(s_j/s)$  and generalizing to a  $(2s+1) \times (2s+1)$  matrix under renormalization group. With no loss of generality, for each  $\langle ij \rangle$ , the same constant is subtracted from all terms  $E(s_i, s_j)$ , so that the largest energy  $E(s_i, s_j)_{max}$  of the spin-spin interaction is zero (and all other  $E(s_i, s_i) < 0$ ). This formulation makes it possible to follow global renormalization-group trajectories, necessary for the calculation of phase boundaries, Lyapunov exponent, and runaway exponent, without running into numerical overflow problems. As the local renormalization-group transformation, the Migdal-Kadanoff approximate transformation [53–55] and, equivalently, the exact transformation for the d=3hierarchical lattice [56–58] is used (Fig. 2). Recent works using exactly soluble hierarchical models are in Refs. [60–68]. The length rescaling factor of b=3 is used, to preserve under renormalization group the ferromagneticantiferromagnetic symmetry of the system. This local transformation consists in bond moving followed by decimation, with the above-mentioned subtraction after each local bond moving and decimation, giving the local renormalized energies  $E'(s_i, s_i) \leq 0$ . In our notation, all renormalized quantities are designated by a prime.

The quenched randomness is included by keeping, as a distribution, 10000 sets of the nearest-neighbor interaction energies  $E(s_i,s_j)$ . At the beginning of each renormalization-group trajectory, this distribution is formed from the double-delta distribution characterized by interactions  $\pm J$  with probabilities p, (1-p). 10000 local renormalization-group transformations determine each subsequent distribution as explained below.

The local renormalization-group transformation is simply expressed in terms of the transfer matrix  $T(s_i, s_j) = e^{E(s_i, s_j)}$ : Bond moving consists of multiplying elements at the same position of  $b^{d-1} = 9$  transfer matrices randomly chosen from the distribution,

$$\widetilde{T}(s_i, s_j) = \prod_{k=1}^{9} T_k(s_i, s_j), \qquad (3)$$

so that a distribution of 10000 bond-moved transfer matrices is generated. Decimation consists of matrix multiplication of three randomly chosen bond-moved transfer matrices,

$$\mathbf{T}' = \widetilde{\mathbf{T}}_1 \cdot \widetilde{\mathbf{T}}_2 \cdot \widetilde{\mathbf{T}}_3 \,, \tag{4}$$

so that a distribution of 10000 renormalized transfer matrices is generated. Phases are determined by following trajectories to their asymptotic limit: The asymptotic limit transfer matrices of trajectories starting in the ferromagnetic phase all have 1 in the corner diagonals and 0 at all other positions. The asymptotic limit transfer matrices of trajectories starting in the antiferromagnetic phase all have 1 in the corner anti-diagonals and 0 at all other positions. The asymptotic limit transfer matrices of trajectories starting in the spin-glass phase all have 1 in the corner diagonals  $(s_i = s, s_j = s)$  and  $(s_i = -s, s_j = -s)$  or in the corner anti-diagonals

 $(s_i = s, s_j = -s)$  and  $(s_i = -s, s_j = s)$ , and 0 at all other positions. We use (s, s - 1, s - 2, ..., -s + 2, -s + 1, -s)for the order of the rows and columns in the matrix  $E(s_i, s_i)$ . The asymptotic limit transfer matrices of trajectories starting in the disordered phase all have 1 at all other positions. Phase diagrams are obtained by numerically determining the boundaries, in the unrenormalized system, of these asymptotic flows. Thus, to numerically obtain the phase diagram, we determine the asymptotic limit, under repeated renormalization-group transformations, of the distribution of transfer matrices. These asymptotic limits for the ferromagnetic, antiferromagnetic, spin-glass, and disordered phases are distinct, as described here. The boundaries between different asymptotic limits, as a function of starting (unrenormalized) Jand p, yield the phase diagram numerically, as seen in Fig. 1 for different values of the spin s.

# III. RESULTS: GLOBAL S-SEQUENCE PHASE DIAGRAM AND SATURATION

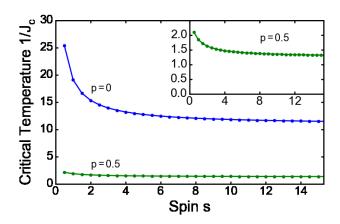


FIG. 3. The calculated ferromagnetic (at p=0) and spinglass (at p=0.5) phase transition temperatures as a function of spin value s. Note that with increasing s both transition temperatures saturate around  $s \simeq 8$ . A similar behavior was found in q-state clock models.[59] The critical temperatures in this figure are the results of our numerical calculations and are accurate to the size of the data points in this figure.

In this and the next Sections, we present the numerical results on spin-s Ising spin-glass systems, calculated exactly on a d=3 hierarchical model and, simultaneously, by the Migdal-Kadanoff approximation on the cubic lattice.

The calculated phase diagrams of the spin-s Ising spin glasses in d=3 are shown in Fig. 1. Fom top to bottom, the phase diagrams are for spin-s = 1/2, 1, 3/2, 2, 5/2, 3, ... to  $s \to \infty$ . There is an accumulation, from above, of the phase diagrams at the lowermost, but still at finite-temperature, phase diagram of the continuum limit  $s \to \infty$ . The phase diagrams shown in Fig.1 are the results of our numerical calculations and

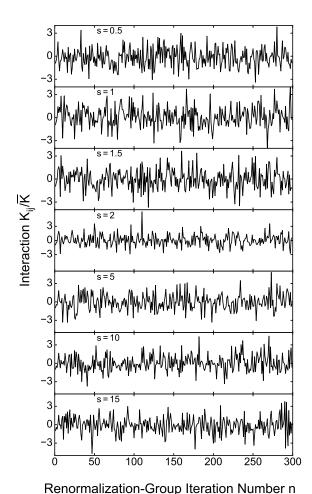


FIG. 4. The chaotic renormalization-group trajectory of the interaction  $K_{ij}$  at a given location < ij>, for various spin s values, at spatial dimension d=3. Note the strong chaotic behavior for all s, as also reflected by the calculated Lyapunov exponent  $\lambda=1.93$  for all s. The calculated runaway exponent is  $y_R=0.24$  for all s, showing simultaneous strong-chaos and strong-coupling behaviors.

are accurate to the thickness of the lines in this figure.

The calculated ferromagnetic (at p=0) and spin-glass (at p=0.5) phase transition temperatures as a function of spin value s are given in Fig. 3. With increasing s both transition temperatures saturate around  $s \simeq 8$ . A similar behavior was found in q-state clock models saturating at the continuum XY model transition temperature.[59] The critical temperatures in Fig. 3 are the results of our numerical calculations and are accurate to the size of the data points in this figure.

### IV. RESULTS: CHAOS FOR ALL SPINS S, LYAPUNOV EXPONENT AND RUNAWAY EXPONENT

For all spin-s, the renormalization-group trajectories starting within the spin-glass phase are asymptotically

chaotic, as seen in Fig. 4, where the consecutively renormalized (combining with neighboring interactions) values at a given location, meaning  $\langle ij \rangle$  then the renormalized  $\langle i'j' \rangle$  that includes  $\langle ij \rangle$ , etc., are followed. For the interaction  $K_{ij}$ , we have used the difference between the largest value (which is 0 by construction) and the lowest value in  $E(s_i, s_j)$ .  $\overline{K}$  is the average of this interaction over the entire distribution at the given renormalization-group step. Since dimensionless interaction K is proportional to inverse temperature as seen from our Eq.(1), chaotic K means chaotic temperature.

The chaotic behavior is strong, as measured by the Lyapunov exponent [69, 70]

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{dx_{k+1}}{dx_k} \right|, \tag{5}$$

where  $x_k = K_{ij}/\overline{K}$  at step k of the renormalization-group trajectory. We obtain the Lyapunov exponents by numerically calculating the logarithmic sum in Eq.(5), eliminating the first 100 renormalization-group steps as crossover from initial conditions to asymptotic behavior and then using the next 1500 steps. This calculation, for all spins s, yields  $\lambda = 1.93$ , numerically correct to the three shown significant figures.

In addition to strong chaos, the renormalization-group trajectories show asymptotic strong coupling behavior,

$$\overline{K'} = b^{y_R} \, \overline{K} \,, \tag{6}$$

where  $y_R > 0$  is the runaway exponent [40]. Again using 1500 renormalization-group steps after discarding 100 steps, we find  $y_R = 0.24$  for all spins s, numerically correct to the two shown significant figures. We thus find that  $y_R$  and  $\lambda$  are universal with respect to spin s.

Note that here is a "weak" strong coupling behavior, as the stronger runaway exponent of the ferromagnetic and antiferromagnetic phases is  $y_R = d-1 = 2$ . In fact, the runaway exponent  $y_R$  occurs, under renormalization group, in all ordered phases and is  $y_R = d-1$  in all conventionally ordered phases. However, in the spin-glass phase, since deep into this phase where the renormalization-goup flows lead (at the sink of this phase), we do not have the rescaling of compactly ordered renormalization-group cells [22],  $y_R < d-1$ , as also seen in Refs.[40, 42], where the runaway exponents  $y_R$  have been studied in detail.

#### V. CONCLUSION

We have calculated the global spin-s sequence of phase diagrams for all spins  $s=1/2,1,3/2,2,5/2,3,...,s\to\infty$  for the Ising spin-glass system in spatial dimension d=3. The phase diagrams, all with a finite-temperature spin-glass phase, for increasing spin s saturate to the limit value of  $s\to\infty$ . For all spins s, the spin-glass phase has renormalization-group trajectories that are chaotic, with calculated Lyapunov exponent  $\lambda=1.93$  and runaway exponent  $y_R=0.24$ , thus simultaneously showing strong chaotic and "weak" strong-coupling behaviors.

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