Discrete Applied Mathematics 306 (2022) 70–82

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Four-searchable biconnected outerplanar graphs

Öznur Yaşar Diner[a,b,*], Danny Dyer[c], Boting Yang[d]

[a] Mathematics Department, Universitat Politècnica de Catalunya, Spain
[b] Computer Engineering Department, Kadir Has University, Turkey
[c] Department of Mathematics and Statistics, Memorial University of Newfoundland, Canada
[d] Department of Computer Science, University of Regina, Canada

ARTICLE INFO

Article history:
Received 5 December 2019
Received in revised form 9 August 2021
Accepted 8 September 2021
Available online 13 October 2021

Keywords:
Graph minors
Edge searching
Outerplanar graphs
Forbidden minors

ABSTRACT

This paper deals with constructing obstruction sets for two subclasses of 4-searchable graphs. We first characterize the 4-searchable biconnected outerplanar graphs by listing all graphs that cannot be their minors; we then give a constructive characterization of such graphs. We also characterize the 4-searchable biconnected generalized wheel graphs by listing all graphs that cannot be their minors.

1. Introduction

Imagine that we want to secure a system of tunnels from a hidden intruder who can move very fast. We model this system as a finite connected graph $G = (V, E)$ where junctions correspond to vertices and tunnels correspond to edges. We assume that G may have multiple edges but no loops. We follow the terminology in [7].

We launch a group of *searchers* into the system in order to catch the intruder. We assume that every edge of G is contaminated initially, and our goal is to clean the whole graph by a sequence of steps. At each step, we are allowed to do one of the following *moves*: (1) place a searcher at a vertex; (2) remove a searcher from a vertex; (3) slide a searcher from a vertex along an edge to an adjacent vertex. Note that placing multiple searchers on the same vertex is allowed. We do not pose any restriction on the number of searchers used.

If a searcher slides along an edge $e = uv$ from u to v, then the edge e is *clean* if either (i) another searcher is stationed at u, or (ii) all other edges incident to u are already clean. We define a *clean vertex* (or alternatively, say a vertex is *clean*) if all edges incident with the vertex are *clean*.

An *edge search strategy* is a sequence of moves that ends with all edges being simultaneously clean, in which case we say that the graph is *cleaned*.

If a searcher is stationed at a vertex v, then we say that v is *guarded*. If a path does not contain any searcher, then it is called an *unguarded path*. If there is an unguarded path that contains one endpoint of a contaminated edge and one endpoint of a clean edge e, then e gets *recontaminated*. Hence, a clean edge remains clean as long as every path from it to a contaminated edge is blocked by at least one searcher.

We measure a search strategy by the maximum number of searchers used over all search steps. For a given graph, it is a natural question to ask what is the smallest value of k with which we can clean the graph. Over all search strategies

* Corresponding author at: Computer Engineering Department, Kadir Has University, Turkey.
E-mail addresses: oznur.yasar@khas.edu.tr (Ö.Y. Diner), dyer@mun.ca (D. Dyer), boting.yang@uregina.ca (B. Yang).

https://doi.org/10.1016/j.dam.2021.09.011

the fewest number of searchers needed to clean the graph is the *edge search number* of the graph and is denoted $s(G)$. A graph G is said to be *k-searchable* if $s(G) \leq k$.

The edge search problem has been extensively studied. Its origin dates back to the late 1960s in the work of Breisch [4]. It was first faced by a group of spelunkers who were trying to find a person lost in a system of caves. They were interested in the minimum number of people they needed in the searching team. Parsons [20] was the first one to formalize it as a mathematical problem. He defined it as a continuous problem where the searchers and the intruder are allowed to move according to continuous functions. Golovach [13] proved the equivalence of the continuous problem and the discrete version that we are considering. A survey of edge searching results is available at [11].

The edge search problem is closely related with layout problems and due to this connection there are various inequalities showing the relationship between width parameters and the search number. Some of these parameters are pathwidth [9], cutwidth [6], bandwidth [10,18], and topological bandwidth [6].

The decision version of the problem is called the EDGE SEARCH problem: Given a graph G and an integer k, decide whether G is *k-searchable*. It has been shown that EDGE SEARCH is NP-hard for general graphs [19].

One of the major problems of edge search is to give structural theorems on k-searchable graphs for a fixed positive integer k. One way to characterize the k-searchable graphs is to give a complete set of graphs that are forbidden as minors.

Consider two operations: *edge deletion*, which deletes an edge e, and *edge contraction*, which deletes an edge $e = uv$ and identifies the vertices u and v. The second operation replaces an edge $e = uv$ with a new vertex v' which is adjacent to all of the former neighbors of u and v.

Given a graph G, a graph H is called a *minor* of G if a graph isomorphic to H can be obtained from G by edge contractions applied to a subgraph of G. Notice that G is a minor of itself.

If H is a minor of G, then we say that H is less than or equal to G in the minor order and we write $H \preceq_m G$. A family $\mathcal{G}$ of graphs is said to be *closed under minor ordering* if for any $G \in \mathcal{G}$, we have $H \in \mathcal{G}$ for every $H \preceq_m G$. A graph property is said to be *hereditary* or *inherited by minors* if it defines a minor closed family.

Some hereditary graph properties are being cycle free, being series parallel, being embeddable in any fixed surface and being linklessly embeddable. In particular the edge search number is inherited by minors; that is, if H is a minor of G, then $s(H) \leq S(G)$. Some variants of edge search are also inherited by minors, such as mixed search and weighted search; however fast search is not inherited by minors [8].

The crucial graph minor theory was developed by Robertson and Seymour in a series of more than 20 papers. One of the implications for any minor closed graph class is that there are only a finite number of forbidden minors (given in Graph Minors XX, 2004 [22]). This is equivalent to the following result: in every infinite class of graphs, there are two such that one is a minor of the other.

Furthermore, once the obstruction set is known, we can decide in polynomial time whether a given arbitrary graph is contained in the graph family $\mathcal{G}$. This is an immediate consequence of the following theorem by Robertson and Seymour [21]: for every fixed graph H, the problem that takes as input a graph G and determines whether H is a minor of G is solvable in polynomial time. Thus testing membership in $\mathcal{G}$ can be done in polynomial time.

In 1930 Kuratowski [16] gave the two forbidden *topological subgraphs* for planar graphs: a graph G is planar, if it does not contain $K_5, K_{3,3}$ or any of their subdivisions. Later on, Wagner [25] showed these graphs are the exact list of forbidden minors for graphs that have a planar embedding. This is considered the first result of topological graph theory and the two graphs are called Kuratowski graphs. Since then, many people have worked on similar issues for other surfaces, though complete obstruction sets are known for few such surfaces. In 1981, it was shown by Archdeacon and Huneke that there are 35 forbidden minors for the projective planar graphs [2]. Glover and Huneke conjectured in 1995 that there are more than 1000 forbidden minors for toroidal graphs [1]. Later, it was shown that this number is much larger, with Chambers [5] and Myrvold showing that there are more than 16629 forbidden minors for toroidal graphs by giving the complete list of forbidden minors that have at most 11 vertices, the ones that are 3-regular and have at most 24 vertices, the disconnected minors, and those minors with a cut vertex. More recently, Gagarin, Myrvold and Chambers have shown that there are four forbidden minors for toroidal graphs that are $K_{3,3}$-free [12].

Similar types of results have been given for several width parameters. Kinnersley [14] showed that there are at least 110 forbidden minors for graphs with pathwidth at most 2, and later on in 1994, together with Langston [15], they showed that this list is complete. The complete list containing 57 obstructions for graphs with linearwidth at most 2 is given by Thilikos [24]. Arnborg et al. showed that there are four forbidden minors for graphs with tree-width at most three and Sanders [23] showed that there are more than 75 forbidden minors for graphs with treewidth at most 4.

A *subdivision* of a graph is obtained by inserting one or more vertices of degree two to some of its edges. The reverse operation is called *reduction*. Let V' be the set of vertices of degree 2 in G. The *reduction* of a graph G is the graph obtained by removing each vertex $v_i \in V'$ and joining its former neighbors by an edge. Thus, the reduced graph does not contain any vertex of degree two. We say that two graphs are *homeomorphic* if they have the same reduced graph. One can observe that homeomorphic graphs will have the same search number. Thus, it is sufficient to consider only the reduced graphs.

Given a graph G, a graph H is called an *S-minor* of G if H is a minor of a subdivision of G. In particular, G is an S-minor of itself. An S-minor of G is called a *proper S-minor* if it is not isomorphic to G.

Note that S-minor is usually stronger than the classical minor. If H is a minor of a graph G, then it is also an S-minor of G; but the opposite may not be true. For example, the third graph in Fig. 3 is not a minor of the graph in Fig. 1, though it is an S-minor.

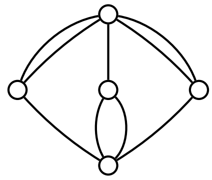

Fig. 1. A graph that is not 3-searchable.

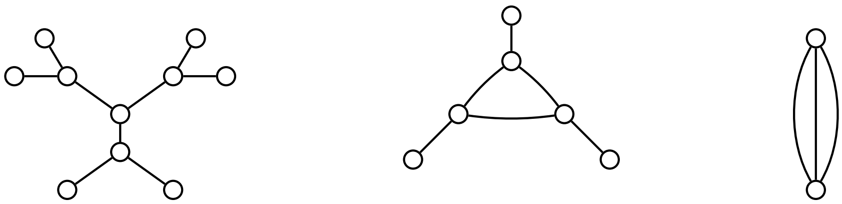

Fig. 2. A complete set of obstructions for 2-searchable graphs [19].

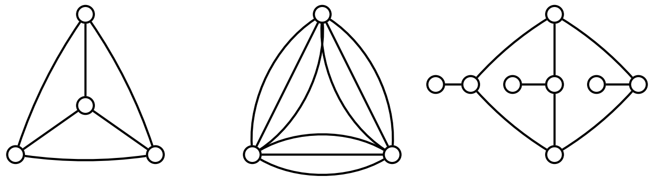

Fig. 3. A complete set of obstructions for biconnected 3-searchable graphs [19].

Let $\mathcal{H}_k$ be the set of k-searchable graphs. For a fixed k, the *obstructions* for k-searchable graphs are those graphs that are S-minor minimal in the complement of $\mathcal{H}_k$. Let $\mathcal{F}_k$ denote the set of all obstructions for k-searchable graphs. Thus $H \in \mathcal{F}_k$ whenever H is not in $\mathcal{H}_k$ and every proper S-minor of H is in $\mathcal{H}_k$.

As far as edge search is concerned, for fixed k, $\mathcal{F}_k$ is known completely only when k is at most 3. These results are given by Megiddo, Hakimi, Garey, Johnson and Papadimitriou [19]. In Figs. 2 and 3 we give the sets $\mathcal{F}_2$ and $\mathcal{F}_3$, respectively. For $k \geq 4$, determining $\mathcal{F}_k$ is an open problem.

In this paper we partially answer this open problem by giving a complete list of graphs that cannot be contained in any biconnected outerplanar graph as a minor. We give a description of how a 4-searchable biconnected outerplanar graph is constructed by giving its explicit structure. Next, in Section 4, we give a partial list of obstructions for 2-outerplanar graphs by giving a complete list of graphs that cannot be contained in any 4-searchable generalized wheel graph as a minor.

2. Preliminaries

A complete graph with n vertices, denoted by K_n, is a graph in which every pair of distinct vertices is connected by a unique edge. Similarly, a complete bipartite graph $K_{m,n}$ is a bipartite graph (V_1, V_2, E), where $|V_1| = m$ and $|V_2| = n$ such that every vertex of V_1 is connected to every vertex of V_2 and E contains only these edges. A multigraph is a graph that may have multiple edges (also called parallel edges). We will simply use graphs instead of multigraphs if there is no confusion from the context. The *degree of a vertex* is the number of edges that are incident to this vertex. The *multiplicity of an edge* is the number of parallel edges that have the same endpoints. An edge with endpoints u and v is denoted as uv. We also simply use uv to denote all parallel edges between u and v if there is no ambiguity from the context.

Recall that a graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by edge contractions. A minor of G is *proper* if it is not isomorphic to G. In the remainder of this paper, we only consider minors under this definition. So the definition of obstructions is changed accordingly: an *obstruction* for k-searchable graphs is a graph that is minor minimal in the set of graphs with search number strictly more than k.

We first show that all 4-searchable graphs are planar.

Theorem 1. *If $s(G) \leq 4$, then G is a planar graph.*

Proof. Note that $s(K_5) = s(K_{3,3}) = 5$. If G contains $K_{3,3}$, K_5 or their subdivisions, then $s(G) \geq 5$. This is a contradiction. Hence from Kuratowski's Theorem, G must be a planar graph. ∎

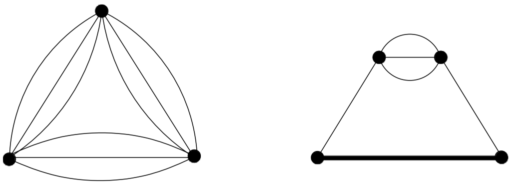

Fig. 4. On the left: A tent. On the right: A house with thick base edge.

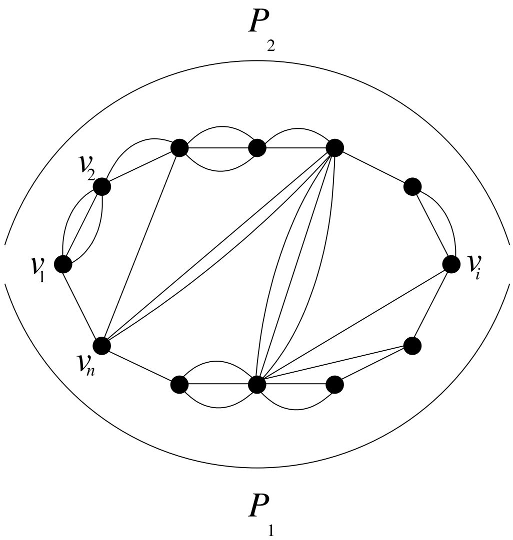

Fig. 5. A biconnected generalized bipolar graph with poles v_1 and v_i.

For brevity, we consider *reduced* graphs, which are obtained by successive reduction operations defined above. Thus a reduced graph has no vertex of degree two. Notice that reduction does not change the search number. In the remainder of the paper, all of the graphs that we consider are reduced multigraphs. Therefore, we hereafter omit the description "reduced multigraph"; reduced multigraphs will simply be called graphs.

A graph is said to be *outerplanar*, or *1-outerplanar*, if it can be drawn in the plane in such a way that all vertices are on the boundary of the unbounded (or "outer") face. A k-outerplanar graph is defined recursively. For $k > 1$, a graph is *k-outerplanar* if there exists a planar embedding of G which has an outer face so that by removing the vertices of the outer face, we obtain a $(k-1)$-outerplanar graph. Note that not every planar graph is outerplanar. For instance, K_4 is not outerplanar but is 2-outerplanar.

A *tent* is a multigraph $3C_3$, i.e., each pair of vertices of C_3 are connected with three parallel edges. A *house* is a multigraph $H = (V, E)$ where $V = V(C_4) = \{v_0, v_1, v_2, v_3\}$ and $E = E(C_4) \cup \{e_5 = v_0v_1, e_6 = v_0v_1\}$. Given a house H, the edge whose two endpoints each is incident with two edges, is called the *base* of the house (Fig. 4).

It is known that the boundary of the outer face of a biconnected outerplanar graph is a spanning cycle [7]. Assume that G is a biconnected outerplanar graph. We fix an outerplanar embedding of G and label its vertices as $v_1, v_2, \ldots, v_n$ so that they consecutively lie on the boundary of the outer face, ending with v_n being adjacent to v_1. We denote the graph induced by the vertices $\{v_i, v_{i+1}, \ldots, v_j\}$ as $P_{i,j}$, where the indices (except n) are modulo n. Thus $P_{i,j}$ denotes the boundary path from v_i to v_j together with all the chords between the vertices in $\{v_i, v_{i+1}, \ldots, v_j\}$. Note that v_i and v_j are the only common vertices shared by the induced subgraphs $P_{i,j}$ and $P_{j,i}$.

If there are two vertices v_i and v_j such that neither $P_{i,j}$ nor $P_{j,i}$ has a tent or a house as a minor, with the base of the house as a chord of $P_{i,j}$ or $P_{j,i}$, then we say that G is a *generalized bipolar graph*. The vertices v_i and v_j are then called the *poles* of G. An example of a generalized bipolar graph is given in Fig. 5.

3. Four-searchable outerplanar graphs

In this section we present a characterization of 4-searchable biconnected outerplanar graphs. A search strategy in a graph G is *monotonic* if the searchers must move in such a way to keep an edge clean once it has been cleaned. That is, the searchers can allow no edge to be recontaminated.

The following lemma is the main result in [3,17].

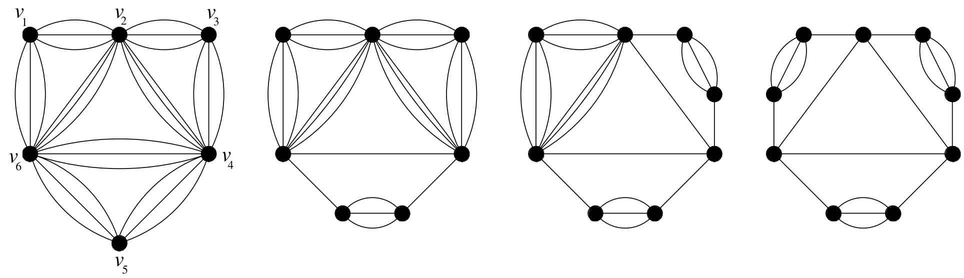

Fig. 6. A complete list of minimal minors that cannot be contained in any outerplanar graph with search number at most 4.

Lemma 2 ([3,17]). *For any graph G, there always exists a monotonic search strategy that cleans G using s(G) searchers.*

Theorem 3. *The following are equivalent for a reduced biconnected outerplanar graph G:*

1. $s(G) \leq 4$.
2. *G does not contain any of the graphs in Fig. 6 as a minor.*
3. *G is a generalized bipolar graph.*

Proof. Consider a planar embedding of G and label the vertices of G as $v_1, v_2, \ldots, v_n$ such that they clockwisely lie on the boundary of the outer face. We will show that (1)⇒(2)⇒(3)⇒(1).

(1)⇒(2). To see this, we use proof by contrapositive. Assume that G contains one of the graphs in Fig. 6 as a minor. If we show that all of the graphs in Fig. 6 have search number strictly greater than 4, then we can conclude that $s(G) > 4$, because edge search number is inherited by minors.

Let us show that $s(H) > 4$ for the leftmost graph H in Fig. 6. Assume that $s(H) = 4$. From Lemma 2, there is a monotonic search strategy that cleans H using $s(H)$ searchers. The graph H has 6 vertices and 27 edges. There are two types of vertices in H: those with $|N(v)| = 4$ and those with $|N(v)| = 2$. Label the vertices as in Fig. 6. If the first vertex to become clean in a monotonic search strategy has four neighbors, then it will use at least 6 searchers. Since the graph is symmetric, we can start with any vertex v with $|N(v)| = 2$. Hence, without loss of generality, let the first clean vertex be v_1. To clean v_1 we need at least 4 searchers. When v_1 becomes clean, we must keep one searcher on each of v_2 and v_6. Hence there are two free searchers. Observe that these two free searchers do not suffice to clean any other vertex. Hence a second vertex cannot be cleaned using only 4 searchers. Thus $s(H) > 4$.

Also notice that deleting any vertex or edge from H or contracting any edge will reduce the search number to 4. Hence H is a minimal minor with $s(H) > 4$. Similar arguments suffice for the other graphs in Fig. 6.

(2)⇒(3). Assume that G does not contain any of the graphs in Fig. 6 as a minor. Let $P := P_{i,j}$ be the subgraph induced by a maximal length boundary path with end vertices v_i and v_j such that P does not contain a tent or a house as a minor. If P contains all the vertices in $V(G)$, then G is a generalized bipolar graph where v_i and v_j are the poles. Otherwise, suppose that v_{i-1} and v_{j+1} do not lie on P. We denote the subgraph induced by $V(P) \cup V'$, for $V' \subseteq V(G)$, as $P + V'$.

Since P is maximal, the induced subgraph $P + \{v_{i-1}\}$ contains a tent (or a house) as a minor. Similarly, the induced subgraph $P + \{v_{j+1}\}$ contains a tent (or a house) as a minor. Since G is outerplanar, these two tents (or two houses or a tent and a house) are edge disjoint.

Let $P' := P_{j+1,i-1}$ be the subgraph induced by the boundary path from v_{j+1} to v_{i-1} which is vertex-disjoint from P. Then, P' cannot contain any tent or house as a minor since otherwise G would have one of the graphs in Fig. 6 as a minor. To see this assume that P' contains a house or a tent. Since the subgraph $P + \{v_{i-1}, v_{j+1}\}$ contains two tents (or two houses or a tent and a house) which are edge disjoint, G will contain one of the graphs in Fig. 6. But this leads to a contradiction with the assumption that G does not contain any graph in Fig. 6 as a minor. Hence, P' does not contain a house or a tent as a minor.

Assume that $P' + \{v_i\}$ has a tent or a house, say H, as a minor. Then, $v_i \in V(H)$, because P' does not contain a house or a tent as a minor. Furthermore, there exists a vertex $u \in \{v_{j+1}, v_{j+2}, \ldots, v_{i-2}\}$ such that $uv_i \in E(H)$. Thus v_{i-1} has no neighbor in P other than v_i, since G is outerplanar. Hence $P + \{v_{i-1}\}$ contains a longer boundary path without any tent or a house as a minor, contradicting the maximality of P. Therefore $P' + \{v_i\}$ does not have a tent or a house as a minor.

Similarly, $P' + \{v_j\}$ cannot have a tent or a house as a minor either. Furthermore, it is impossible for $P' + \{v_i, v_j\}$ to contain any tent or a house as a minor. Thus none of the boundary paths connecting v_i and v_j, namely, neither P nor $P' + \{v_i, v_j\}$, contains a tent or a house. Therefore, G is a generalized bipolar graph with v_i and v_j as poles.

(3)⇒(1). Suppose that v_1 and v_i are the poles of G. Let $P_1 := P_{1,i}$ and $P_2 := P_{i,1}$ be the subgraphs induced by the boundary paths from v_1 to v_i and from v_i to v_1, respectively. First we put two searchers σ_1 and σ_2 on v_1.

During the search process we always keep σ_1 on P_1 and σ_2 on P_2. The other two searchers, σ_3 and σ_4, are used to clean boundary edges and chords in P_1 and P_2; they are also used to clean cross chords of G connecting a vertex in P_1 with a vertex in P_2. By the outerplanarity of G, a cross chord can always be cleaned using one searcher while its end vertices are guarded by σ_1 and σ_2.

We will show how to clean P_1 using σ_1, σ_3 and σ_4 where the vertices are cleaned consecutively from v_1 to v_i. Similarly, we can use σ_2, σ_3 and σ_4 to clean the vertices of P_2 consecutively from v_1, then v_n and down to v_i.

Assume that $v_1, v_2, \ldots, v_{j-1}$ are cleaned and that currently v_j is occupied by σ_1. We have two cases regarding cross chords incident to v_j.

Case 1. There are no contaminated cross chords incident to v_j. We consider three subcases based on the number of contaminated edges incident to v_j.

Case 1.1. There are at most two contaminated edges incident to v_j, say $e_1 = v_jv_{j+1}$ and $e_2 = v_jv_k$, where $j+1 \leq k \leq i$. We put σ_3 on v_j and we clean v_j by sliding σ_1 along e_1 and σ_3 along e_2. If $k = j+1$, both searchers are on v_{j+1} and we proceed to clean v_{j+1}. If $k = j+2$, we put σ_4 on v_{j+1} and clean all the edges between v_{j+1} and v_{j+2} by σ_4. There are no cross chords incident to v_{j+1} due to the outerplanarity. Hence v_{j+1} is clean. If $j+3 \leq k \leq i$, since P_1 does not contain a house as a minor, the boundary path from v_{j+1} to v_k can be cleaned by σ_1 and σ_4.

Case 1.2. There are at least three contaminated edges incident to v_j and these edges are incident to exactly two vertices on P_1 other than v_j. Let these two vertices be v_{j+1} and some v_k ($j+1 \leq k \leq i$). We show that the subgraph $P_{j,k}$ induced by $\{v_j, v_{j+1}, \ldots, v_k\}$ can be cleaned by σ_1, σ_3 and σ_4. Note that there are no contaminated cross chords incident to v_j. If $k = j+1$, then σ_1 remains on v_j, we place σ_3 on v_{j+1} while σ_4 cleans all edges between v_j and v_{j+1}.

If $j+2 \leq k \leq i$, since $P_{j,k}$ does not contain a tent as a minor, it contains at most two edges of multiplicity at least three. If it contains only one such edge, call it uv, with u having a subscript less than v, then either $u = v_j$ or $v = v_k$, as otherwise $P_{j,k}$ would contain a house as a minor. In the former case, place σ_4 on $u = v_j$, then use σ_1 and σ_2 to clean the vertices of $P_{j,k}$ sequentially until v is reached. Let σ_2 sit on v, and then σ_1 cleans the edges uv. Then σ_1 and σ_2 may clean the remaining vertices in $P_{j,k}$, eventually reaching v_k. A similar argument suffices for when $v = v_k$.

Next, we consider when $P_{j,k}$ contains exactly two edges of multiplicity at least three. Call these edges uv and $u'v'$. Notice that $u, v, u', v' \in \{v_j, v_{j+1}, \ldots, v_k\}$. Without loss of generality, suppose the subscript of u is less than that of v, the subscript of u' is less than that of v', and the subscript of u is less than or equal to that of u'. Since $P_{j,k}$ does not contain a house as a minor, we know that $u = v_j$, and either $v = v_k$ or $v' = v_k$. Thus there are four possible cases for the endpoints of uv and $u'v'$.

Case 1.2.1: $u = u'$ and $v \neq v'$. In this case, $u = u' = v_j$, and $v = v_k$ or $v' = v_k$. Without loss of generality, suppose $v = v_k$. Since $P_{j,k}$ does not contain a house or a tent as a minor, $P_{j,k}$ can be cleaned as follows: (1) Put σ_4 on v_j and use σ_1, σ_3 to clean the subgraph induced by the vertices on the boundary between v_j and v' except the edge v_jv'. After that, v_j is occupied by σ_4 and v' is occupied by σ_1, σ_3 and we clean the parallel edges between v_j and v'. (2) Let σ_4 stay on v_j and use σ_1 and σ_3 to clean the subgraph induced by the vertices on the boundary between v' and v_k. (3) Use σ_3 to clean the parallel edges between v_j and v_k while v_1 is occupied by σ_4 and v_k is occupied by σ_1.

Case 1.2.2: $u \neq u'$, $v \neq u'$ and $v \neq v'$. Since $P_{j,k}$ does not contain a house as a minor, we know that $u = v_j$ and $v' = v_k$; and further, that the subscript of u' must be greater than that of v, since otherwise we would violate the outerplanarity of G. Then $P_{j,k}$ can be cleaned in the following way: (1) While σ_1 is on u, use σ_3 and σ_4 to clean the subgraph induced by the vertices on the boundary between u and v except the edge uv. After that, u is occupied by σ_1 and v is occupied by σ_3, σ_4 and we can easily clean the parallel edges between u and v. (2) Then, with the only contaminated edges incident with $v_j = u$ being uv', use σ_1 and s_3 to clean all edges uv', ending with σ_1 on v'. (3) Use σ_3, σ_4 to clean the subgraph induced by the vertices on the boundary between v and u'. After that, u' is occupied by σ_3, σ_4 and v' is occupied by σ_1. (4) Clean the parallel edges between u' and v'. (5) Then σ_1 remains on v', and use σ_3, σ_4 to clean the subgraph induced by the vertices on the boundary between u' and v' other than $u'v'$.

Case 1.2.3: $u \neq u'$, $v = u'$ and $v \neq v'$. Similarly to Case 1.2.2, we have $u = v_j$ and $v' = v_k$, and $P_{j,k}$ can be cleaned in the same way as that in Case 1.2.2 omitting the unnecessary step (3).

Case 1.2.4: $u \neq u'$ and $v = v'$. Similarly to Case 1.2.1, we have $u = v_j$ and $v = v' = v_k$. In this case, $P_{j,k}$ can be cleaned by the following steps: (1) Use $\sigma_1, \sigma_3, \sigma_4$ to clean the parallel edges between v_j and v_k. After that, v_j is occupied by σ_3 and σ_4 and v_k is occupied by σ_1. (2) Use σ_3 and σ_4 to clean the subgraph induced by the vertices on the boundary between v_j and u'. (3) While u' is occupied by σ_3 and v_k is occupied by σ_1, use σ_4 to clean the parallel edges between u' and v_k. (5) Use σ_3 and σ_4 to clean the subgraph induced by the vertices on the boundary between u' and v_k except the edge $u'v_k$.

Case 1.3. There are at least three contaminated edges incident to v_j and these edges are incident to at least three distinct vertices on P_1 other than v_j, say v_{j+1}, $v_{k'}$ and v_k, where $j+1 < k' < k$. Since $P_{j,k}$ does not contain a house or a tent as a minor, the only edges in $P_{j,k}$ with multiplicity at least three are at most two of the v_jv_{j+1}, v_jv_k and $v_{k-1}v_k$. If there are at most one edge in $P_{j,k}$ with multiplicity at least three, it is easy to see that $P_{j,k}$ can be cleaned by σ_1, σ_3 and σ_4. Suppose that there are two edges in $P_{j,k}$ with multiplicity at least three. Then we have three cases for these two edges.

Case 1.3.1: v_jv_{j+1} and v_jv_k have multiplicity at least three. Similarly to Case 1.2.1, we can clean $P_{j,k}$ using σ_1, σ_3 and σ_4.

Case 1.3.2: v_jv_{j+1} and $v_{k'}v_k$ have multiplicity at least three. Similarly to Case 1.2.3, we can clean $P_{j,k}$ using σ_1, σ_3 and σ_4.

Case 1.3.3: v_jv_k and $v_{k'}v_k$ have multiplicity at least three. Similarly to Case 1.2.4, we can clean $P_{j,k}$ using σ_1, σ_3 and σ_4.

Case 2. There is a contaminated cross chord v_lv_j incident to v_j, where $v_l \in V(P_2)\backslash\{v_1, v_i\}$. Similarly to the above strategy for cleaning $P_{j,k}$, we can clean P_2 from v_1 down to v_l. After that, v_l is occupied by σ_2 and v_j is occupied by σ_1. Then we use σ_3 to clean all parallel edges between v_l and v_j.

We repeat the above procedure until P_1 and P_2 are cleaned, meanwhile, all cross chords of G between P_1 and P_2 are also cleaned. Therefore, the graph G can be cleaned using at most 4 searchers. ■

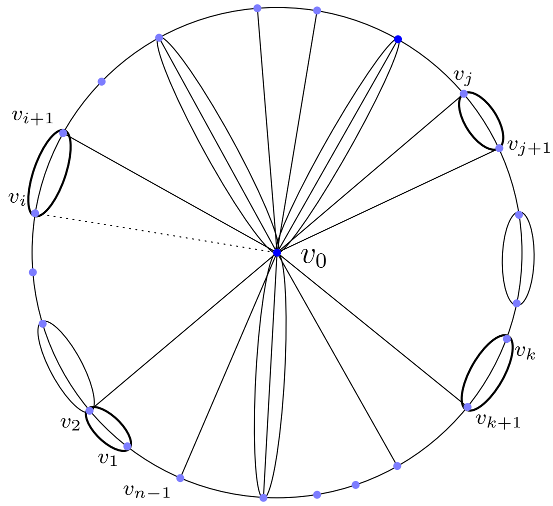

Fig. 7. A graph formed by the solid edges is a 4-searchable generalized wheel graph, while the graph formed by the solid and dotted edges is not.

Now that we know the biconnected components, we may comment on how to construct 4-searchable graphs. Extending the definition of a pinched graph in [19], we define a 3-*pinched graph* as a graph obtained by identifying two opposing poles of a bipolar graph. Observe that every 3-pinched graph is 4-searchable. The *endpoint* of a 2-searchable graph is defined in [19] and we define the endpoint of a 3-searchable biconnected graph as an end vertex of one of its poles.

The construction of an outerplanar 4-searchable graph allows joining such biconnected components at pole vertices, and adding (a) an arbitrary number of 1-, 2- and 3-searchable graphs at their end points, or, (b) at most a constant number of pinched graphs and 3-pinched graphs at selected vertices of the result. We omit this result as the characterization of which graphs can be added to which vertices is straightforward to see but extremely tedious to write.

4. Four-searchable generalized wheel graphs

A *generalized wheel graph* is a 2-outerplanar graph that consists of a *center vertex* and vertices on the boundary of the outer face, called *boundary vertices*, such that after removing the center vertex, the remaining graph is a cycle which may have multiple edges. A *spoke* is an edge that connects the center vertex with a boundary vertex. Note that multiple edges are allowed for spokes.

Let W be a biconnected generalized wheel graph with n vertices. In this section we fix a planar embedding of W and label its vertices as $v_0, v_1, \ldots, v_{n-1}$, where v_0 is the center vertex and $v_1, \ldots, v_{n-1}$ are boundary vertices in a clockwise order lying on the boundary of the outer face so that v_{n-1} is adjacent to v_1.

Consider the graph in Fig. 7. That graph, omitting the dotted line, is 4-searchable, as we will show directly by giving a monotonic search strategy using four searchers, σ_i, $1 \le i \le 4$. Place σ_1 on v_k and σ_2 on v_{k+1}. Then use σ_3 to clean the edges between them. Using searchers σ_1, σ_3, and σ_4, continue to clean edges sequentially from v_k until v_{j+1} is reached. Leaving s_4 on v_{j+1}, move σ_1 to v_j, and then use σ_3 to clean the edges v_jv_{j+1}. At this point, all edges incident with v_{j+1} have been cleaned except for the spoke v_0v_{j+1}. Move s_4 along v_0v_{j+1} (cleaning it). Then use σ_3 to clean the spokes v_0v_j and v_0v_{k+1}. From here on, the σ_1 will continue to move from v_j to v_{i+1}. When it reaches an uncleaned spoke, it will stop, and then σ_3 will clean that spoke. This will continue, until σ_1 reaches v_{i+1}, with σ_3 then cleaning v_0v_{i+1}. Then, σ_2 will duplicate this motion, moving from v_{k+1} towards v_{n-1}. Again, when spokes are encountered, σ_3 will clean them. Eventually, σ_2 reaches v_{n-1}, and σ_3 cleans v_0v_{n-1}. At this point, the only edge incident with v_0 that has not been cleaned is v_0v_2. The searcher σ_4 can slide along that spoke, cleaning it. Then σ_2 can move to v_1 from v_{n-1}, and subsequently σ_3 can clean all the edges v_1v_2. Leaving σ_4 on v_2, σ_1 can move to v_i, and σ_3 can clean the edges v_iv_{i+1}. Then s_1 can move along the outer edges until it is adjacent to v_2, and σ_3 can clean the remaining edges.

However, by adding the dotted edge (the spoke v_0v_i) we see that the "end game" of this strategy would not be possible, since, for instance, σ_4 would not be free to move along the spoke v_0v_2. In fact, no strategy using only four searchers is possible, as the graph then contains the graph W_{11} from Fig. 8 as a minor, and $s(W_{11}) > 4$.

Theorem 4. *A reduced biconnected generalized wheel graph is 4-searchable if and only if it does not contain any of the graphs in Fig. 8 as a minor.*

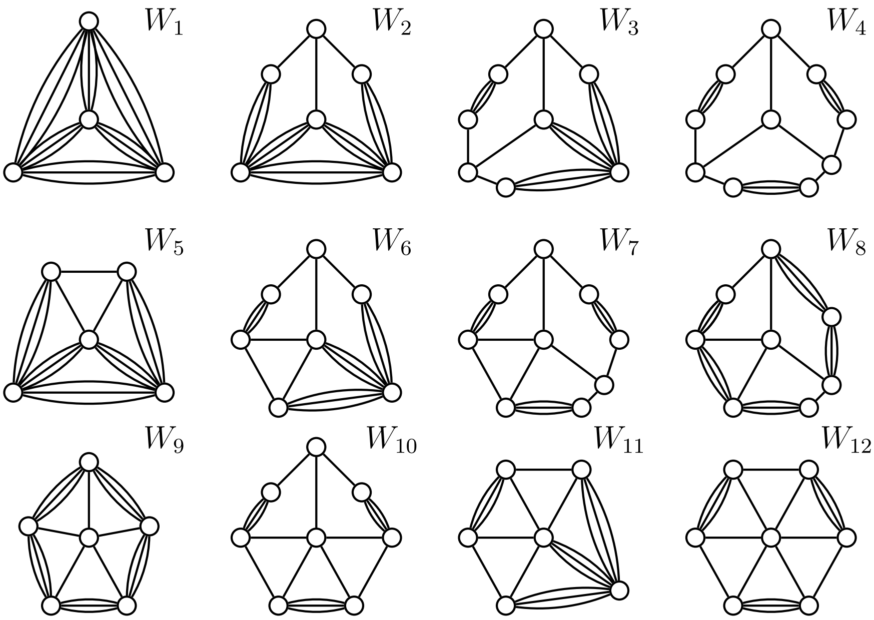

Fig. 8. A complete list of minimal minors $W_1, W_2, \ldots, W_{12}$ that cannot be contained in any biconnected generalized wheel graph with search number at most 4.

Proof. Label the graphs in Fig. 8 as $W_1, W_2, \ldots, W_{12}$.

"$\Rightarrow$". Let W be a reduced biconnected generalized wheel graph with $s(W) \leq 4$. Observe that for each W_i, $1 \leq i \leq 12$, in Fig. 8, we have $s(W_i) > 4$. Thus W cannot contain any of them as a minor.

"$\Leftarrow$". Let W be a reduced biconnected generalized wheel graph that does not contain any W_i, $1 \leq i \leq 12$, in Fig. 8 as a minor. Consider a planar embedding of W and label the vertices of W as $v_0, v_1, \ldots, v_{n-1}$, where v_0 is the center vertex and $v_1, \ldots, v_{n-1}$ clockwisely lie on the boundary of the outer face. Denote the graph induced by the vertices $\{v_i, v_{i+1}, \ldots, v_j\}$ as $P_{i,j}$, where the indices except $n-1$ are modulo $n-1$. We also call $P_{i,j}$ the boundary path from v_i to v_j. Let E_S be the set of all non-parallel spokes in W. We have the following cases regarding the structure of W.

Case 1: The multiplicity of each edge on the boundary of the outer face of W is at most two. From this structure, we place the searcher σ_1 on v_0 and the other three searchers on v_1. The searcher σ_3 can clean all edges between v_0 and v_1. Since W does not contain any outer $3K_2$, σ_3 and σ_4 can move along the boundary of the outer face to clean all boundary edges and spokes easily. Thus, in the cases that remain, we assume that the multiplicity of some boundary edge is 3 or more.

Case 2: $|E_S| \leq 2$. If $|E_S| = 1$, it is easy to see that $s(W) \leq 4$. Suppose $E_S = \{v_0v_i, v_0v_j\}$. We first place four searchers on v_i. While σ_4 stays still on v_i, the other three searchers can move along $P_{i,j}$ from v_i to v_j to clean all edges one by one along the path. Then σ_1 slides from v_j to v_0. So σ_2 can clean all edges between v_0 and v_i and also clean all edges between v_0 and v_j. After that, σ_1 and σ_2 move back to v_j, and then three searchers can move along $P_{j,i}$ from v_j to v_i to clean all remaining edges.

Case 3: $|E_S| = 3$. Let $E_S = \{v_0v_i, v_0v_j, v_0v_k\}$, where $1 \leq i < j < k \leq n-1$. Depending on the multiplicity of each spoke in E_S, we have the following four subcases.

Case 3.1: Each spoke in E_S has multiplicity of at least three. Since W_1 is not contained in W as a minor, among the three paths $P_{i,j}$, $P_{j,k}$ and $P_{k,i}$, there is at least one of them, say $P_{i,j}$, on which every edge has multiplicity of at most two. We first place σ_1 on v_j and move the other three searchers along $P_{j,k}$ from v_j to v_k to clean all edges on the path. After that, σ_3 and σ_4 slide to v_0 and then σ_3 cleans all spokes between v_0 and v_k and all spokes between v_0 and v_j. Then σ_1 and σ_4 move along $P_{i,j}$ from v_j to v_i to clean all edges on this path. While σ_1 stays still on v_i, σ_4 cleans all spokes between v_0 and v_i. Finally, σ_2, σ_3 and σ_4 move along $P_{k,i}$ from v_k to v_i to clean all edges on this path.

Case 3.2: Only one spoke in E_S, say v_0v_k, has multiplicity of at most two and the other two spokes have multiplicity of at least three. If there is at least one of the three paths $P_{i,j}$, $P_{j,k}$ and $P_{k,i}$, on which every edge has multiplicity at most two, then similarly to Case 3.1, we can clean W using four searchers. Suppose that each of $P_{i,j}$, $P_{j,k}$, $P_{k,i}$ contains an edge of multiplicity at least three. We first place σ_1 on v_i and move the other three searchers along $P_{i,j}$ from v_i to v_j to clean all edges on the path. After that, σ_3 and σ_4 slide to v_0 and then σ_3 cleans all spokes between v_0 and v_i and all spokes between v_0 and v_j. Then σ_3 and σ_4 slide to v_k. Since W_2 is not contained in W as a minor, at least one of $P_{j,k}$ and $P_{k,i}$ contains exactly one edge of multiplicity at least three, and this edge is incident with v_k. Without loss of generality, let $v_{k-1}v_k$ be this edge. While σ_1 stays on v_i and σ_4 stays on v_k, σ_2 and σ_3 move along $P_{j,k-1}$ from v_j to v_{k-1} to clean all edges on this path. Then σ_2 cleans all edges between v_{k-1} and v_k. Finally, σ_2, σ_3 and σ_4 move along $P_{k,i}$ from v_k to v_i to clean all remaining edges.

Case 3.3: Only one spoke in E_S, say v_0v_i, has multiplicity of at least three and the other two spokes have multiplicity of at most two. If there is at least one of $P_{i,j}$, $P_{j,k}$ and $P_{k,i}$, on which every edge has multiplicity at most two, then similarly to Case 3.1, we can clean W using four searchers. Assume that all of $P_{i,j}$, $P_{j,k}$, $P_{k,i}$ contain an edge of multiplicity at least three. Place σ_1 on v_i and move the other three searchers along $P_{i,j}$ from v_i to v_j to clean all edges on the path. Since W_3 is not contained in W as a minor, $P_{j,k}$ contains at most two edges of multiplicity at least three, each of which is incident with v_j or v_k. Without loss of generality, suppose that v_jv_{j+1} and $v_{k-1}v_k$ have multiplicity of at least three. Slide σ_3 to v_{j+1} and use σ_4 to clean the edges between v_j and v_{j+1}. Move σ_3 and σ_4 along $P_{j+1,k-1}$ to clean all edges on this path. Use σ_2 and σ_3 to clean the spokes v_0v_j, v_0v_i and v_0v_k. Then clean the edges between v_{k-1} and v_k. Finally, move σ_2, σ_3 and σ_4 along $P_{k,i}$ from v_k to v_i to clean all remaining edges.

Case 3.4: Each spoke in E_S has multiplicity at most two. Suppose that each of $P_{i,j}$, $P_{j,k}$, $P_{k,i}$ contains an edge of multiplicity at least three. Since W_4 is not contained in W as a minor, one of $P_{i,j}$, $P_{j,k}$ and $P_{k,i}$, say $P_{j,k}$, contains only one edge of multiplicity at least three which is incident to v_j or v_k, or contains exactly two non-parallel edges of multiplicity at least three. Similarly to Case 3.3, we can first clean $P_{i,j}$, then all spokes and $P_{j,k}$, and finally clean $P_{k,i}$.

Case 4: $|E_S| = 4$. Let $E_S = \{v_0v_i, v_0v_j, v_0v_k, v_0v_l\}$, where $1 \le i < j < k < l \le n-1$. Depending on the multiplicity of spokes in E_S, we have the following five subcases.

Case 4.1: Each spoke in E_S has multiplicity of at least three. Since W_1 is not contained in W as a minor, among the four paths $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$, there are at least two of them on which every edge has multiplicity of at most two. We have two cases for these two paths: they either share a terminal vertex, or they do not. We only prove the latter; the former can be proved similarly. So assume that every edge of $P_{j,k}$ and $P_{l,i}$ has multiplicity of at most two. Place σ_1 on v_i and move the other three searchers along $P_{i,j}$ from v_i to v_j to clean all edges on $P_{i,j}$. Slide σ_2 from v_j to v_0 and use σ_4 to clean the spokes v_0v_j and v_0v_i. Move σ_3 and σ_4 along $P_{j,k}$ from v_j to v_k and then move σ_1 and σ_4 along $P_{l,i}$ from v_i to v_l to clean all edges on these two paths. Use σ_4 to clean the spokes v_0v_k and v_0v_l. Finally, move σ_2, σ_3 and σ_4 along $P_{k,l}$ from v_k to v_l to clean all remaining edges.

Case 4.2: Only one spoke in E_S has multiplicity of at most two and the other three spokes have multiplicity of at least three. This case can be proved in a way similar to Case 4.1.

Case 4.3: Two spokes in E_S have multiplicity of at most two and the other two have multiplicity of at least three. We have two subcases regarding the relative positions of the two kinds of spokes.

Case 4.3.1: Spokes v_0v_i and v_0v_k have multiplicity of at least three and v_0v_j and v_0v_l have multiplicity of at most two. We have six subcases regarding the distribution of the outer edges with multiplicity at least three.

Case 4.3.1.1: One of $P_{i,k}$ and $P_{k,i}$ contains no edge of multiplicity at least three. Suppose each edge of $P_{i,k}$ has multiplicity of at most two. Place σ_4 on v_i and move the other three searchers along $P_{l,i}$ from v_i to v_l to clean all edges on $P_{l,i}$. Move σ_2 and σ_3 from v_l to v_0 to clean the spoke(s) between v_0 and v_l. Use σ_3 to clean the spokes between v_0 and v_i. Move σ_3 and σ_4 along $P_{i,j}$ from v_i to v_j to clean all edges on the path. Use σ_3 to clean the spoke(s) between v_0 and v_j. Move σ_3 and σ_4 along $P_{j,k}$ from v_j to v_k to clean all edges on $P_{j,k}$. Use σ_3 to clean the spokes between v_0 and v_k. Finally, move σ_2, σ_3 and σ_4 along $P_{k,l}$ from v_k to v_l to clean all remaining edges.

Case 4.3.1.2: One of $P_{j,l}$ and $P_{l,j}$ contains no edge of multiplicity at least three. Suppose each edge of $P_{l,j}$ has multiplicity of at most two. Similarly to Case 4.3.1.1, we first place one searcher on v_j and use the other three searchers to clean $P_{j,k}$. Then clean the spoke(s) v_0v_j and move two searchers along $P_{i,j}$ from v_j to v_i to clean it. Clean the spokes v_0v_i, v_0v_k and v_0v_l. Move two searchers along $P_{l,i}$ from v_i to v_l to clean it. Finally, move three searchers along $P_{k,l}$ from v_l to v_k to clean all remaining edges.

Case 4.3.1.3: $P_{i,j}$ and $P_{k,l}$ contain no edge of multiplicity at least three. Similarly to Case 4.3.1.2, we first clean $P_{j,k}$, and then clean $P_{i,j}$. After that, clean all the spokes. Then move two searchers along $P_{k,l}$ from v_k to v_l to clean it. Finally, move three searchers along $P_{l,i}$ from v_l to v_i to clean all remaining edges.

Case 4.3.1.4: $P_{j,k}$ and $P_{l,i}$ contain no edge of multiplicity at least three. This case can be proved in a way similar to Case 4.3.1.3.

Case 4.3.1.5: Exactly one of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains no edge of multiplicity at least three. Suppose $P_{j,k}$ is such a path. Since W_2 is not contained in W as a minor, at least one of $P_{k,l}$ and $P_{l,i}$ contains exactly one edge of multiplicity at least three, and this edge is incident with v_l. Without loss of generality, suppose that v_lv_{l+1} is the only edge on $P_{l,i}$ with multiplicity at least three. Similarly to Case 4.3.1.2 or Case 4.3.1.3, we first clean $P_{i,j}$ and then clean $P_{j,k}$. After that, clean all the spokes. Then move two searchers along $P_{l+1,i}$ from v_i to v_{l+1} to clean it. Then clean edges between v_l and v_{l+1}. Finally, move three searchers along $P_{k,l}$ to clean all remaining edges.

Case 4.3.1.6: Each of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains at least one edge of multiplicity at least three. Since W_2 is not contained in W as a minor, W cannot have either of the following properties: each of $P_{i,j}$ and $P_{j,k}$ contains an edge of multiplicity at least three which is not incident with v_j; or each of $P_{k,l}$ and $P_{l,i}$ contains an edge of multiplicity at least three which is not incident with v_l. Thus, at least one of $P_{i,j}$ and $P_{j,k}$ contains exactly one edge of multiplicity at least three which is incident with v_j, and at least one of $P_{k,l}$ and $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Without loss of generality, suppose $P_{j,k}$ contains exactly one edge of multiplicity at least three which is incident with v_j, that is, v_jv_{j+1}. If $P_{k,l}$ contains exactly one edge of multiplicity at least three which is incident with v_l, that is, v_lv_{l-1}, then we first clean $P_{i,j}$ and then clean the edges between v_j and v_{j+1} and further clean $P_{j+1,k}$. After that, clean all the spokes.

Then move two searchers along $P_{k,l-1}$ from v_k to v_{l-1} to clean it. Then clean edges between v_l and v_{l-1}. Finally, move three searchers along $P_{l,i}$ from v_l to v_i to clean all remaining edges. If $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l, that is, $v_l v_{l+1}$, then similarly, we clean $P_{i,j}$, $v_j v_{j+1}$, $P_{j+1,k}$, and clean all the spokes. Then move two searchers along $P_{l+1,i}$ from v_i to v_{l+1} to clean this path, then the edges between v_l and v_{l+1}, and finally, clean $P_{k,l}$.

CASE 4.3.2: Spokes $v_0 v_i$ and $v_0 v_j$ have multiplicity of at least three and $v_0 v_k$ and $v_0 v_l$ have multiplicity of at most two.

For this and subsequent (sub)cases, as the reader is now familiar with the style of proof employed, we will simplify a monotonic cleaning strategy to a *cleaning order* if the searchers' positions and sliding directions can be determined from the context without ambiguity. For example, given two paths, $P_{i,j}$ and $P_{j,k}$, the first containing at least one edge with multiplicity three and the second containing no edge with multiplicity greater than two, and having specified that there are 4 searchers on vertex v_i, we might give a cleaning order of $\langle P_{i,j}, P_{j,k} \rangle$, to indicate that three searchers would proceed to clean the path $P_{i,j}$ (one remaining on v_i). Then with another remaining on v_j, the two remaining searchers would clean $P_{j,k}$.

CASE 4.3.2.1: Two of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contain no edge of multiplicity at least three. Similarly to CASE 4.3.1.1 – CASE 4.3.1.4, we can clean W.

CASE 4.3.2.2: Exactly one of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains no edge of multiplicity at least three. Since W_5 is not contained in W as a minor, $P_{k,l}$ must contain at least one edge of multiplicity at least three. Then we have three cases.

CASE 4.3.2.2.1: Each edge of $P_{i,j}$ has multiplicity of at most two. If $P_{k,l}$ does not contain an edge of multiplicity at least three which is not incident with v_k or v_l, then W can be cleaned in the following cleaning order $\langle P_{j,k}, v_k v_{k+1}, v_0 v_k, v_0 v_j, P_{i,j}, v_0 v_i, v_0 v_l, P_{k+1,l-1}, v_l v_{l-1}, P_{l,i} \rangle$. If $P_{k,l}$ contains an edge of multiplicity at least three which is not incident with v_k or v_l, since W_7 is not contained in W as a minor, either $P_{j,k}$ contains exactly one edge of multiplicity at least three which is incident with v_k, or $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Without loss of generality, suppose $P_{j,k}$ contains exactly one edge of multiplicity at least three which is incident with v_k. Then W can be cleaned in the cleaning order $\langle P_{k,l}, v_k v_{k-1}, P_{j,k-1}, v_0 v_k, v_0 v_j, P_{i,j}, v_0 v_i, v_0 v_l, P_{l,i} \rangle$.

CASE 4.3.2.2.2: Each edge of $P_{j,k}$ has multiplicity of at most two. Since $P_{i,j}$ contains an edge of multiplicity at least three and W_2 is not contained in W as a minor, either $P_{k,l}$ contains exactly one edge of multiplicity at least three which is incident with v_l, or $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Without loss of generality, suppose $P_{k,l}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Then W can be cleaned in the cleaning order $\langle P_{i,j}, v_0 v_j, v_0 v_i, P_{j,k}, v_0 v_k, v_0 v_l, P_{k,l-1}, v_l v_{l-1}, P_{l,i} \rangle$.

CASE 4.3.2.2.3: Each edge of $P_{l,i}$ has multiplicity of at most two. Similarly to CASE 4.3.2.2.2, we can clean W.

CASE 4.3.2.3: Each of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains an edge of multiplicity at least three. Since W_5 is not contained in W as a minor, this case cannot occur for W.

CASE 4.4: Only one spoke in E_S has multiplicity of at least three and the other three spokes have multiplicity of at most two. Let $v_0 v_i$ be the spoke in E_S has multiplicity of at least three. We have three subcases regarding the distribution of the outer edges with multiplicity at least three.

CASE 4.4.1: Two of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contain no edge of multiplicity at least three. The strategies to clean W in this case are similarly to those in CASE 4.3.1.1 – CASE 4.3.1.4.

CASE 4.4.2: Exactly one of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains no edge of multiplicity at least three. By symmetry, we only need to consider the following two subcases.

CASE 4.4.2.1: Each edge of $P_{i,j}$ has multiplicity of at most two. If $P_{k,l}$ does not contain an edge of multiplicity at least three which is not incident with v_k or v_l, then W can be cleaned in the same way as that in CASE 4.3.2.2.1. If $P_{k,l}$ contains an edge of multiplicity at least three which is not incident with v_k or v_l, Since W_7 is not contained in W as a minor, either $P_{j,k}$ contains exactly one edge of multiplicity at least three which is incident with v_k, or $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. So W can be cleaned in the same way as that in CASE 4.3.2.2.1.

CASE 4.4.2.2: Each edge of $P_{j,k}$ has multiplicity of at most two. Since $P_{i,j}$ contains an edge of multiplicity at least three and W_6 is not contained in W as a minor, either $P_{k,l}$ contains exactly one edge of multiplicity at least three which is incident with v_l, or $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Thus W can be cleaned in the same way as that in CASE 4.3.2.2.2.

CASE 4.4.3: Each of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains an edge of multiplicity at least three. Since W_3 is not contained in W as a minor, either $P_{i,j}$ contains exactly one edge of multiplicity at least three which is incident with v_j, or $P_{l,i}$ contains exactly one edge of multiplicity at least three which is incident with v_l. Then we have the following three subcases.

CASE 4.4.3.1: If $v_{j-1} v_j$ is the only edge on $P_{i,j}$ with multiplicity at least three and $v_l v_{l+1}$ is the only edge on $P_{l,i}$ with multiplicity at least three, then W can be cleaned in the cleaning order $\langle P_{j,k}, v_{j-1} v_j, P_{i,j-1}, v_0 v_j, v_0 v_i, v_0 v_k, v_0 v_l, P_{l-1,i}, v_l v_{l+1}, P_{k,l} \rangle$.

CASE 4.4.3.2: If $v_{j-1} v_j$ is the only edge on $P_{i,j}$ with multiplicity at least three, and $P_{l,i}$ contains an edge of multiplicity at least three which is not incident with v_l, then $v_{l-1} v_l$ is the only edge on $P_{k,l}$ with multiplicity at least three because W_6 is not contained in W as a minor. So W can be cleaned in the cleaning order $\langle P_{j,k}, v_{j-1} v_j, P_{i,j-1}, v_0 v_j, v_0 v_i, v_0 v_k, v_0 v_l, P_{j,l-1}, v_{l-1} v_l, P_{l,i} \rangle$.

CASE 4.4.3.3: If $v_l v_{l+1}$ is the only edge on $P_{l,i}$ with multiplicity at least three, and $P_{i,j}$ contains an edge of multiplicity at least three which is not incident with v_j, then similarly to CASE 4.4.3.2., we can prove this case.

CASE 4.5: All four spokes in E_S have multiplicity of at most two. We have the following three subcases regarding the distribution of the outer edges with multiplicity at least three.

CASE 4.5.1: Two or fewer of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contain edges with multiplicity at least three. By symmetry, we only need to consider two cases: (1) If all edges of $P_{k,l}$ and $P_{l,i}$ have multiplicity at most two, then clean W in the order $\langle P_{i,j}, P_{l,i}, v_0v_i, v_0v_j, v_0v_l, v_0v_k, P_{k,l}, P_{j,k}\rangle$; (2) if all edges of $P_{j,k}$ and $P_{l,i}$ have multiplicity at most two, then clean W in the order $\langle P_{i,j}, P_{j,k}, v_0v_j, v_0v_i, v_0v_k, v_0v_l, P_{l,i}, P_{k,l}\rangle$.

CASE 4.5.2: Exactly one of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains no edge of multiplicity at least three. Without loss of generality, suppose that each edge of $P_{i,j}$ has multiplicity of at most two. We have the following three subcases.

CASE 4.5.2.1: If $P_{j,k}$ contains only one edge with multiplicity at least three which is incident with v_k, then W can be cleaned in the order $\langle P_{k,l}, v_{k-1}v_k, P_{j,k-1}, v_0v_k, v_0v_l, v_0v_j, v_0v_i, P_{i,j}, P_{l,i}\rangle$.

CASE 4.5.2.2: If $P_{l,i}$ contains only one edge with multiplicity at least three which is incident with v_l, then W can be cleaned in the order $\langle P_{k,l}, v_lv_{l+1}, P_{i,l+1}, v_0v_l, v_0v_k, v_0v_i, v_0v_j, P_{i,j}, P_{j,k}\rangle$.

CASE 4.5.2.3: Suppose that $P_{j,k}$ contains an edge with multiplicity at least three which is not incident with v_k and $P_{l,i}$ contains an edge with multiplicity at least three which is not incident with v_l. Since W_7 is not contained in W as a minor, we have the following two subcases.

CASE 4.5.2.3.1: $P_{k,l}$ contains only one edge with multiplicity at least three which is incident with one of v_k or v_l, say v_k. Then W can be cleaned in the order $\langle P_{j,k}, v_kv_{k+1}, P_{k+1,l}, v_0v_k, v_0v_l, v_0v_j, v_0v_i, P_{i,j}, P_{l,i}\rangle$.

CASE 4.5.2.3.2: $P_{k,l}$ contains exactly two edges with multiplicity at least three such that one is incident with v_k and the other is incident with v_l. Then W can be cleaned in the order $\langle P_{j,k}, v_kv_{k+1}, P_{k+1,l-1}, v_0v_k, v_0v_j, P_{i,j}, v_0v_i, v_0v_l, v_{l-1}v_l, P_{l,i}\rangle$.

CASE 4.5.3: Each of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contains an edge of multiplicity at least three. Since W_8 is not contained in W as a minor, at least two of $P_{i,j}$, $P_{j,k}$, $P_{k,l}$ and $P_{l,i}$ contain exactly one edge with multiplicity at least three. By symmetry, we only need to consider the following two subcases.

CASE 4.5.3.1: $P_{i,j}$ and $P_{j,k}$ contain exactly one edge with multiplicity at least three.

CASE 4.5.3.1.1: The edge with multiplicity at least three in $P_{i,j}$ is not incident with v_i and v_j, and the edge with multiplicity at least three in $P_{j,k}$ is not incident with v_j and v_k. If $P_{k,l}$ has an edge with multiplicity at least three which is not incident with v_k or $P_{l,i}$ has an edge with multiplicity at least three which is not incident with v_i, then W_4 is a minor of W. This is a contradiction. Thus, there is only one edge on $P_{k,l}$ whose multiplicity is at least three and which is incident with v_k, and there is only one edge on $P_{l,i}$ whose multiplicity is at least three and which is incident with v_i. Then W can be cleaned in the order $\langle P_{i,j}, v_{i-1}v_i, P_{l,i-1}, v_0v_i, v_0v_j, v_0v_l, v_0v_k, P_{k+1,l}, v_kv_{k+1}, P_{j,k}\rangle$.

CASE 4.5.3.1.2: The edge with multiplicity at least three in $P_{j,k}$ is incident with v_j. Then we have the following three subcases.

CASE 4.5.3.1.2.1: There is an edge with multiplicity at least three on $P_{k,l}$ which is not incident with v_k and v_l. If there is an edge with multiplicity at least three on $P_{l,i}$ which is not incident with v_l and v_i, then W_4 is a minor of W, which is a contradiction; if there is an edge with multiplicity at least three on $P_{l,i}$ which is not incident with v_l, then W_7 is a minor of W, which is a contradiction; otherwise, the edge with multiplicity at least three on $P_{l,i}$ must be incident with v_l, which implies that W can be cleaned in the order $\langle P_{i,j}, v_jv_{j+1}, P_{j+1,k}, v_0v_j, v_0v_k, v_0v_i, v_0v_l, P_{l+1,i}, v_lv_{l+1}, P_{k,l}\rangle$.

CASE 4.5.3.1.2.2: There are only two edges with multiplicity at least three on $P_{k,l}$, one is incident with v_k and the other is incident with v_l. If there is an edge with multiplicity at least three on $P_{l,i}$ which is not incident with v_l, then W_8 is a minor of W, which is a contradiction; otherwise, the edge with multiplicity at least three on $P_{l,i}$ must be incident with v_l; so W can be cleaned in the same cleaning order as that in CASE 4.5.3.1.2.1.

CASE 4.5.3.1.2.3: There is only one edge with multiplicity at least three on $P_{k,l}$ which is incident with v_k or v_l. Similarly to CASE 4.5.3.1.2.1, we can prove this case.

CASE 4.5.3.2: $P_{j,k}$ and $P_{l,i}$ contain exactly one edge with multiplicity at least three. This case can be proved using the analysis in CASE 4.5.3.1.

CASE 5: $|E_S| = 5$. Let $E_S = \{v_0v_i, v_0v_j, v_0v_k, v_0v_l, v_0v_m\}$, where $1 \le i < j < k < l < m \le n-1$. Depending on the multiplicity of spokes in E_S, we have the following six subcases.

CASE 5.1: Each spoke in E_S has multiplicity of at least three. Since W_1 is not contained in W as a minor, among the five paths $P_{i,j}$, $P_{j,k}$, $P_{k,l}$, $P_{l,m}$ and $P_{m,i}$, there are at most two of them which contain edges of multiplicity at least three. These two paths either share a terminal vertex, or not. We consider the latter case. Without loss of generality, suppose that only $P_{i,j}$ and $P_{k,l}$ contain edges of multiplicity at least three. Then W can be cleaned in the order $\langle P_{i,j}, v_0v_i, v_0v_j, P_{j,k}, v_0v_k, P_{m,i}, v_0v_m, P_{l,m}, v_0v_l, P_{k,l}\rangle$. The former case is proved similarly.

CASE 5.2: Only one spoke in E_S has multiplicity of at most two and the other four spokes have multiplicity of at least three. Let v_0v_l be the spoke that has multiplicity of at most two. Since W_1 is not contained in W as a minor, among the four paths $P_{i,j}$, $P_{j,k}$, $P_{k,m}$ and $P_{m,i}$, there are at most two of them which contain edges of multiplicity at least three. By symmetry, we have four subcases.

CASE 5.2.1: Only $P_{j,k}$ and $P_{m,i}$ contain edges of multiplicity at least three. In this case we can clean W in the order $\langle P_{j,k}, v_0v_k, P_{k,l}, v_0v_l, P_{l,m}, v_0v_m, v_0v_j, P_{i,j}, v_0v_i, P_{m,i}\rangle$.

CASE 5.2.2: Only $P_{i,j}$ and $P_{m,i}$ contain edges of multiplicity at least three. We can clean W similarly to CASE 5.2.1.

CASE 5.2.3: Only $P_{i,j}$ and $P_{k,m}$ contain edges of multiplicity at least three. Regarding the positions of edges of multiplicity at least three on $P_{k,l}$ and $P_{l,m}$, we have two subcases.

CASE 5.2.3.1: Only one of $P_{k,l}$ and $P_{l,m}$ contains edges of multiplicity at least three. Without loss of generality, suppose that all edges on $P_{l,m}$ have multiplicity at most two. Then we can clean W in the same order as that in CASE 5.1.

CASE 5.2.3.2: Both $P_{k,l}$ and $P_{l,m}$ contain edges of multiplicity at least three. If both $P_{k,l}$ and $P_{l,m}$ contain edges of multiplicity at least three which are not incident with v_l, then W_2 is a minor, a contradiction. Thus one of $P_{k,l}$ and $P_{l,m}$, say

$P_{k,l}$, contains only one edge of multiplicity at least three which is incident with v_l. Then W can be cleaned in the order $\langle P_{i,j}, v_0v_i, v_0v_j, P_{j,k}, v_0v_k, P_{m,i}, v_0v_m, v_0v_l, P_{k,l-1}, v_{l-1}v_l, P_{l,m}\rangle$.

Case 5.2.4: Only $P_{j,k}$ and $P_{k,m}$ contain edges of multiplicity at least three. This case follows in a similar fashion as Case 5.2.3.

Case 5.3: Two spokes in E_S have multiplicity of at most two and the other three have multiplicity of at least three. By symmetry, we have two subcases.

Case 5.3.1: Spokes v_0v_j and v_0v_l have multiplicity of at most two. Since W_1 is not contained in W as a minor, among the three paths $P_{i,k}$, $P_{k,m}$ and $P_{m,i}$, there are at most two of them which contain edges of multiplicity at least three. By symmetry, we have two subcases.

Case 5.3.1.1: $P_{i,k}$ and $P_{k,m}$ contain edges of multiplicity at least three. If both $P_{k,l}$ and $P_{l,m}$ contain edges of multiplicity at least three which is not incident with v_l, then W_2 is a minor. This is a contradiction. Thus one of $P_{k,l}$ and $P_{l,m}$, say $P_{k,l}$, contains only one edge of multiplicity at least three which is incident with v_l. Similarly, one of $P_{i,j}$ and $P_{j,k}$, say $P_{i,j}$, contains only one edge of multiplicity at least three which is incident with v_j. Then W can be cleaned in the order $\langle P_{j,k}, v_{j-1}v_j, v_0v_j, v_0v_k, P_{i,j-1}, v_0v_i, P_{m,i}, v_0v_m, v_0v_l, P_{k,l-1}, v_{l-1}v_l, P_{l,m}\rangle$.

Case 5.3.1.2: $P_{i,k}$ and $P_{m,i}$ contain edges of multiplicity at least three. If both $P_{i,j}$ and $P_{j,k}$ contain edges of multiplicity at least three which is not incident with v_j, then W_2 is a minor. This is a contradiction. Thus one of $P_{i,j}$ and $P_{j,k}$, say $P_{i,j}$, contains only one edge of multiplicity at least three which is incident with v_j. Then W can be cleaned in the order $\langle P_{m,i}, v_0v_i, v_0v_m, P_{l,m}, v_0v_l, P_{k,l}, v_0v_k, v_0v_j, P_{i,j-1}, v_{j-1}v_j, P_{j,k}\rangle$.

Case 5.3.2: Spokes v_0v_k and v_0v_l have multiplicity of at most two. Since W_1 is not contained in W as a minor, among the three paths $P_{i,j}$, $P_{j,m}$ and $P_{m,i}$, at most two contain edges of multiplicity at least three. By symmetry, we have two subcases.

Case 5.3.2.1: $P_{i,j}$ and $P_{j,m}$ contain edges of multiplicity at least three. If both $P_{j,k}$ and $P_{l,m}$ contain edges of multiplicity at least three, then W_5 is a minor, a contradiction. Thus only one of $P_{j,k}$ and $P_{l,m}$, say $P_{j,k}$, contains edges of multiplicity at least three. If both $P_{j,k}$ and $P_{k,l}$ contain edges of multiplicity at least three which are not incident with v_k, then W_2 is a minor. Again, a contradiction. Thus only one of $P_{j,k}$ and $P_{k,l}$, say $P_{j,k}$, contains only one edge of multiplicity at least three which is incident with v_k. Then W can be cleaned in the order $\langle P_{k,l}, v_{k-1}v_k, v_0v_k, v_0v_l, P_{j,k-1}, v_0v_j, P_{l,m}, v_0v_m, P_{m,i}, v_0v_i, P_{i,j}\rangle$.

Case 5.3.2.2: $P_{i,j}$ and $P_{m,i}$ contain edges of multiplicity at least three. If $P_{l,k}$ contains at least one edge of multiplicity at least three, then W has W_{11} as a minor, a contradiction. If $P_{j,k}$ (or $P_{l,m}$) contain at least one edge of multiplicity at least 3, then W has a W_2 minor, another contradiction. Thus, all edges in $P_{j,k}$, $P_{k,l}$, and $P_{l,m}$ may be assumed to have multiplicity two or less. Then W can be cleaned in the order $\langle P_{i,j}, v_0v_i, v_0v_j, P_{j,k}, v_0v_k, P_{k,l}, v_0v_l, P_{l,m}, v_0v_m, P_{i,m}\rangle$.

Case 5.4: Three spokes in E_S have multiplicity of at most two and the other two have multiplicity of at least three. This case follows the same pattern as Case 4.4.

Case 5.5: Four spokes in E_S have multiplicity of at most two and one has multiplicity of at least three. Without loss of generality, suppose that spoke v_0v_i has multiplicity of at least three. If each of the paths $P_{i,j}$, $P_{k,l}$ and $P_{m,i}$ contains an edge with multiplicity at least three, then W contains W_9 as a minor, a contradiction. Then the rest of this case follows similarly to Case 4.5.

Case 5.6: All five spokes in E_S have multiplicity of at most two. Since W_{10} is not contained in W as a minor, among the five paths $P_{i,j}$, $P_{j,k}$, $P_{k,l}$, $P_{l,m}$ and $P_{m,i}$, there are at most four which contain edges of multiplicity at least three. Suppose that among the five paths $P_{i,j}$, $P_{j,k}$, $P_{k,l}$, $P_{l,m}$ and $P_{m,i}$, there are at least three of them, say $P_{i,j}$, $P_{k,l}$ and $P_{l,m}$, which contains edges of multiplicity at least three. Since W_{11} is not contained in W as a minor, one of $P_{k,l}$ and $P_{l,m}$ contains only one edge of multiplicity at least three. The rest of this case can be proved similarly to Case 4.5.

Case 6: $|E_S| \geq 6$. We only consider the case where $|E_S| = 6$ because the proof for the case where $|E_S| > 6$ is similar. Let $E_S = \{v_0v_h, v_0v_i, v_0v_j, v_0v_k, v_0v_l, v_0v_m\}$, where $1 \leq h < i < j < k < l < m \leq n-1$. Among the six paths $P_{h,i}$, $P_{i,j}$, $P_{j,k}$, $P_{k,l}$, $P_{l,m}$ and $P_{m,h}$, if there exist three vertex-disjoint paths which contains an edge of multiplicity at least three, then W_{12} is a minor of W. This is a contradiction. By symmetry, we only consider the following two subcases; the other cases are either similar or simpler.

Case 6.1: Each of $P_{h,i}$, $P_{i,j}$, $P_{j,k}$ and $P_{k,l}$ contains an edge of multiplicity at least three while $P_{l,h}$ does not contain an edge of multiplicity at least three. If there are at least four spokes with multiplicity at least three, then W_1 is a minor, a contradiction. If one of v_0v_i, v_0v_j and v_0v_k has multiplicity at least three, then W_3 is a minor, another contradiction. If three of $P_{h,i}$, $P_{i,j}$, $P_{j,k}$ and $P_{k,l}$ each contains an edge of multiplicity at least three which is not incident with any terminal vertices of these paths, then W_4 is a minor, which is again a contradiction. Without loss of generality, assume that each of $P_{h,i}$ and $P_{j,k}$ contains an edge of multiplicity at least three which is not incident with any terminal vertices, while any edge with multiplicity at least three on $P_{i,j}$ and $P_{k,l}$ is incident with some terminal vertex. Under this assumption, W_{11} is a minor. This is a contradiction. The rest of this case can be proved as previously, or there is a simple cleaning strategy.

Case 6.2: Each of $P_{h,i}$, $P_{i,j}$, $P_{k,l}$ and $P_{l,m}$ contains an edge of multiplicity at least three while $P_{j,k}$ and $P_{m,h}$ do not contain an edge of multiplicity at least three. If one of the six spokes has multiplicity at least three, then W_3, W_6 or W_9 is a minor, a contradiction. If three of $P_{h,i}$, $P_{i,j}$, $P_{j,k}$ and $P_{k,l}$ each contains an edge of multiplicity at least three which is not incident with any terminal vertices, then W_4 is a minor, another contradiction. If three of $P_{h,i}$, $P_{i,j}$, $P_{j,k}$ and $P_{k,l}$ each contains an edge of multiplicity at least three such that two of them are not incident with the same terminal vertex, then W_{11} is a minor, a contradiction. The remaining versions of this case can be proved as previously, or there is a simple cleaning strategy. ■

5. Conclusion

We have constructively characterized when a biconnected outerplanar graph is 4-searchable. Observe that each graph in Fig. 6 is a combination of houses and tents. The gluing rules are as follows: (1) A combination of exactly three of the houses and tents suffice to form a graph in Fig. 6; and (2) the houses are glued at the two end vertices of their base edges. Thus, these graphs are composed of smaller sub-structures that can be called *base* graphs. One future direction is to characterize forbidden minors for k-searchable graphs constructively using a set of base graphs. In the case of outerplanar graphs, the base graphs correspond to the house and tent, thus, there are only 2 base graphs needed.

Ideally, the result given in Section 3 would be extended to any 4-searchable outerplanar graph. This could be done by giving the exact structure of such a graph by giving (1) the rules for attaching biconnected components that have a smaller search number to a biconnected 4-searchable outerplanar component and (2) the rules for glueing two biconnected 4-searchable outerplanar components to each other. In this paper, we only gave the structure of the biconnected components for the graph classes we are interested in. The extensions of these results to general outerplanar graphs can be done by the analysis mentioned above.

As another extension of Section 4, one can obtain the obstructions for 4-searchable 2-outerplanar graphs in $\mathcal{F}_4$. Although we do not give them here, by analyzing such obstructions that we have obtained, we notice the difficulty of the characterization of graphs in $\mathcal{F}_4$ in general. If we can find each member of this family, we may be able to observe the transition between the obstructions for 4-searchable k-outerplanar graphs and 4-searchable $(k+1)$-outerplanar graphs.

CRediT authorship contribution statement

Öznur Yaşar Diner: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Danny Dyer:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Boting Yang:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which improved the presentation of this paper.

Öznur Yaşar Diner is partially supported by the EU Marie Curie International Reintegration, European Union Grant [grant number PIRG07/GA/2010/268322] and by Kadir Has University BAP, Turkey [grant number 2011-BAP-07]. Danny Dyer is partially supported by NSERC, Canada. Boting Yang is partially supported by an NSERC, Canada Discovery Research Grant [grant number RGPIN-2018-06800].

References

[1] D. Archdeacon, Problems in Topological Graph Theory, 1995, Compiled by D. Archdeacon, 10.1.1.26.8877 (from: citeseerx.ist.psu.edu) Taken on 01.03.2019, fmf.uni-laf.si, Takan on 29.09.2021.
[2] D. Archdeacon, P.A. Huneke, Kuratowski theorem for the projective plane, J. Graph Theory 5 (1981) 243–246.
[3] D. Bienstock, P. Seymour, Monotonicity in graph searching, J. Algorithms 12 (1991) 239–245.
[4] R.L. Breisch, An intuitive approach to speleotopology, Southwest. Cavers 6 (1967) 72–78.
[5] J. Chambers, Hunting for torus obstructions, (M.Sc. thesis), Dept. of Comput. Sci. University of Victoria, British Columbia, 2002.
[6] F. Chung, On the cutwidth and the topological bandwidth of a tree, SIAM J. Algebr. Discrete Methods 6 (1985) 268–277.
[7] R. Diestel, Graph Theory, in: Heidelberg Graduate Texts in Mathematics, Vol. 173, Springer-Verlag, 2016.
[8] D. Dyer, B. Yang, Ö. Yaşar, On the Fast Searching Problem, in: Lecture Notes in Computer Science, Vol. 5034, Springer-Verlag, 2008, pp. 143–154.
[9] J.A. Ellis, I.H. Sudborough, J.S. Turner, The vertex separation and search number of a graph, Inform. and Comput. 113 (1994) 50–79.
[10] F.V. Fomin, P. Heggernes, J.A. Telle, Graph searching, elimination trees, and a generalization of bandwidth, Algorithmica 41 (2004) 73–87.
[11] F.V. Fomin, D.M. Thilikos, An annotated bibliography on guaranteed graph searching, Theoret. Comput. Sci. 399–3 (2008) 236–245.
[12] A. Gagarin, W. Myrvold, J. Chambers, Forbidden minors for troidal graphs with no $K_{3,3}$, Electron. Notes Discrete Math. 22 (2005) 151–156.
[13] P.A. Golovach, Equivalence of two formalizations of a search problem on a graph (in Russian), Vestnik Leningrad. Univ. Mat. Mekh. Astronom. 10–14. Translation in Vestnik Leningrad Univ. Math. 22-1 (1989) 13–19.
[14] N.G. Kinnersley, Obstruction set isolation for layout permutation problems, (Ph. D. Thesis), Washington State University, 1989.
[15] N.G. Kinnersley, M.A. Langston, Obstruction set isolation for the gate matrix layout problem, Discrete Appl. Math. 54 (1994) 169–213.
[16] C. Kuratowski, Sur le probleme der courbes gauches en topologie, Fund. Math. 15 (1930) 271–283.
[17] A.S. LaPaugh, Recontamination does not help to search a graph, J. ACM 40 (1993) 224–245.
[18] F. Makedon, H. Sudborough, Minimizing width in linear layout, in: Lecture notes in Computer Science, Vol. 154, Springer-Verlag, 1983, pp. 478–490.
[19] N. Megiddo, S.L. Hakimi, M.R. Garey, D.S. Johnson, C.H. Papadimitriou, The complexity of searching a graph, J. ACM 35 (1988) 18–44.
[20] T. Parsons, Pursuit-evasion in a graph, in: Theory and Applications of Graphs, in: Lecture Notes in Mathematics, Springer-Verlag, 1976, pp. 426–441.
[21] N. Robertson, P.D. Seymour, Graph minors XVI. excluding a non-planar graph, J. Combin. Theory Ser. B 89 (2003) 43–76.
[22] N. Robertson, P.D. Seymour, Graph minors XX. Wagner's Conjecture, J. Combin. Theory Ser. B 92 (2004) 325–357.
[23] D.P. Sanders, On linear recognition of tree-width at most four, 1992, Manuscript.
[24] D. Thilikos, Algorithms and obstructions for linear-width and related search parameters, Discrete Appl. Math. 105 (2000) 239–271.
[25] K. Wagner, Über eine Erweiterung des Satzes von Kuratowski, Deutsche Math. 2 (1937) 280–285.