

Measurement of bivariate risks by the north–south quantile points approach



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ABSTRACT

This paper attempts to determine the Value at Risk (VaR) and Conditional Value at Risk (CVaR) measures for the sum of bivariate risks under dependence. The computation of these risk measures is performed by the north–south quantile points of bivariate distributions. The Farlie–Gumbel–Morgenstern (FGM) copula model is chosen to express dependence of bivariate risks. The behaviors of VaR and CVaR are examined by varying dependence parameter values of the copula model and probability levels of the risk measures. The findings are interpreted from the view point of portfolio risk management.

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1. Introduction

Risk, in general, is defined as a possible and harmful loss occurrence event and risk measures are required to reflect the vulnerability of those who are exposed to it. Risk measures are essential elements of risk management. Several quantitative risk measures that are based on the quantiles of risk distributions have been proposed and used by many authors. Artzner et al. [1], Jorion [2], Kaas et al. [3], Szegő [4] and Dhaene et al. [5] lucidly present and discuss the features and risk measurement capacities of such measures. Following them, this paper considers Value at Risk (VaR) and Conditional Value at Risk (CVaR) as measures for bivariate risks. In the finance and insurance areas there is an ample use of VaR and CVaR measures where portfolio losses, in particular, are the core subjects for decision making and risk modeling [6–8].

Claim amounts in excess of endurable limits are possible losses to cite for portfolios of insurance policies outstanding in a given time interval. Similarly, failures about returns on portfolios of assets, in a given time period, are possible losses for financial establishments. Determination of composition of such insurance and finance portfolios, pricing of insurance policies at individual or collective levels, determination of required return rates and setting of risk reserves and premiums are the decision making outputs that can be mentioned just to give some examples of the use of risk measures for.

In many circumstances it is imperative to take account of dependencies in financial and insurance portfolios: same people may exist in two different portfolios of policies of an insurer or two or more policy owners in an insurance portfolio may cause defaults interdependently. Similarly, by virtue of the presence of same sets of assets in two different financial portfolios, interdependence of return failures in portfolios comes into existence. Dependencies that arise, as exemplified, must always be expressed in risk modeling through risk measurement under dependence. Recently, copula models have been in the use of dependence modeling. Copulas are useful analytical tools that separate joint distribution of risk variables (loss amounts) into two parts: marginal distributions of individual risk variables and an expression about their joint dependence structure. Joe [9], Nelsen [10,11], Cherubini et al. [12] in a financial context, and Denuit et al. [13] in an actuarial context present a detailed discussion of copula functions.

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There is a conceptual and analytical relation between VaR, CVaR, quantiles of distributions and copulas. Chen and Welsh [14] introduce bivariate quantiles through the use of distribution functions and give multivariate extensions of quantile points. Embrechts et al. [15] use copulas to put bounds on VaR functions for sums of dependent risks and present some applications of them. Cheng et al. [16], Belzunce et al. [17], Gebizlioglu and Yagci [18], Fantazzini [19], He and Gong [20] and Halbleib and Pohlmeier [21] present some results, relevant to this paper, that connect risk measures, quantiles and copula functions.

In the following sections, CVaR function for the sum of dependent bivariate risks is determined by making use of the analytical relation between CVaR and VaR, the north–south quantile points of bivariate distributions are used up to this aim. Dependence parameters of copula functions and percentile levels of quantiles directly affect VaR and CVaR values. The VaR and CVaR functions are examined, in this regard, and interpreted by some numerical examples.

2. Bivariate north–south quantile points and risk measures

The use of bivariate north–south quantile points (BNSQP) makes it easy to perform the risk measurement by using copula functions for distributions of dependent risk variables. BNSQP approach concentrates on the quantiles of the bivariate distributions for the conduct of the inference under the bivariate analysis [14]. In this regard, we firstly refer to risk measures and copulas. Then we show the determination of the bivariate VaR and CVaR functions by using BNSQP.

Let X be a continuous random risk variable, as an individual or collective loss amount, with distribution function F . VaR at p -level for this random variable is defined as

$$\text{VaR}[X; p] = F_X^{-1}(p) \quad (1)$$

which is actually a p -th order quantile of F distribution. VaR is not a convex measure. In this sense we consider CVaR as a risk measure that is superior to VaR [1].

CVaR estimates the average loss amount on the tail region of a loss amount distribution beyond the VaR value. Tail risks are important for assessing excessive loss amounts. In addition, CVaR is a coherent risk measure. Any risk measure that satisfies the axioms of sub-additivity, monotonicity, positive homogeneity and translation invariance is a coherent risk [1]. It is well known that VaR is not a coherent risk measure. So it may discourage portfolio diversification. However, it is coherent for $p > 0.5$ under the assumption of elliptically distributed loss amounts when portfolio losses are linear functions of the loss values under concern [5].

CVaR is defined as follows:

$$\text{CVaR}[X; p] = E[X - \text{VaR}[X; p] | X > \text{VaR}[X; p]].$$

CVaR is also defined by the associated mean-excess function (*mef*) as follows,

$$\text{CVaR}[X; p] = m_X(\text{VaR}[X; p])$$

where m_X is defined as

$$\begin{aligned} m_X(x) &= E[X - x | X > x], \quad x > 0 \\ &= \frac{\Pi_X(x)}{\bar{F}_X(x)} \end{aligned}$$

and Π and \bar{F} stand for stop-loss function and survival function of random loss variable X , respectively [13].

Now, let X and Y be continuous random loss amount variables as dependent risks, with distribution functions F_1 and F_2 , respectively. Sklar [22] states that if X and Y are random variables with joint distribution function F and marginal distribution functions F_1 and F_2 , a copula, C , uniquely determines $F(x, y) = C(F_1(x), F_2(y))$ on $\text{Range } F_1 \times \text{Range } F_2$ for all x, y . Nelsen [10] presents copula techniques and several copula function families. Nelsen [11] and Rodriguez-Lallena and Ubeda-Flores [23] extend these to multivariate copula functions for more than two random variables in dependence.

The minimum and maximum copulas C^- and C^+ are named as Fréchet lower and upper bounds and the inequalities $C^- \leq C \leq C^+$, openly expressed below

$$\max(F_1(x) + F_2(y) - 1, 0) \leq C(F_1(x), F_2(y)) \leq \min(F_1(x), F_2(y))$$

are known as the inequality of Fréchet–Hoeffding for distribution functions.

The bivariate copula C is restricted to the support $I^2 = [0, 1]^2$ of the bivariate distribution function $F(x, y)$ with uniform margins on $I = [0, 1]$ such that

$$C(t, 0) = C(0, t) = 0, \quad C(t, 1) = C(1, t) = t, \quad t \in I$$

and

$$C(\tilde{u}_2, \tilde{v}_2) - C(\tilde{u}_2, \tilde{v}_1) - C(\tilde{u}_1, \tilde{v}_2) + C(\tilde{u}_1, \tilde{v}_1) > 0$$

for all $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2$ in I , $\tilde{u}_1 \leq \tilde{u}_2$ and $\tilde{v}_1 \leq \tilde{v}_2$. So, a bivariate copula with support $[0, 1]^2$ is $C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)) = F(x, y)$ where u_1 and u_2 are uniformly distributed random variables.

Quantiles of continuous distribution functions are connected to VaR functions at p -levels as expressed in expression (1). In this connection Chen and Welsh [14] define the bivariate north–south quantile points as follows:

The north–south bivariate quantile point (p_1, p_2) is the vector $\xi(p_1, p_2) = (F_{12}^{-1}(p_1, p_2), F_2^{-1}(p_1 + p_2))$ that satisfies

$$F_2^{-1}(p_1 + p_2) = \inf \{y : F_2(y) \geq p_1 + p_2\} \tag{2}$$

and

$$F_{12}^{-1}(p_1, p_2) = \inf \{x : F(x, F_2^{-1}(p_1 + p_2)) \geq p_1\} \tag{3}$$

for $p_1, p_2 \geq 0$ and $p_1 + p_2 \leq 1$. The p -th north–south bivariate quantile point is $\xi(p) = \xi(\frac{1}{2}p, \frac{1}{2}p)$, $0 \leq p \leq 1$.

In order to build CVaR for bivariate dependent risks, we use the bivariate north–south quantile points in VaR and CVaR expressions:

$$\text{VaR}(p_1, p_2) = (F_{12}^{-1}(p_1, p_2), F_2^{-1}(p_1 + p_2)) \tag{4}$$

where the first component is the p -th quantile of X when $p_1 = 1 - p_2 = p$ and the second component is the $p = p_1 + p_2$ quantile of Y .

By utilizing the VaR (p_1, p_2) function above we can express the bivariate CVaR values. The (X, Y) variate may have any continuous joint distribution with continuous marginals. Without loss of the generality of the following results to any continuous distribution, we let $(X, Y) \equiv (U_1, U_2) = U$ such that U is a random loss amounts vector with uniform components. The bivariate survival function and the mean excess function, that are needed to construct the bivariate CVaR function, are $S_U(\mathbf{u}) = P(U_1 > u_1, U_2 > u_2)$ and $m_U(\mathbf{u}) = (m_U^1(\mathbf{u}), m_U^2(\mathbf{u}))$, respectively, where $m_U^i(\mathbf{u}) = E[U_i - u_i | U_j > u_j]$ for $i, j = 1, 2$ and $i \neq j$. Further, there is a need for a stop-loss transformation vector $\pi_U(\mathbf{u}) = (\pi_U^1(\mathbf{u}), \pi_U^2(\mathbf{u}))$ such that

$$\pi_U^1(\mathbf{u}) = \int_x^1 S_U(t, y) dt \quad \text{and} \quad \pi_U^2(\mathbf{u}) = \int_y^1 S_U(x, t) dt, \tag{5}$$

where $\pi_U^i(\mathbf{u}) = S_U(\mathbf{u}) m_U^i(\mathbf{u})$ for $i = 1, 2$. The bivariate VaR and CVaR are now expressed as $\text{VaR}(p_1, p_2) = \text{VaR}_p(\mathbf{U}) \equiv (\text{VaR}_p(X), \text{VaR}_p(Y))$ and $\text{CVaR}_p(\mathbf{U}) = (\text{CVaR}_p(X), \text{CVaR}_p(Y)) \equiv (\text{CVaR}_p^1(\mathbf{U}), \text{CVaR}_p^2(\mathbf{U}))$ respectively, with $\text{CVaR}_p(\mathbf{U})$ components computed as

$$\begin{aligned} \text{CVaR}_p^i(\mathbf{U}) &= m_X^i[\text{VaR}_p(\mathbf{U})] \\ &= \frac{\pi_U^i[\text{VaR}_p(\mathbf{U})]}{S_U[\text{VaR}_p(\mathbf{U})]}, \quad i = 1, 2. \end{aligned} \tag{6}$$

CVaR is coherent only under suitable conditions such as the continuity of the probability distribution function of (X, Y) [24]. CVaR is a sub-additive risk measure. This is depicted for the sum of two loss amounts $T = X + Y \equiv U_1 + U_2$ as

$$\text{CVaR}_p(T|\mathbf{U}) \leq \sum_{i=1}^2 \text{CVaR}_p^i(\mathbf{U}). \tag{7}$$

Under the comonotonicity conditions for $\mathbf{U} = (U_1, U_2)$, the conditional value at risk for the aggregate loss amount becomes [13,25],

$$\text{CVaR}_p(T|\mathbf{U}) = \sum_{i=1}^2 \text{CVaR}_p^i(\mathbf{U}).$$

In this set-up, the bivariate VaR and CVaR functions are derived, below, for the aggregate loss amount T under the Farlie–Gumbel–Morgenstern (FGM) copula model.

3. The bivariate VaR and CVaR for the FGM distribution by using the BNSQP approach

The FGM distribution for the dependent loss amounts X and Y allows the assessment of dependence through the dependence parameters. Dependence properties for the FGM distributions are closely related with the correlation coefficient as a measure of association under certain conditions [25,26].

The FGM copula function for the FGM distribution proposed by Morgenstern [27], has the form

$$C_\alpha(x, y) = xy(1 + \alpha(1 - x)(1 - y)), \quad -1 \leq \alpha \leq 1 \tag{8}$$

where X and Y are each uniformly distributed over $[0, 1]$ interval and α is the dependence parameter. If the dependence parameter α equals zero, then independence of X and Y variables exists. In general, given (8), the joint distribution function for the bivariate loss amounts is

$$F(x, y) = F_1(x)F_2(y) [1 + \alpha(1 - F_1(x))(1 - F_2(y))]$$

for continuous F_1 and F_2 . The FGM copula function is attractive due to its simplicity and practically. Prieger [28] uses it in the modeling of health insurance. Cossette et al. [29] use the FGM copulas for the modeling of compound sums of dependent

insurance portfolio risks by introducing a generalized FGM copula model. Their model, in our terms, has the form of $C_\alpha(x, y) = xy + \alpha x^a(1-x)^b y^c(1-y)^d$; $a, b, c, d \geq 1$, which turns out to be the copula model in Eq. (8) above when $a = b = c = d = 1$. This extension is shown by them to improve the dependence association range, in terms of Kendall's tau or Spearman's rho, between the X and Y variables. Bairamov and Bayramoglu [30] discuss and review the recent developments and extensions on a modified FGM distribution, namely Huang–Kotz FGM distribution, and develop a new bivariate distribution. Thus, the interest on the FGM distribution or copula family is current and growing over several directions in theory and practice. Focusing on the Morgenstern's FGM copula, we develop VaR and CVaR functions below.

The bivariate CVaR with the FGM copula given in (8) are derived for $-1 \leq \alpha < 0$ and $0 < \alpha \leq 1$ by using BNSQP. For $-1 \leq \alpha < 0$, the bivariate VaR is

$$\text{VaR}(p_1, p_2) = \left(\frac{1}{2} \left(-b_1 + \sqrt{b_1^2 - 4c_1} \right), p_1 + p_2 \right) \quad (9)$$

and thus the $\text{CVaR}_p(T)$ for the aggregate loss amount $T = X + Y$ is found, by using Eq. (7), as follows:

$$\begin{aligned} \text{CVaR}_p(T) &\leq \text{CVaR}_p(X) + \text{CVaR}_p(Y) \\ &= \frac{1}{6 \left(\alpha \left(\frac{1}{2} \sqrt{b_1^2 - 4c_1} - \frac{1}{2} b_1 \right) (p_1 + p_2) + 1 \right)} \left(\alpha (p_1 + p_2) - \alpha \left(b_1 - \sqrt{b_1^2 - 4c_1} \right) (p_1 + p_2) + 3 \right) \\ &\quad \times \left(\frac{1}{2} b_1 - \frac{1}{2} \sqrt{b_1^2 - 4c_1} + 1 \right) + \frac{1}{6 \left(\alpha \left(\frac{1}{2} \sqrt{b_1^2 - 4c_1} - \frac{1}{2} b_1 \right) (p_1 + p_2) + 1 \right)} (p_1 + p_2 - 1) \\ &\quad \times \left(\frac{1}{2} \alpha \left(b_1 - \sqrt{b_1^2 - 4c_1} \right) + \alpha \left(b_1 - \sqrt{b_1^2 - 4c_1} \right) (p_1 + p_2) - 3 \right) \end{aligned} \quad (10)$$

where $0 < p_1 + p_2 < 1$, $p_1, p_2 \geq 0$, and b_1 and c_1 are

$$b_1 = \frac{((p_1 + p_2) + \alpha(p_1 + p_2) - \alpha(p_1 + p_2)^2)}{(\alpha(p_1 + p_2)^2 - \alpha(p_1 + p_2))} \quad \text{and} \quad c_1 = -\frac{p_1}{(\alpha(p_1 + p_2)^2 - \alpha(p_1 + p_2))}$$

respectively.

Similarly, VaR (p_1, p_2) and $\text{CVaR}_p(T)$ for $0 < \alpha \leq 1$ are found as

$$\text{VaR}(p_1, p_2) = \left(\frac{1}{2} \left(-b_1 - \sqrt{b_1^2 - 4c_1} \right), p_1 + p_2 \right) \quad (11)$$

and

$$\begin{aligned} \text{CVaR}_p(T) &\leq \text{CVaR}_p(X) + \text{CVaR}_p(Y) \\ &= \frac{1}{6 \left(\frac{1}{2} \alpha (p_1 + p_2) \left(-b_1 - \sqrt{b_1^2 - 4c_1} \right) + 1 \right)} \left(\frac{1}{2} b_1 + \frac{1}{2} \sqrt{b_1^2 - 4c_1} + 1 \right) \\ &\quad \times \left(\alpha (p_1 + p_2) - \alpha (p_1 + p_2) \left(b_1 + \sqrt{b_1^2 - 4c_1} \right) + 3 \right) \\ &\quad + \frac{1}{6 \left(\frac{1}{2} \alpha \left(-b_1 - \sqrt{b_1^2 - 4c_1} \right) (p_1 + p_2) + 1 \right)} (p_1 + p_2 - 1) \\ &\quad \times \left(\frac{1}{2} \alpha \left(b_1 + \sqrt{b_1^2 - 4c_1} \right) + \alpha (p_1 + p_2) \left(b_1 + \sqrt{b_1^2 - 4c_1} \right) - 3 \right). \end{aligned} \quad (12)$$

Note that the condition $\alpha^2(p-1)^2 > \frac{2\alpha(p-2p_2)(1-p)}{p} - 1$ holds for $p = p_1 + p_2$, for all the above.

4. Behavior of risk functions and risk management

This section presents some numerical results and interpretations for the bivariate risk functions VaR and CVaR under the FGM copula representation of dependence. VaR and CVaR are obtained according to the BNSQP approach as explained in the previous section. The numerical results are given for different cases in the tables where VaR(Y) indicates the infimum (greatest lower bound) y value for which $F_2(y)$ value is greater than or equal to $(p_1 + p_2)$. For the case that X and Y are dependent, corresponding to a VaR(Y) value, there exists a VaR(X) which indicates the infimum x value such that the joint distribution of (X, Y) has the condition that $F(x, F_y^{-1}(p_1 + p_2)) \geq p_1$.

Table 1

$(p_1 = p_2 = q) ; \alpha = (-1 : 0.25 : 1) ; VaR(X), VaR(Y), \mathbf{u} \in (0, 1)^2$.

VaR(Y)	0.30	0.40	0.50	0.60	0.70	0.80	0.90
α	VaR(X)						
-1.00	0.6576	0.6385	0.6180	0.5963	0.5734	0.5495	0.5249
-0.75	0.6233	0.6073	0.5907	0.5734	0.5556	0.5373	0.5187
-0.50	0.5850	0.5734	0.5616	0.5495	0.5373	0.5249	0.5125
-0.25	0.5434	0.5373	0.5311	0.5249	0.5187	0.5125	0.5062
0.25	0.4566	0.4627	0.4689	0.4751	0.4813	0.4875	0.4938
0.50	0.4150	0.4266	0.4384	0.4505	0.4627	0.4751	0.4875
0.75	0.3767	0.3927	0.4093	0.4266	0.4444	0.4627	0.4813
1.00	0.3424	0.3615	0.3820	0.4037	0.4266	0.4505	0.4751

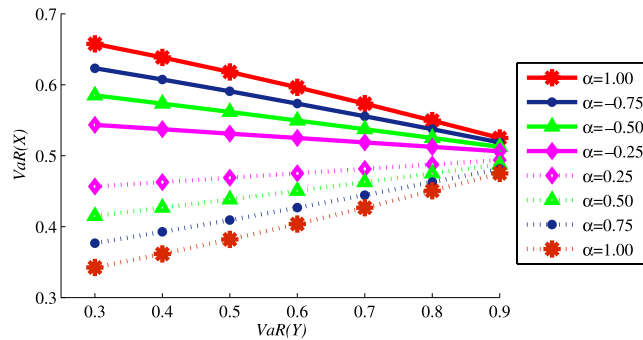


Fig. 1. $\alpha ; VaR(X), VaR(Y)$.

Table 2

$(p_1 = p_2 = q) ; \alpha = (-1 : 0.25 : 1) ; CVaR(T), \mathbf{u} \in (0, 1)^2$.

p	0.30	0.40	0.50	0.60	0.70	0.80	0.90
α	CVaR(T)						
-1.00	0.4470	0.4176	0.3861	0.3517	0.3134	0.2705	0.2217
-0.75	0.4877	0.4535	0.4175	0.3794	0.3390	0.2958	0.2496
-0.50	0.5266	0.4870	0.4463	0.4043	0.3610	0.3163	0.2700
-0.25	0.5640	0.5191	0.4737	0.4276	0.3811	0.3339	0.2864
0.25	0.6343	0.5799	0.5257	0.4719	0.4183	0.3649	0.3119
0.50	0.6666	0.6086	0.5509	0.4933	0.4361	0.3792	0.3000
0.75	0.6964	0.6359	0.5754	0.5146	0.4538	0.3930	0.3327
1.00	0.7238	0.6620	0.5991	0.5355	0.4711	0.4065	0.3420

Table 1 and Fig. 1 present some VaR values for X and Y as functions of several α dependence parameter levels and p 's:

It is seen that VaR(X) declines as VaR(Y) increase when X and Y are in negative dependence. On the other hand, for the case of positive dependence, the relationship between VaR(X) and VaR(Y) is positive. This means that VaR(X) increases as the dependence parameter tends to decrease from $\alpha = 1$ to $\alpha = -1$ for each VaR(Y) value at a $p_1 + p_2$ level. For $p_1 = p_2 = 0.15, p_1 + p_2 = 0.30$, the largest VaR(X) values occurs at $\alpha = -1$ for (X, Y) in negative dependence, and at $\alpha = 0.25$ for (X, Y) in positive dependence. Whereas, the smallest values of VaR(X) under negative and positive dependence cases, for the same VaR(Y), occurs at $\alpha = -0.25$ and $\alpha = 1$, respectively. Therefore, it can be said that the risk reserve for the portfolio represented by Y should be at least 0.6576 units (high risk) for $\alpha = -1$ and at least 0.3424 units (low risk) for $\alpha = 1$, in case of $p_1 = p_2 = 0.15$. Similar interpretations from Table 1 can be made for the other $p = p_1 + p_2$ values.

As to the behavior of CVaR values for total loss amount $T = X + Y$ with respect to p_1, p_2 and α , some numerical illustrations are shown in Table 2 and Fig. 2.

It is observed that CVaR(T) increases as $p = p_1 + p_2$ decreases for both of negative and positive dependency cases. It is also observed here that CVaR(T) values decrease as dependency between X and Y decreases from the extreme positive dependence level of $\alpha = 1$ to the extreme negative dependence level of $\alpha = -1$. The highest risk situation occurs when $p = p_1 + p_2 = 0.15 + 0.15 = 0.30$, so that the expected risk reserve about the aggregate loss T must be 0.5640 and 0.7238 units for the risks in excess of VaR in cases of $\alpha = -0.25$ and $\alpha = 1$, respectively. The lowest risk situation occurs for $p = p_1 + p_2 = 0.45 + 0.45 = 0.90$ such that the expected reserves for the risks that are in excess of VaR must be 0.2217 and 0.3119 units in cases of $\alpha = -1$ and $\alpha = 0.25$, respectively.

Further cases about some selected α and (p_1, p_2) values are summarized in Table 3.

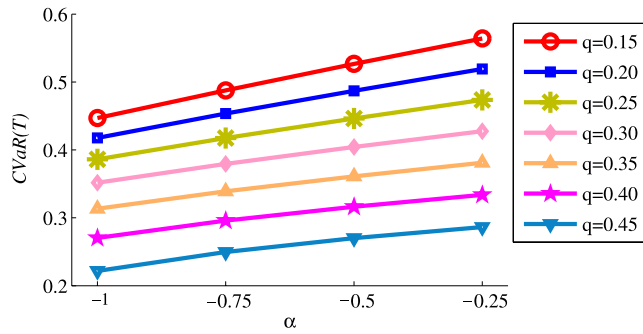


Fig. 2a. (α_-); CVaR(T).

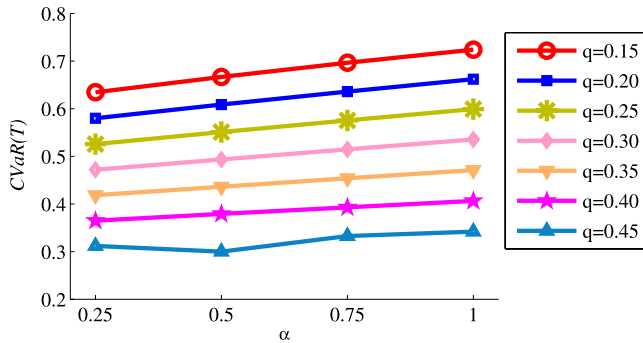


Fig. 2b. (α_+); CVaR(T).

Table 3
VaR and CVaR values for several α and (p_1, p_2) levels.

	$\alpha = -1$					$\alpha = 1$				
	VaR(X)	VaR(Y)	CVaR(X)	CVaR(Y)	CVaR(T)	VaR(X)	VaR(Y)	CVaR(X)	CVaR(Y)	CVaR(T)
$p_1 = p_2$										
0.15	0.6576	0.3000	0.1639	0.2831	0.4470	0.3424	0.3000	0.3484	0.3754	0.7238
0.45	0.5249	0.9000	0.1734	0.0483	0.2217	0.4751	0.9000	0.2914	0.0506	0.3420
$p_1 < p_2$										
0.10–0.20	0.5083	0.3000	0.2316	0.3010	0.5326	0.2151	0.3000	0.4214	0.3665	0.7829
0.20–0.30	0.5247	0.5000	0.2121	0.2204	0.4325	0.2958	0.5000	0.3881	0.2607	0.6488
0.40–0.50	0.4694	0.9000	0.1921	0.0486	0.2407	0.4201	0.9000	0.3265	0.0505	0.3770
$p_1 > p_2$										
0.20–0.10	0.7849	0.3000	0.1045	0.2662	0.3707	0.4917	0.3000	0.2654	0.3850	0.6504
0.30–0.20	0.7042	0.5000	0.1366	0.2047	0.3413	0.4753	0.5000	0.2809	0.2660	0.5469
0.50–0.40	0.5799	0.9000	0.1547	0.0480	0.2027	0.5306	0.9000	0.2571	0.0506	0.3077

At any level of dependence contained in the FGM copula, the order of quantiles for the loss amounts, p_1 and p_2 , may be conceived in three general cases: $p_1 = p_2$, $p_1 < p_2$ and $p_1 > p_2$. Precautions about these cases are expressed below for the extreme negative and positive dependence situations for the loss amount variables X and Y .

Case 1, Equal p -levels; $p_1 = p_2 = 0.15$: By the VaR measures, there is a need of reserves of 0.9576 and 0.6424 units, at least, in cases of extreme negative and extreme positive measures, respectively. According to the CVaR values, expected reserve should be 0.4470 and 0.7238 units for the risks in excess of VaR levels under the full negative and positive dependency, respectively.

Case 2, $p_1 < p_2$: When $p_1 = 0.10$ and $p_2 = 0.20$, at least 0.8083 and 0.5151 units of reserve amounts are required in cases of full negative and full positive dependency under VaR, respectively. According to the CVaR values, there is a need for the reserve amounts of 0.5326 and 0.7879 units on the average for the risks in excess of VaR levels under the extreme negative and positive dependency, respectively.

Case 3, $p_1 > p_2$: Under the VaR measure, when $p_1 = 0.20$ and $p_2 = 0.10$, reserves of at least 1.0849 and 0.7917 units must be allocated in cases of full negative and full positive dependency, respectively. According to the CVaR values, the expected reserve amounts should be 0.3707 and 0.6504 units for the risks that are in excess of VaR values under the full negative and full positive dependency, respectively.

Similar precautionary interpretations can be made for different probability and dependency levels. Considering the extreme dependency levels of $\alpha = -1$ and $\alpha = 1$, the following results are noticeable:

- (a) The highest risks for any probability level occurs in case of $\alpha = 1$, according to CVaR.
- (b) According to VaR and CVaR, the highest risks for any α occurs in cases of $p_1 > p_2$ and $p_1 < p_2$, respectively.
- (c) According to CVaR, the lowest risks for any probability level occurs in case of $\alpha = -1$.
- (d) According to VaR and CVaR, the lowest risks for any α occurs in cases of $p_1 < p_2$ and $p_1 > p_2$, respectively.

Several risk management strategies can be comprised according to the α , p_1 , p_2 values and VaR and CVaR as functions of them shown above. As the most basic risk management decisions, valuations of policy premiums for insurance portfolios or required rate of returns for financial portfolios need to be determined with due regard to eliminating or reducing possible losses. Obviously, they should be higher as the risk of excessive loss gets higher.

5. Conclusions

The north–south quantile points, NSQP, approach for the computation of the quantiles of random loss variables can be efficiently utilized in order to model bivariate dependent risks through the VaR and CVaR measures. The choice of dependence models and the definition of risks are important factors in this respect.

The FGM family of distributions or the FGM copula family are used in the current work as the dependence models. Nevertheless, the NSQP approach and the computational procedures shown here can certainly be extended to the other types of dependence models without difficulty. On the other hand, the FGM copula model in its extended forms, as referred to in the paper, can be taken under study along the same lines for more detailed analysis with a larger set of copula parameters and coefficients.

Aggregate loss amounts, as the defined risks, have been a major risk quantity of many risk modeling endeavors. That is the reason that the sum of losses arising from different but dependent portfolios has been the concern here. It is obvious that, several linear functions of aggregate loss amounts can also be tried along the same lines to accommodate various needs of applications.

The VaR and CVaR measures are considered for the NSQP approach in the presented work; but, the approach itself can be extended easily to the problem of determination and computation of other risk measures as long as they are quantile based ones.

Appendix. Supplementary data

Supplementary material related to this article can be found online at <http://dx.doi.org/10.1016/j.cam.2013.04.050>.

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