Lower-Critical Dimension of the Random-Field XY Model and the Zero-Temperature Critical Line

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The random-field XY model is studied in spatial dimensions $d = 3$ and 4, and in-between, as the limit $q \to \infty$ of the q-state clock models, by the exact renormalization-group solution of the hierarchical lattice or, equivalently, the Migdal-Kadanoff approximation to the hypercubic lattices. The lower-critical dimension is determined between $3.81 < d_c < 4$. When the random-field is scaled with q, a line segment of zero-temperature criticality is found in $d = 3$. When the random-field is scaled with q^2, a universal phase diagram is found at intermediate temperatures in $d = 3$.

I. INTRODUCTION: ISING AND XY LOWER-CRITICAL DIMENSIONS

Quenched randomness strongly affects the occurrence of order at low spatial dimension d, reflected as the lower-critical dimension d_c below which no ordering occurs for a given class of systems. In the random-magnetic-field $n = 1$ component spin Ising model, after a strong experimental and theoretical controversy between $d_c = 2$ claims [1–3] and $d_c = 3$ claims [4], the issue was settled for $d_c = 2$.[5, 6] The fact that d_c is not 3 fell in contradiction with the prediction of a dimensional shift of 2 due to random fields coming from all-order field-theoretic expansions from $d = 6$ down to $d = 1$ [7], which indeed is a considersble distance to expand upon for a small-parameter expansion of $\epsilon = 6 - d$. In this study, the logically next model, namely the $n = 2$ components spin XY model under random fields is examined and surprising results are obtained, this time in near-agreement with the dimensional shift of 2, but also with an interesting zero-temperature critical line segment and a universal scaled finite-temperature phase diagram.

Random-field Ising results supporting $d_c = 2$ were obtained [5, 6] by the Migdal-Kadanoff [8, 9] renormalization-group calculations in $d = 2$ (no random-field order), $d = 2.32$ (random-field order), and $d = 3$ (more random-field order). In the same vein, for the random-field XY model, Migdal-Kadanoff renormalization-group calculations are done here in $d = 3$ and 4, and in between. The Migdal-Kadanoff renormalization-group calculation (Fig. 1) is a highly successful, flexible, and therefore most used todate and today, physically motivated approximation for hypercubic lattices and, simultaneously, an exact calculation for d-dimensional hierarchical lattices [10–12]. The hierarchical lattice connection makes the Migdal-Kadanoff procedure a physically realizable approximation. For recent work using hierarchical lattices, see Refs. [14–21]. Migdal-Kadanoff-hierarchical-lattices correctly give the lower-critical dimensions of $d_c = 1$ of the Ising model [8, 9], $d_c = 2$ of the XY [22, 23] and ($n = 3$ spin components) Heisenberg [24] models in the absence of quenched randomness. For the much more complex system with competing quenched-random interactions, Migdal-Kadanoff gives the non-integer $d_c = 2.46$ for the Ising spin-glass system.[25–31] In addition to giving the lower-critical temperatures, it yields such diverse results as, e.g., the low-temperature algebraic order of the $d = 2$ XY model [22, 23], the chaotic nature [32–34] of the ferromagnetic-antiferromagnetic [35] and left-right chiral [36] Ising spin glasses, and the changeover from second- to first-order phase transitions of q-state Potts models in $d = 2$ and 3.[37]

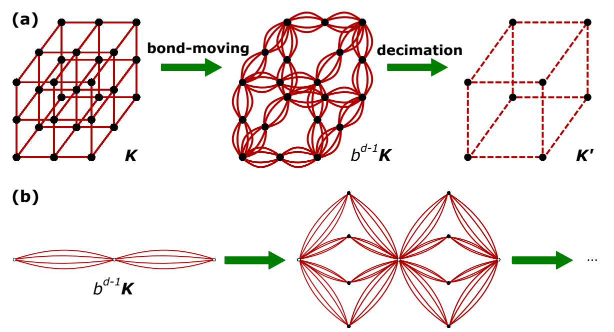

FIG. 1. From Ref.[38]: (a) Migdal-Kadanoff approximate renormalization-group transformation for the $d = 3$ cubic lattice with the length-rescaling factor of $b = 2$. (b) Construction of the $d = 3, b = 2$ hierarchical lattice for which the Migdal-Kadanoff recursion relation is exact. For general spatial dimension d, the bond-moving is (b^{d-1})-fold. The renormalization-group solution of a hierarchical lattice proceeds in the opposite direction of its construction.

II. MODEL AND METHOD

The XY model is approached as the $q \to \infty$ limit of the q-state clock models. In the q-state clock models, at each site i of the lattice, a planar unit spin $\vec{s}_i$ can

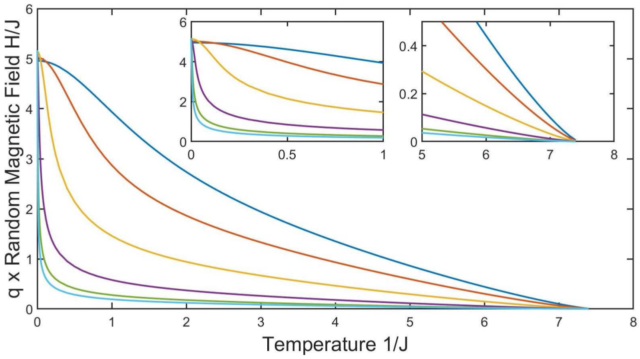

FIG. 2. Phase diagrams for $(q = 7, 10, 20, 50, 100, 150)$-state random-field clock models in $d = 3$, occurring in the figure respectively from high field to low field. Disordered and ferromagnetic phases occur at high temperature-high field and low temperature-low field, respectively. It is seen that the ferromagnetic phase, in random field, disappears as $q \to \infty$, indicating that no ferromagnetic phase occurs in the random-field XY model at non-zero temperature in $d = 3$. However, for high q, the ordered phase extends to $qH/J = 5.1$ at zero temperature, as also seen in the left box. For high q, the zero-field ferromagnetic transition temperature saturates, as also seen in Ref.[38] and in the left box in this figure.

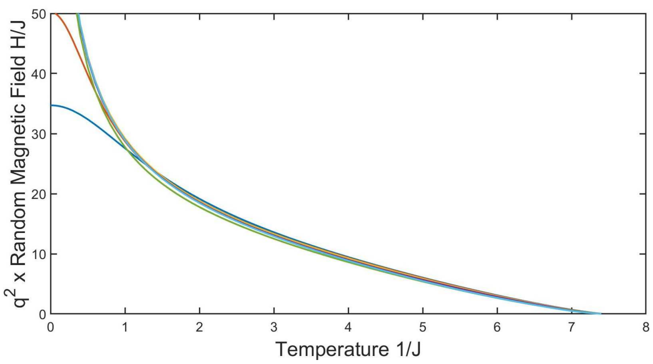

FIG. 3. Phase diagrams for $(q = 7, 10, 20, 50, 100, 150)$-state random-field clock models in $d = 3$. At low temperature, the curves are, from low field to high field, $q = 7, 10$ and indistinguishably $q = 20, 50, 100, 150$. It is seen that, when the random field is scaled with q^2, a universal phase diagram is found above low temperature for high q.

point in one of q directions in the plane, namely with the angle $\theta_k = k(2\pi/q)$, where $k = 0, 1, ..., q - 1$. A detailed renormalization-group study on the phase transitions and thermodynamics of the q-state clock models, without quenched randomness, has been done.[38] The currently studied q-state clock model, with quenched random fields, is defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{<ij>} (J\vec{s}_i \cdot \vec{s}_j + \vec{s}_i \cdot \vec{H}_i + \vec{s}_j \cdot \vec{H}_j), \qquad (1)$$

where $\beta = 1/k_B T$ and sum is over all nearest-neighbor pairs of spins. In each term in the sum, the random-fields $\vec{H}_i, \vec{H}_j$ have magnitude H and each randomly points along one of the allowed directions θ_k.

We solve this model using the Migdal-Kadanoff renormalization group. The local renormalization-group transformation is given in Fig. 1 and is simple to implement in systems without quenched randomness. With our currently studied quenched random-field model, the renormalization-group evolution of quenched random distributions has to be pursued. Initially, 5,000 nearest-neighbor Hamiltonians are created, with 10,000 randomly chosen magnetic field directions as described above. From this distribution, b^d nearest-neighbor Hamiltonians are randomly chosen, to effect the local Migdal-Kadanoff transformation and obtain a renormalized nearest-neighbor Hamiltonian. This is repeated 5,000 times and the renormalized distribution is obtained. Each nearest-neighbor Hamiltonian in the distribution is exponentiated and thus kept as a transfer matrix.[35, 38] To conserve, in this distribution, the $(ij) \leftrightarrow (ji)$ and the random-field direction symmetries, each transfer matrix is replicated by its transpose and by the simultaneous cyclic permutations of the rows and columns. Of the resulting $2q \times 5000$ matrices, 5,000 are randomly chosen. Thus, the distribution continues as 5,000 $q \times q$ matrices.

The flows of the distributions determine the phase diagram: Renormalization-group trajectories starting in the ferromagnetic phase flow to the strong-coupling sink of $J_{ij} \to \infty, H_i = 0$. Renormalization-group trajectories starting in the disordered phase flow to the decoupled sink of $J_{ij}, H_i = 0$. The boundaries between these flow basins are the phase boundaries.

III. $d = 3$ DIMENSIONS AND ZERO-TEMPERATURE CRITICALITY SEGMENT

Our calculated phase diagrams for $(q = 7, 10, 20, 50, 100, 150)$-state random-field clock models in $d = 3$ are in Fig. 2, occurring in the figure respectively from high field to low field. Disordered and ferromagnetic phases occur at high temperature-high field and low temperature-low field, respectively. The H/J values on the vertical axis are multiplied with q, originally for better graphical visibility, but eventually leading to a physical result, as seen here. Firstly, note that the ferromagnetic region under random fields recedes and disappears as q is increased. This result is even more evident, when we recall that the vertical axis values are amplified by a factor of q for better pictorial visibility. The ferromagnetic phase, in random field, disappearing as $q \to \infty$ indicates that no ferromagnetic phase occurs in the random-field XY model at non-zero temperature in $d = 3$.

Secondly and quite interestingly, given our choice of vertical axis values, it revealed that the ordered phase extends at very low temperatures, for the high q to the universal value of $qH/J = 5.1$. This is more visible in the left inset box of Fig. 2. Thus, at $q \to \infty$, a line segment of zero-temperature critical points occurs between $qH/J = 0$ and $qH/J = 5.1$. Zero-temperature critical segments and multicritical points have been found before, under exact renormalization-group treatment, in the $d = 1$ Blume-Emery-Griffiths model [39].

Thirdly, for high q, the zero-field ferromagnetic transition temperature saturates, as also seen in Ref.[38] and in detail in the right inset box in Fig. 2. Furthermore, when the vertical axis is scaled, not by q, but by q^2, a universal phase diagram emerges above low temperature for high q, as seen in Fig. 3.

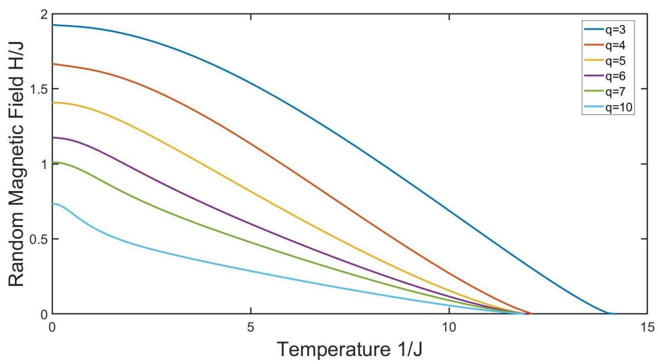

FIG. 4. Phase diagrams for $(q = 3, 4, 5, 6, 7, 10)$-state random-field clock models in $d = 3.32$, occurring in the figure respectively from high field to low field.

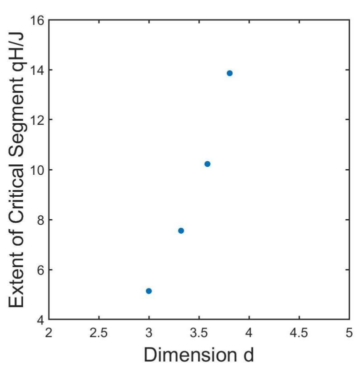

FIG. 5. The critical line segment, at zero temperature, is between $qH/J = 0$ and the qH/J values shown in this figure for each dimension d. The values are consistent with a divergence as $d = 4$ is approached.

IV. $d = 4$ DIMENSIONS AND THE LOWER-CRITICAL DIMENSION

The phase diagrams for $(q = 3, 4, 5, 6, 7, 10)$-state random-field clock models in $d = 3.32$ are shown in Fig. 4. It is again seen that the ferrromagnetic phase, under random fields, recedes and disappears as $q \to \infty$. Thus, no ferromagnetic phase occurs under random fields in the XY model in $d = 3.32$. However, our calculation again gives the zero-temperature critical segment, between $qH/J = 0$ and $qH/J = 7.6$ universally for all q in $d = 3.32$.

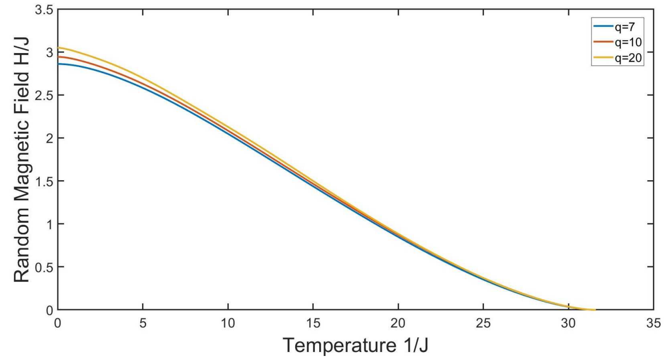

FIG. 6. Phase diagrams for $(q = 7, 10, 20)$-state random-field clock models in $d = 4$, occurring in the figure respectively from low field to high field.

The same results are obtained for $d = 3.58$ and 3.81, with the zero-temperature critical segment expanding, reaching $qH/J = 10.2$ and 13.9, respectively.

A qualitatively different picture occurs in the phase diagrams for $d = 4$, seen in Fig. 5. Going from $q = 7$ to $q = 10$, the ferromagnetic phase slightly expands in the random field, as opposed to drastically receding as in the lower dimensions. Going from $q = 10$ to $q = 20$, a much larger q interval, the ferromagnetic phase even more slightly expands in the random field. Thus, the ferromagnetic phase occurs, under random fields, for $q \to \infty$ and for the XY model in $d = 4$.

We thus see that the lower-critical dimension for the random-field XY model is between $d = 3.81$ and $d = 4$, namely $3.81 < d_c < 4$.

V. CONCLUSION

In order to investigate the random-field XY model, we have studied the random-field q-state clock models for increasing q, for dimensions $d = 3, 3.32, 3.58, 3.81, 4$. We find that for the random-field XY model, the lower-critical dimension is between $d = 3.81$ and $d = 4$, namely $3.81 < d_c < 4$. At $d < d_c$, we find a zero-temperature segment of criticality, stretching from zero to a value of qH/J that is q-independent for large q and that increases as d_c is approached.

ACKNOWLEDGMENTS

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[1] D. P. Belanger, A. R. King, and V. Jaccarino, Random-field effects on critical behavior of diluted Ising antiferromagnets, Phys. Rev. Lett. **48**, 1050 (1982).

[2] P.-Z. Wong and J. W. Cable, Hysteretic behavior of the diluted random-field Ising system $Fe_{0.70}Mg_{0.30}Cl_2$, Phys. Rev. B **28**, 5361 (1983).

[3] A. N. Berker, Ordering under random fields: Renormalization-group arguments, Phys. Rev. B **29**, 5243 (1984).

[4] H. Yoshizawa, R. A. Cowley, G. Shirane, R. J. Birgeneau, H. J. Guggenheim, and H. Ikeda, Random-field effects in two- and three-dimensional Ising antiferromagnets, Phys. Rev. Lett. **48**, 438 (1982).

[5] M. S. Cao and J. Machta, Migdal-Kadanoff study of the random-field Ising model, Phys. Rev. B **48**, 3177 (1993).

[6] A. Falicov, A. N. Berker, and S. R. McKay, Renormalization-group theory of the random-field Ising model in 3 dimensions, Phys. Rev. B **51**, 8266 (1995).

[7] A. Aharony, Y. Imry, and S.-k. Ma, Lowering of dimensionality in phase transitions with random fields, Phys. Rev. Lett. **37**, 1364 (1976).

[8] A. A. Migdal, Phase transitions in gauge and spin lattice systems, Zh. Eksp. Teor. Fiz. **69**, 1457 (1975) [Sov. Phys. JETP **42**, 743 (1976)].

[9] L. P. Kadanoff, Notes on Migdal's recursion formulas, Ann. Phys. (N.Y.) **100**, 359 (1976).

[10] A. N. Berker and S. Ostlund, Renormalisation-group calculations of finite systems: Order parameter and specific heat for epitaxial ordering, J. Phys. C **12**, 4961 (1979).

[11] R. B. Griffiths and M. Kaufman, Spin systems on hierarchical lattices: Introduction and thermodynamic Limit, Phys. Rev. B **26**, 5022R (1982).

[12] M. Kaufman and R. B. Griffiths, Spin systems on hierarchical lattices: 2. Some examples of soluble models, Phys. Rev. B **30**, 244 (1984).

[13] K. Jiang, J. Qiao, and Y. Lan, Chaotic renormalization flow in the Potts model induced by long-range competition, Phys. Rev. E **103**, 062117 (2021).

[14] G. Mograby, M. Derevyagin, G. V. Dunne, and A. Teplyaev, Spectra of perfect state transfer Hamiltonians on fractal-like graphs, J. Phys. A **54**, 125301 (2021).

[15] I. Chio, R. K. W. Roeder, Chromatic zeros on hierarchical lattices and equidistribution on parameter space, Annales de l'Institut Henri Poincaré D, **8**, 491 (2021).

[16] B. Steinhurst and A. Teplyaev, Spectral analysis on Barlow and Evans' projective limit fractals, J. Spectr. Theory **11**, 91 (2021).

[17] A. V. Myshlyavtsev, M. D. Myshlyavtseva, and S. S. Akimenko, Classical lattice models with single-node interactions on hierarchical lattices: The two-layer Ising model, Physica A **558**, 124919 (2020).

[18] M. Derevyagin, G. V. Dunne, G. Mograby, and A. Teplyaev, Perfect quantum state transfer on diamond fractal graphs, quantum information processing, **19**, 328 (2020).

[19] S.-C. Chang, R. K. W. Roeder, and R. Shrock, q-Plane zeros of the Potts partition function on diamond hierarchical graphs, J. Math. Phys. **61**, 073301 (2020).

[20] C. Monthus, Real-space renormalization for disordered systems at the level of large deviations, J. Stat. Mech. - Theory and Experiment, 013301 (2020).

[21] O. S. Sarıyer, Two-dimensional quantum-spin-1/2 XXZ magnet in zero magnetic field: Global thermodynamics from renormalisation group theory, Philos. Mag. **99**, 1787 (2019).

[22] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in 2-dimensional planar Model, Phys. Rev. B **16**, 1217 (1977).

[23] A. N. Berker and D. R. Nelson, Superfluidity and phase separation in Helium films, Phys. Rev. B **19**, 2488 (1979).

[24] E. Tunca and A. N. Berker, Renormalization-group theory of the Heisenberg model in d dimensions, arXiv:2202.06049 [cond-mat.stat-mech] (2022).

[25] S. Franz, G. Parisi, and M.A. Virasoro, Interfaces and lower critical dimension in a spin-glass model, J. Physique I **4**, 1657 (1994).

[26] C. Amoruso, E. Marinari, O. C. Martin, and A. Pagnani, Scalings of domain wall energies in two dimensional Ising spin glasses, Phys. Rev. Lett. **91**, 087201 (2003).

[27] J.-P. Bouchaud, F. Krzakala, and O. C. Martin, Energy exponents and corrections to scaling in Ising spin glasses, Phys. Rev. B **68**, 224404 (2003).

[28] S. Boettcher, Stiffness of the Edwards-Anderson model in all dimensions, Phys. Rev. Lett. **95**, 197205 (2005).

[29] M. Demirtaş, A. Tuncer, and A. N. Berker, Lower-critical spin-glass dimension from 23 sequenced hierarchical models, Phys. Rev. E 92, 022136 (2015).

[30] A. Maiorano and G. Parisi, Support for the value 5/2 for the spin glass lower critical dimension at zero magnetic field, Proc. Natl. Acad. Sci. USA **115**, 5129 (2018).

[31] B. Atalay and A. N. Berker, A lower lower-critical spin-glass dimension from quenched mixed-spatial-dimensional spin glasses, Phys. Rev. E **98**, 042125 (2018).

[32] S. R. McKay, A. N. Berker, and S. Kirkpatrick, Spin-glass behavior in frustrated Ising models with chaotic renormalization-group trajectories, Phys. Rev. Lett. **48**, 767 (1982).

[33] S. R. McKay, A. N. Berker, and S. Kirkpatrick, Amorphously packed, frustrated hierarchical models: Chaotic rescaling and spin-glass behavior, J. Appl. Phys. **53**, 7974 (1982).

[34] A. N. Berker and S. R. McKay, Hierarchical models and chaotic spin glasses, J. Stat. Phys. **36**, 787 (1984).

[35] S.E. Gürleyen and A.N. Berker, Asymmetric phase diagrams, algebraically ordered BKT phase, and peninsular Potts flow structure in long-range spin glasses, S.E. Gürleyen and A.N. Berker, Phys. Rev. E **105**, 024122 (2022).

[36] T. Çağlar and A. N. Berker, Chiral Potts spin glass in d = 2 and 3 dimensions, Phys. Rev. E **94**, 032121 (2016).

[37] H. Y. Devre and A. N. Berker, First-order to second-order phase transition changeover and latent heats of q-state Potts models in d=2,3 from a simple Migdal-Kadanoff adaptation, arXiv:2202.01528 [cond-mat.stat-mech] (2022).

[38] E. C. Artun and A. N. Berker, Complete density calculations of q-state Potts and clock models: Reentrance of interface densities under symmetry breaking, Phys. Rev. E **102**, 062135 (2020).

[39] S. Krinsky and D. Furman, Exact renormalization group exhibiting tricritical fixed point for a spin-one Ising model

in one dimension, Phys. Rev. B **11**, 2602 (1975).