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'Level grading' a new graded algebra structure on differential polynomials: application to the classification of scalar evolution equations

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Abstract

We define a new grading, which we call the 'level grading', on the algebra of polynomials generated by the derivatives u_{k+i} over the ring $\mathcal{K}^{(k)}$ of C^∞ functions of $x, t, u, u_1, \ldots, u_k$, where $u_j = \frac{\partial^j u}{\partial x^j}$. This grading has the property that the total derivative and the integration by parts with respect to x are filtered algebra maps. In addition, if u satisfies the evolution equation $u_t = F[u]$, where F is a polynomial of order $m = k + p$ and of level p, then the total derivative with respect to t, D_t, is also a filtered algebra map. Furthermore, if the separant $\frac{\partial F}{\partial u_m}$ belongs to $\mathcal{K}^{(k)}$, then the canonical densities $\rho^{(i)}$ are polynomials of level $2i+1$ and $D_t\rho^{(i)}$ is of level $2i + 1 + m$. We define 'KdV-like' evolution equations as those equations for which all the odd canonical densities are non-trivial. We use the properties of level grading to obtain a preliminary classification of scalar evolution equations of orders $m = 7, 9, 11, 13$ up to their dependence on x, t, u, u_1 and u_2. These equations have the property that the canonical density $\rho^{(-1)}$ is $(\alpha u_3^2 + \beta u_3 + \gamma)^{1/2}$, where α, β and γ are functions of x, t, u, u_1, u_2. This form of $\rho^{(-1)}$ is shared by the essentially nonlinear class of third order equations and a new class of fifth order equations.

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1. Introduction

The classification of evolution equations has been a long-standing problem in the literature on evolution equations. The existence of higher symmetries or conserved densities and the existence of a recursion operator or the Painleve property of reduced equations have been proposed as integrability tests. Among these, we follow the 'formal symmetry' method of Mikhailov–Shabat–Sokolov [8], which is based on the remark that if the evolution equation admits a recursion operator, than its expansion as a formal series satisfies an operator equation that has to be solved in the class of local functions. This requirement gives a sequence of conserved density conditions, called the 'canonical conserved densities' $\rho^{(i)}$ and their existence is proposed as an integrability test [8].

The Korteweg–deVries (KdV) hierarchy [4] consists of the symmetries of the well-known third order KdV equation at every odd order. This hierarchy is characterized by the existence of conserved densities that are quadratic in the highest derivative at each order. In fact the canonical even and odd densities $\rho^{(2i)}$ and $\rho^{(2i-1)}$ have the same order, but it is well known that even canonical densities are trivial while the odd ones are non-trivial [8]. At fifth order, there are two basic hierarchies that start at this order; these are the Sawada–Kotera [15] and Kaup [7] hierarchies, which are derived from a third order Lax operator, and their symmetries give integrable equations at odd orders that are not divisible by 3. The recursion operators of these hierarchies have order 6 and generate the flows of orders $3k + 1$ and $3k + 2$ for each hierarchy [3]. These equations are characterized by the triviality of the canonical densities of orders divisible by 3.

The non-existence of integrable hierarchies starting at higher orders was studied by Wang and Sanders, who proved that scale homogeneous scalar integrable evolution equations of orders greater than or equal to 7 are symmetries of lower order equations [13]. In subsequent papers, these results were extended to the cases where negative powers are involved [14] but the case where F is arbitrary remained open.

The general case where the functional form of F is arbitrary was studied in the [2] and [10]. The first result in this direction was obtained in [2], where the canonical densities $\rho^{(i)}$, $i = 1, 2, 3, 4$ were computed for evolution equations of arbitrary order m. It was first proven that, up to total derivatives, conserved densities of order $n > m$ are at most quadratic in the highest derivative. Then assuming that an evolution equation $u_t = F(x, t, u, \dots, u_m)$ admits a conserved density $\rho^{(1)} = Pu_{m+1}^2 + Qu_{m+1} + R$, where P, Q, R are functions independent of u_{m+1}, it has been shown that for $m \geqslant 7$, $PF_{mm} = 0$, where $F_{mm} = \frac{\partial^2 F}{\partial u_m^2}$ [2]. Finally, it was shown that the coefficient P in the canonical density $\rho^{(1)}$, has the form $P = F_m$, where $F_m = \frac{\partial F}{\partial u_m}$ [2], hence it was concluded that evolution equations of order $m \geqslant 7$ that admit the canonical density $\rho^{(1)}$ are quasi-linear. In [10], the same scheme was applied to quasi-linear equations and it was proven that if the canonical densities $\rho^{(i)}$, $i = 1, 2, 3$ are conserved then the evolution equation has to be polynomial in the derivatives u_{m-1} and u_{m-2}. The existence of essentially nonlinear third order equations is well known [5, 6]. Recently we have shown that there are also candidates for integrable fifth order equations that are not quasi-linear (work in progress).

In these derivations we have observed that the partial differential equations leading to a classification had a hierarchical structure with respect to the orders of the derivatives of the unknown function u. This observation led to the definition of a graded algebra structure [9], the 'level grading', which is the main subject of this paper. Roughly speaking, if a function f depends on the derivatives of u up to order k, then its jth derivative $D^j f$ depending on u_{k+i}, $i \leqslant j$ has a certain homogeneity reflecting the order of differentiation j. For example, $D^3 f$ is a linear combination of $u_{k+3}, u_{k+2}u_{k+1}, u_{k+1}^3, u_{k+2}, u_{k+1}^2, u_{k+1}$ with coefficients depending on functions of u_i, $i \leqslant k$. According to definition 3.2, the first three terms are of level 3, the next two are of level 2 and the last one is of level 1. We have called this graded algebra structure as 'level grading' above the 'base level k'. The crucial property of the level grading is its invariance under integrations by parts which allows us to perform conserved density computations for each level separately, starting from the higher levels that give simpler equations.

We introduce the notation and the terminology in section 2. Section 3 is devoted to the description of level grading and the proofs of the properties that will be used in conserved density computations. In section 4, we prove that, essentially, the canonical densities of level homogeneous equations are also level homogeneous and we apply this to the classification of evolution equation of orders $m = 7, 9, 11, 13$ admitting non-trivial conserved densities at all orders. We obtain the dependence of these equations on u_k for $k \geqslant 3$, i.e, up to their dependences in x, t, u, u_1 and u_2. We have seen that at all orders the candidates for integrable equations are characterized by the common form of

$$\rho^{(-1)} = \left(\frac{\partial F}{\partial u_m}\right)^{-1/m} = \left(\alpha u_3^2 + \beta u_3 + \gamma\right)^{1/2},$$

where α, β and γ are functions of x, t, u, u_1, u_2. As discussed in section 5, this form of $\rho^{(-1)}$ is, up to a function of x, t, u, u_1, u_2, the same as the form of $\rho^{(-1)}$ for the essentially nonlinear class of third order equations. Furthermore, fifth order KdV-like equations with a nonconstant separant, recently classified again up to their dependences on u, u_1 and u_2 [12], and a non-quasi-linear candidate of integrable fifth order equation, admit the same canonical density. We discuss these relations and possible directions for future work in section 5.

2. Notation and terminology

Let $u = u(x, t)$, where x and t are spatial and temporal variables respectively. A function φ of x, t, u and the derivatives of u up to a fixed but finite order, denoted by $\varphi[u]$, will be called a 'differential function' [11]. We shall assume that φ has partial derivatives of all orders. For notational convenience, we shall denote indices by subscripts or superscripts in parentheses such as in $\alpha_{(i)}$ or $\rho^{(i)}$ and reserve subscripts without parentheses for partial derivatives, i.e., for $u = u(x, t)$,

$$u_0 = u, \quad u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x}, \quad u_k = \frac{\partial^k u}{\partial x^k}$$

and for $\varphi = \varphi(x, t, u, u_1, \ldots, u_n)$,

$$\varphi_t = \frac{\partial \varphi}{\partial t}, \quad \varphi_x = \frac{\partial \varphi}{\partial x}, \quad \varphi_k = \frac{\partial \varphi}{\partial u_k}, \quad \varphi_{k,j} = \frac{\partial^2 \varphi}{\partial u_k \partial u_j}.$$

We will use either the explicit form or the short form as appropriate.

Algebraic structures such as rings or modules will be denoted by calligraphic letters such as $\mathcal{K}$ or $\mathcal{M}$. For such symbols, to simplify the notation subscripts will always denote indices.

If φ is a differential function, the total derivative with respect to x is denoted by $D\varphi$ and it is given by

$$D\varphi = \sum_{i=0}^{n} \varphi_i u_{i+1} + \varphi_x. \tag{2.1}$$

Higher order derivatives can be computed by applying the binomial formula as given below

$$D^k\varphi = \sum_{i=0}^{n} \left[\sum_{j=0}^{k-1} \binom{k-1}{j} (D^j\varphi_i) u_{i+k-j}\right] + D^{k-1}\varphi_x. \tag{2.2}$$

If $u_t = F[u]$, then the total derivative of φ with respect to t is given by

$$D_t\varphi = \sum_{i=0}^{n} \varphi_i D^i F + \varphi_t. \tag{2.3}$$

The 'order' of a differential function $\varphi[u]$, denoted by $ord(\varphi) = n$ is the order of the highest derivative of u present in $\varphi[u]$. The total derivative with respect to x increases the order by 1. From the expression of the total derivative with respect to t given by (2.3) it can be seen that if u satisfies an evolution equation of order m, D_t increases the order by m. Equalities up to total derivatives with respect to x will be denoted by $\cong$, i.e.,

$$\varphi \cong \psi \quad \text{if and only if} \quad \varphi = \psi + D\eta.$$

The effect of the integration by parts on monomials is described as follows. Note that if a monomial is nonlinear in its highest derivative we cannot integrate by parts and reduce the order. Let $k < p_1 < p_2 < \ldots < p_l < s-1$ and φ be a function of $x, t, u, u_1, \ldots, u_k$. Then a product of φ and powers of u_{p_i}, denoted by $u_{p_i}^{a_i}$ can be integrated by parts provided that the power of the highest derivative is 1, as shown below

$$\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s \cong -D\big(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\big) u_{s-1},$$
$$\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_{s-1}^{p} u_s \cong -frac1p + 1D\big(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\big) u_{s-1}^{p+1}.$$

The integrations by parts are repeated successively until one encounters a 'non-integrable monomial' of the following form:

$$u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s^{p}, \quad p > 1.$$

The order of a differential monomial is not invariant under integration by parts, but we will show in the next section that its level decreases by 1 under integration by parts [9]. This will be the rationale and the main advantage of using the level grading.

3. The ring of polynomials and 'level grading'

The scaling symmetry and scale homogeneity are well-known properties of polynomial integrable equations. We recall that scaling symmetry is the invariance of an equation under the transformation $u \to \lambda^a u$, $x \to \lambda^{-1} x$, $t \to \lambda^{-b} t$. If $a = 0$, then scale invariant quantities may be non-polynomial and the scaling weight is just the order of differentiation. In the early stages of our investigations we have noticed that if F is a function of the derivatives of u up to order k, and we differentiate F j times, the resulting expressions are polynomial in $u_{k+1}, \ldots, u_{k+j}$. Furthermore, the sum of the order of differentiations exceeding k has some type of invariance. This remark led us to the definition of 'level grading' as a generalization of the scaling symmetry for the case $a = 0$, as a graded algebra structure.

We consider the ring of functions of $x, t, u, \ldots, u_k$ the algebra generated by the derivatives $u_{k+1}, \ldots$. This set up is given a graded algebra structure as described below. Let $\mathcal{M}$ be an algebra over a ring $\mathcal{K}$. If we can write $\mathcal{M} = \oplus_{i\in\mathbb{N}} \mathcal{M}_i$, as a direct sum of its submodules $\mathcal{M}_i$, with the property that $\mathcal{M}_i \mathcal{M}_j \subseteq \mathcal{M}_{i+j}$, then we have a 'graded algebra' structure on $\mathcal{M}$. For example if $\mathcal{K} = \mathcal{R}$ and $\mathcal{M}$ is the algebra of polynomials in x and y, then the $\mathcal{M}_i$ may be chosen as the submodule consisting of homogeneous polynomials of degree i. In the same example, we may also consider the submodules consisting of polynomials of degree i (not necessarily homogeneous) that we denote by $\tilde{\mathcal{M}}_i$. Then $\tilde{\mathcal{M}}_i$ is the direct sum of the submodules $\mathcal{M}_j$, j ranging from zero to i. It follows that the full algebra $\mathcal{M}$ can be written as a union of the submodules $\tilde{\mathcal{M}}_j$. This structure is called a 'filtered algebra'. The formal definitions are given below.

Definition 3.1. *Let $\mathcal{K}$ be a ring, and $\mathcal{M}$ be an algebra over $\mathcal{K}$. $\mathcal{M}$ is called a 'graded algebra', if there exists a decomposition of $\mathcal{M}$*

$$\mathcal{M} = \oplus_{i\in\mathbb{N}} \mathcal{M}_i \quad \text{and} \quad \mathcal{M}_i \mathcal{M}_j \subseteq \mathcal{M}_{i+j}$$

where the $\mathcal{M}_i$, $i \in \mathbb{N}$ are submodules of $\mathcal{M}$.

Given a graded algebra $\mathcal{M}$, we can obtain an associated 'filtered algebra' $\tilde{\mathcal{M}}$ [16], by defining $\tilde{\mathcal{M}}_i = \oplus_{j=0}^{i}\mathcal{M}_j$. Then we have

$$\tilde{\mathcal{M}} = \cup_{i\in\mathbb{N}}\tilde{\mathcal{M}}_i.$$

If the algebra $\mathcal{M}$ is characterized by a set of generators, then the submodules $\mathcal{M}_i$ can also be characterized similarly.

We now give the setup for the definition of the level grading. Let $\mathcal{K}^{(k)}$ be the ring of C^∞ functions of $x, t, u, \ldots, u_k$, and $\mathcal{M}^{(k)}$ be the polynomial algebra over $\mathcal{K}^{(k)}$ generated by the set

$$\mathcal{S}^{(k)} = \{u_{k+1}, u_{k+2}, \ldots\}.$$

A monomial in $\mathcal{M}^{(k)}$ is a product of a finite number of elements of $\mathcal{S}^{(k)}$. We define the 'level above k' of a monomial as follows.

Definition 3.2. *Let $\mu = u_{k+j_1}^{a_1}u_{k+j_2}^{a_2}\ldots u_{k+j_n}^{a_n}$ be a monomial in $\mathcal{M}^{(k)}$. The level of μ above k is defined by*

$$lev_k(\mu) = a_1 j_1 + a_2 j_2 + \cdots + a_n j_n.$$

The level of the differential operator D is defined to be 1. *The level of a pseudo-differential operator φD^j, $j \in Z$, is thus $lev_k(\varphi D^j) = lev_k(\varphi) + j$.*

Let $\mathcal{B}_p^{(k)}$ be the set of monomials of level p above the base level k. Then, the module generated by the set $\mathcal{B}_p^{(k)}$ is denoted by $\mathcal{M}_p^{(k)}$; $\mathcal{M}_0^{(k)} = \mathcal{K}^{(k)}$ and $\tilde{\mathcal{M}}_p^{(k)} = \bigoplus_{j=0}^{p}\mathcal{M}_j^{(k)}$.

It can be seen that for any two monomials μ and $\tilde{\mu}$,

$$lev_k(\mu\tilde{\mu}) = lev_k(\mu) + lev_k(\tilde{\mu}),$$

hence the 'level above k' gives a graded algebra structure to $\mathcal{M}^{(k)}$. The elements of $\tilde{\mathcal{M}}_p^{(k)}$ are called 'polynomials of level p' (in analogy with polynomials of a given order). If a is a polynomial in $\tilde{\mathcal{M}}_p^{(k)}$, $a = \sum_{j=0}^{p} a_j$ where $a_j \in \mathcal{M}_j^{(k)}$, is called a 'homogeneous component' of a of level j above k. In particular the image of a under the natural projection

$$\pi : \tilde{\mathcal{M}}_p^{(k)} \to \mathcal{M}_p^{(k)}$$

denoted by $\pi(a)$ is called the 'top level part of a'.

We will now present certain results that demonstrate the importance of the level grading. We will prove that partial derivatives with respect to u_i, total derivatives with respect to x, total derivatives with respect to t and the integration by parts, hence the conserved density conditions, are filtered algebra maps.

Proposition 3.1. *Let φ be a polynomial of level $p > 0$ above k.*

(i) If $\frac{\partial\varphi}{\partial u_{k+j}} \neq 0$, then the partial derivative of φ with respect to u_{k+j} decreases the level by j.

(ii) The total derivative with respect to x, D is a filtered algebra map $\mathcal{M}_p^{(k)} \to \mathcal{M}_{p+1}^{(k)} \bigoplus \mathcal{M}_p^{(k)}$, i.e, increases the level by at most 1.

Proof.

(i) If φ is level homogeneous above k of level p, i.e, in $\mathcal{M}_p^{(k)}$, then it is a linear combination of monomials

$$\mathcal{B}_p^{(k)} = \left\{u_{k+p}, u_{k+p-1}u_{k+1}, u_{k+p-2}u_{k+2}, u_{k+p-2}u_{k+1}^2, \ldots, u_{k+1}^p\right\}.$$

Clearly if $\frac{\partial\varphi}{\partial u_{k+j}} \neq 0$, the effect of differentiation with respect to $\frac{\partial\varphi}{\partial u_{k+j}}$ decreases the level by j.

(ii) Let φ be as above and A be a function of $x, t, u, \ldots, u_k$, i.e, in $\mathcal{K}^{(k)}$. The effect of the total derivative operator D on $A\varphi$ is $D(A\varphi) = DA\varphi + A\, D\varphi$. If φ has level p, then $D\varphi$ has level exactly $p+1$ while $DA = \frac{\partial A}{\partial u_k}u_{k+1} + \frac{\partial A}{\partial u_{k-1}}u_k + \cdots + \frac{\partial A}{\partial u}u_1$, hence has level at most 1. Thus the action of D on $\mathcal{K}^{(k)}$ increases the level by at most 1, while its action on the generators of $\mathcal{M}_p^{(k)}$ increases the level by exactly 1. □

We now study the effect of integration by parts. The subset of the generating set $\mathcal{S}_p^{(k)}$ of the module $\mathcal{M}_p^{(k)}$, consisting of the monomials that are nonlinear in the highest derivative and the submodule that it generates are denoted by $\bar{\mathcal{S}}_p^{(k)}$ and $\bar{\mathcal{M}}_p^{(k)}$, respectively. If a monomial is nonlinear in its highest derivative it cannot be integrated. If it is linear, one can proceed with the integrations by parts until a term that is nonlinear in its highest derivative is encountered. By virtue of the propositions above, these operations will be filtered algebra maps.

Proposition 3.2. *Let α be a polynomial in $\tilde{\mathcal{M}}_p^{(k)}$. Then $\int\alpha = \beta - \int\gamma$ where β belongs to $\tilde{\mathcal{M}}_{p-1}^{(k)}$ and γ belongs to $\tilde{\mathcal{M}}_p^{(k)}$.*

Proof. Let $\mu = u_{k+i_1}^{a_1}u_{k+i_2}^{a_2}\ldots u_{k+i_j}^{a_j}$, $i_1 > i_2 > \cdots > i_j$, $i_1a_1 + i_2a_2 + \cdots + i_ja_j = p$. We have the following three mutually exclusive cases.

(i) When $a_1 > 1$, the monomial is not a total derivative and $\mu \in \bar{\mathcal{S}}_{p,n}^{(k)}$. We cannot proceed with integration by parts.

(ii) When $a = 1$ the term $\varphi\mu$ where $\varphi \in \mathcal{K}^{(k)}$ can be integrated. For $i_2 < i_1 - 1$

$$\begin{aligned}\int\varphi\mu &= \int\varphi u_{k+i_1}^{a_1}u_{k+i_2}^{a_2}\ldots u_{k+i_j}^{a_j}\\ &= \varphi u_{k+i_1-1}^{a_1}u_{k+i_2}^{a_2}\ldots u_{k+i_j}^{a_j} - \int u_{k+i_1-1}^{a_1}D\big(\varphi u_{k+i_2}^{a_2}\ldots u_{k+i_j}^{a_j}\big).\end{aligned}$$

(iii) When $a = 1$ but $i_2 = i_1 - 1$ then

$$\begin{aligned}\int\varphi\mu &= \int\varphi u_{k+i_1}^{a_1}u_{k+i_1-1}^{a_2}u_{k+i_3}^{a_3}\ldots u_{k+i_j}^{a_j}\\ &= \frac{u_{k+i_1-1}^{a_2+1}}{a_2+1}u_{k+i_3}^{a_3}\ldots u_{k+i_j}^{a_j} - \int\frac{u_{k+i_1-1}^{a_2+1}}{a_2+1}D\big(\varphi u_{k+i_3}^{a_3}\ldots u_{k+i_j}^{a_j}\big).\end{aligned}$$

In (i) and (ii), the level of the term that has been integrated decreases by 1 while the terms under the integral sign have levels p or lower. □

We will now give an example that illustrates the effect of total derivatives and integration by parts.

Example 3.1. Let $R = \varphi u_8 + \psi u_7u_6 + \eta u_6^3$, where $\varphi, \psi, \eta \in \mathcal{K}^{(5)}$ be a polynomial in $\mathcal{M}_3^{(5)}$. It can easily be seen that DR is a sum of polynomials in $\mathcal{M}_4^{(5)}$ and $\mathcal{M}_3^{(5)}$

$$\begin{aligned}DR = {}&\big[\varphi u_9 + (\varphi_5 + \psi)u_8u_6 + \psi u_7^2 + (\psi_5 + 3\eta)u_7u_6^2 + \eta_5u_6^4\big]\\ &+ \big[(\varphi_4u_5 + \cdots + \varphi_x)u_8 + (\psi_4u_5 + \cdots + \psi_x)u_7u_6 + (\eta_4u_5 + \cdots + \eta_x)u_6^3\big]\end{aligned}$$

where the first and second groups of terms in the brackets belong to $\mathcal{M}_4^{(5)}$ and $\mathcal{M}_3^{(5)}$ respectively. Note that the projection to $\mathcal{M}_4^{(5)}$ depends only on the derivatives with respect to u_5. In order to write down compactly the effect of integration by parts, we define the operator D_0 by $D_0\varphi = D\varphi - \varphi_ku_{k+1}$ to denote the part of $D\varphi$ depending on lower order derivatives. It follows that $D^2\varphi = \varphi_k u_{k+2} + \varphi_{k,k}u_{k+1}^2 + (D_0\varphi)_k u_{k+1} + D_0(D_0\varphi)$ is a sum of level 2, level 1 and level 0 terms. The integration by parts of R gives

$$\int R dx = \left[\varphi u_7 + \frac{1}{2}(\psi - \varphi_5)u_6^2 - D_0\varphi u_6\right] + \int \left[\frac{1}{2}\varphi_{5,5} - \frac{1}{2}\psi_5 - \eta\right] u_6^3 + \left[\frac{1}{2}D_0\varphi_5 - \frac{1}{2}D_0\psi + (D_0\varphi)_5\right] u_6^2 + D_0(D_0\varphi)u_6,$$

where the first term in the bracket belongs to $\mathcal{M}_2^{(5)}$ while the integrand belongs to $\mathcal{M}_3^{(5)}$.

Now we deal with time derivatives. Given $u_t = F(x, t, u, ..., u_m)$ where F is of order m, if $\rho = \rho(x, t, u, \ldots, u_n)$ is a differential polynomial of order n, then clearly, $D_t\rho$ is of order $n + m$. A similar result holds for level grading.

Proposition 3.3. *Let $u_t = F[u]$, where F is a differential polynomial of order m and of level q above the base level k. Then D_t is a filtered algebra map $\tilde{\mathcal{M}}_p^{(k)} \to \tilde{\mathcal{M}}_{p+q}^{(k)}$.*

Proof. Let ρ be a differential polynomial of order n and of level p above the base level k. Then

$$D_t\rho = \rho_t + \sum_{i=0}^{k} \rho_i D^i F + \sum_{j=1}^{n-k} \rho_{k+j} D^{k+j} F.$$

Note that ρ_t has level at most p. Similarly, the level of ρ_i for $i \leqslant k$, is at most p hence each of the terms in the first sum are of levels at most $p + q + i$, and the sum has level at most $p + q + k$. In the second sum, ρ_{k+j} has level $p - j$, hence the level of $\rho_{k+j}D^{k+j}F$ is $(p - j) + (k + j) + q = p + q + k$. □

We will now prove a very useful proposition stating that the top level depends only on the dependence of the coefficients on u_k.

Proposition 3.4. *Let ρ be a differential polynomial in $\tilde{\mathcal{M}}_p^{(k)}$. Then the projection $\pi(D^j\rho)$ depends only on the dependence of the coefficients in ρ on u_k.*

Proof. Let $\rho = \sum_i \varphi_i P_i$ where $\varphi_i \in \mathcal{K}^{(k)}$ and $P_i \in \mathcal{M}_p^{(k)}$. Without loss of generality we may assume that $\rho = \varphi P$ where φ is of level zero and $P = u_{i_1}^{a_1} \ldots u_{i_n}^{a_n}$.

$$\begin{aligned} D\rho = (D\varphi)P + \varphi(DP) &= \left[\varphi_x + \sum_{i=0}^{k-1} \varphi_i u_{i+1} + \varphi_k u_{k+1}\right] P + \varphi DP \\ &= \underbrace{\left[\varphi_x + \sum_{i=0}^{k-1} \varphi_i u_{i+1}\right] P}_{\mathcal{M}_p^{(k)}} + \underbrace{\varphi_k u_{k+1} P + \varphi DP}_{\mathcal{M}_{p+1}^{(k)}}. \end{aligned} \tag{3.1}$$

It follows that the projection $\pi(D^j\rho)$ is independent of φ_j for $j < k$ and independent of φ_x. It follows that in the conserved density computations, if ρ and $F[u]$ are level homogeneous, then ρ_t up to total derivatives is also level homogeneous. □

We will use the level grading structure in the conserved density computations for the classification of evolution equations. A crucial implication of the level grading is proposition 4.2, which states that the canonical densities of level homogeneous equations are level homogeneous. Thus, we may write the form of the canonical densities without the burden of computing them explicitly. In the next section we shall obtain the classification of evolution equations of orders $m = 7, 9, 11, 13$, up to their dependences on u, u_1 and u_2, under the assumption that there are non-trivial conserved densities of all orders.

4. Preliminary classification of KdV-like evolution equations of orders $m = 7, 9, 11, 13$

In this section we will study the classification of scalar evolution equations of orders $m = 7, 9, 11, 13$ using the level grading structure. In [10] we have shown that if $F = u_t$ is integrable in the sense of admitting a formal symmetry, then it is of the form

$$u_t = F = a^m u_m + Bu_{m-1}u_{m-2} + Cu_{m-2}^3 + Eu_{m-1} + Gu_{m-2}^2 + Hu_{m-2} + K, \tag{4.1}$$

where a, B, C, E, G, H and K are functions of $x, t, u, u_i, i \leqslant m-3$, i.e. they belong to $\mathcal{K}^{m-3}$. Note that F is a sum of level homogeneous terms of levels 3, 2 1 and 0 above the base level $k = m - 3$. We shall assume that the conserved densities $\rho^{(-1)}$, $\rho^{(1)}$ and $\rho^{(3)}$ are non-trivial. We recall that a non-trivial conserved density should be nonlinear in the highest derivative. We have checked that $\rho^{(1)}$ is always non-trivial, but the cases where at least one of the canonical densities $\rho^{(-1)}$, $\rho^{(3)}$ is trivial is not treated in this paper. We characterize such equations as 'KdV-like'. It is well known that the canonical densities of even order are trivial, hence we give the definition as below.

Definition 4.1. *An evolution equation $u_t = F[u]$ is called 'KdV-like' if the sequence of odd numbered canonical densities is non-trivial.*

When we substitute the form of F given by (4.1) in the canonical conserved densities $\rho^{(i)}$ and we integrate by parts we can see that the canonical densities are of the form given below

$$\begin{aligned}
\rho^{(-1)} &= a^{-1}\\
\rho^{(1)} &\cong P^{(1)}u_{m-2}^2 + Q^{(1)}u_{m-2} + R^{(1)}\\
\rho^{(3)} &\cong P^{(3)}u_{m-1}^2 + Q^{(3)}u_{m-2}^4 + R^{(3)}u_{m-2}^3 + S^{(3)}u_{m-2}^2 + T^{(3)}u_{m-2} + V^{(3)}.
\end{aligned} \tag{4.2}$$

where the coefficients are differential functions belonging to $\mathcal{K}^{(k)}$.

4.1. Level homogeneity of the canonical densities

We will now prove a proposition stating, essentially, that if the evolution equation is level homogeneous above a base level k and its separant belongs to $\mathcal{K}^{(m-3)}$, then its canonical densities are also level homogeneous.

Let R be a recursion operator of order n for the evolution equation $u_t = F$. We can express R as a formal series in inverse powers of D, i.e,

$$R = R_{(n)}D^n + R_{(n-1)}D^{n-1} + \cdots + R_{(1)}D + R_{(0)} + R_{(-1)}D^{-1} + R_{(-2)}D^{-2} + \dots.$$

R satisfies the operator equation

$$R_t + [R, F_*] = 0, \tag{4.3}$$

where F_* is the Frechet derivative of F. We will first prove the following proposition on the relation of the levels of F and of the coefficients $R_{(j)}$.

Proposition 4.1. *Let R be a recursion operator of order n for the evolution equation $u_t = F$, where F is of order $m = k + p$. If F is in $\tilde{\mathcal{M}}_p^{(k)}$, and if its separant belongs to $\mathcal{K}^{(k)}$, then the coefficient of D^j in R, $R_{(j)}$, is in $\tilde{\mathcal{M}}_{n-j}^{(k)}$.*

Proof. Let R be a recursion operator of order n. The operator equation (4.3) is of order $n+m-1$ and gives an infinite sequence of partial differential equations. Note that the commutator $[R, F_*]$ is of order $n+m-1$ while R_t is of order at most n. Thus the first m equations are independent of the time derivatives of the functions $R_{(j)}$. In finding R, we solve these equations sequentially, starting from the top order. The coefficient of D^{n+m-1} gives

$$mDR_{(n)}F_m = nDF_mR_{(n)}. \tag{4.4}$$

Using the fact that the separant belongs to the base, we solve (4.4) as

$$R_{(n)} = R_{(n,o)}(F_m)^{n/m} \tag{4.5}$$

where $DR_{(n,o)} = 0$. If we introduce the notation $a = F_m^{1/m}$, and take $R_{(n,o)} = 1$ (assuming no explicit time dependence), we have $R_{(n)} = a^n$, which belongs to $\mathcal{K}^{(k)}$. Then, the coefficient of D^{n+m-2} is an expression of the form

$$DR_{(n-1)} = \frac{n-1}{m}\frac{DF_m}{F_m}R_{(n-1)} + G$$

or

$$DR_{(n-1)} = (n-1)\frac{Da}{a}R_{(n-1)} + G \tag{4.6}$$

where G depends on the F_i and $R_{(n)}$ only. Equation (4.6) is a first order ordinary differential equation that can be integrated easily as

$$R_{(n-1)} = R_{(n-1,o)}a^{n-1} + a^{n-1}\int a^{-n+1}G, \tag{4.7}$$

after making use of the fact that $F_m = a^m$. Here also $DR_{(n-1,o)} = 0$ and we take this integration constant as $R_{(n-1,o)} = 0$. This integrand belongs to $\tilde{\mathcal{M}}_1^{(k)}$, provided that a belongs to $\mathcal{K}^{(k)}$. Iteratively, for each $R_{(j)}$, one obtains first order differential equations whose right-hand sides are level homogeneous. Thus provided that a belongs to $\mathcal{K}^{(k)}$ the solutions belong to $\tilde{\mathcal{M}}_{n-j}^{(k)}$. □

It is well known that the nth root of R and its powers also satisfy the same operator equation, hence they are also recursion operators. Thus by taking roots and powers we can obtain recursion operators of all orders. The canonical density $\rho^{(j)}$ is defined to be the coefficient of D^{-1} in a recursion operator of order j. We now prove the following proposition.

Proposition 4.2. *Let $u_t = F$ be an evolution equation of order $m = k + p$. Assume that F is in $\tilde{\mathcal{M}}_p^{(k)}$ and its separant belongs to $\mathcal{K}^{(k)}$. Then the canonical density $\rho^{(j)}$ belongs to $\tilde{\mathcal{M}}_{j+1}^{(k)}$.*

Proof. In any commutator of pseudo-differential operators, the coefficient of the term D^{-1} is a total derivative (Adler's theorem) [1], hence the coefficient of D^{-1} in R should be a conserved density. The coefficient of D^{-1} in a recursion operator of order j is denoted as $\rho^{(j)}$. Since the coefficient of D^j belongs to $\mathcal{K}^{(k)}$, the level of a recursion operator of order j is just j. By the previous proposition the coefficient of D^{-1} belongs to $\tilde{\mathcal{M}}_{j-(-1)}^{(k)}$, i.e., $\tilde{\mathcal{M}}_{j+1}^{(k)}$. □

4.2. Classification up to the dependences on x, t, u, u_1, u_2

We will outline below the steps leading to the classification of the top level parts of the integrable evolution equations of odd orders $m = 7, \ldots, 13$ for scalar evolution equations admitting non-trivial conserved densities $\rho^{(-1)}$, $\rho^{(1)}$, $\rho^{(3)}$. In particular the non-triviality of $\rho^{(3)}$ will be crucial.

Recall that conserved densities can be given up to total derivatives. Thus a generic conserved density of order $k + j$ is a polynomial in the monomials $\mathcal{M}_{2j}^{(k)}$, as given in appendix B. Over any base level $k \geqslant 3$, the top level parts of these conserved densities will be of the same form:

$$\begin{aligned}
\rho^{(-1)} &= a^{-1}, \\
\rho^{(1)} &\cong P^{(1)} u_{k+1}^2, \\
\rho^{(3)} &\cong P^{(3)} u_{k+2}^2 + Q^{(3)} u_{k+1}^4.
\end{aligned} \tag{4.8}$$

We outline below the solution procedure for $m = 7, \ldots 13$. The generating sets for the modules $\mathcal{M}_j^{(k)}$ that are referred to below are given in appendix B. At each step, the level part of F is denoted by the projection $\pi(F)$ (see proposition 3.4).

Step 1. $k = m-3$, $m = 7, 9, 11, 13$. Note that (4.1) is a polynomial of level 3 above the base level $k = m-3$. Hence it belongs to $\tilde{\mathcal{M}}_3^{(m-3)}$, i.e, it is a polynomial in the generating sets of $\mathcal{M}_j^{(m-3)}$, $j = 3, 2, 1, 0$, with coefficients in $\mathcal{K}^{(m-3)}$. The top level part of F is

$$\pi(F) = a^{k+3} u_{k+3} + B u_{k+2} u_{k+1} + C u_{k+1}^3, \qquad (m \geqslant 7). \tag{4.9}$$

For $m = 7, 9, 11, 13$, we compute the conserved density conditions (4.8), integrate by parts and collect the top level terms. The solutions of these equations determine the coefficients B and C as functions of a and the derivatives of a with respect to u_k of various orders and finally we find that a is independent of u_k. It follows that the top level part of F consists of the linear term $a^m u_m$ only. Since a is independent of u_k, the linear term is of level 4 above the base level $k = m-4$. Then, we use the dependence of a on u_{m-4} to prove that F is a polynomial in u_k and its level is at most 4 above the base level $m-4$. It follows that F belongs to $\tilde{\mathcal{M}}_4^{(m-4)}$.

Step 2. $k = m-4$, $m = 7, 9, 11, 13$. The generic form of the evolution equation for $k = m-4$ is a polynomial in the generators of $\mathcal{M}_j^{(m-4)}$, for $j = 4, \ldots, 0$, with coefficients in $\mathcal{K}^{(k)}$. The top level part is

$$\pi(F) = a^{k+4} u_{k+4} + B u_{k+3} u_{k+1} + C u_{k+2}^2 + E u_{k+2} u_{k+1}^2 + G u_{k+1}^4, \quad (m \geqslant 7). \tag{4.10}$$

The conserved densities have the same form (4.8). Computing the top level parts of the conserved density conditions and integrating by parts we obtain systems of equations for the coefficients in the top level part of F. For $m = 7$, $k = m-4 = 3$ and we see that a satisfies the third order differential equation

$$a_{3,3,3} - 9 a_{3,3} a_3 a^{-1} + 12 a_3^3 a^{-2} = 0. \tag{4.11}$$

For $m > 7$ we obtain $a_{m-4} = 0$ and the top level part of F reduces to the linear term $a^m u_m$, which is of level 5 above $k = m-5$. As in the first step, we use the dependence of a on u_{m-5} to prove that F is polynomial in u_k and of level at most 5 above the base level $m-5$. It follows that F belongs to $\tilde{\mathcal{M}}_5^{(m-5)}$.

Step 3. $k = m-5$, $m = 9, 11, 13$. For $k = m-5$ the generic form of the evolution equation is a polynomial in the generators of $\mathcal{M}_j^{(m-5)}$, $j = 5, \ldots, 0$. The top level part is

$$\begin{aligned}
\pi(F) = {} & a^{k+5} u_{k+5} + B u_{k+4} u_{k+1} + C u_{k+3} u_{k+2} + E u_{k+3} u_{k+1}^2 \\
& + F u_{k+2}^2 u_{k+1} + G u_{k+2} u_{k+1}^3 + H u_{k+1}^5, \quad (m \geqslant 9).
\end{aligned} \tag{4.12}$$

We repeat the computations as described above to obtain $a_{m-5} = 0$ and to prove that for $m = 9, 11, 13$, F belongs to is $\tilde{\mathcal{M}}_6^{(m-6)}$.

Step 4. $k = m-6$, $m = 9, 11, 13$. For $k = m-6$ the generic form of the evolution equation is a polynomial in the generators of $\mathcal{M}_j^{(m-6)}$, $j = 6, \ldots, 0$. The top level part is

$$\begin{aligned}
\pi(F) = {} & a^{k+6} u_{k+6} + B u_{k+5} u_{k+1} + C u_{k+4} u_{k+2} + E u_{k+4} u_{k+1}^2 + G u_{k+3}^2 \\
& + H u_{k+3} u_{k+2} u_{k+1} + K u_{k+3} u_{k+1}^3 + L u_{k+2}^3 + M u_{k+2}^2 u_{k+1}^2 + N u_{k+2} u_{k+1}^4 \\
& + P u_{k+1}^6, \quad m \geqslant 9.
\end{aligned} \tag{4.13}$$

We repeat the computations as described above and for $m = 9$, surprisingly we find that a satisfies the same equation as (4.11). For $m > 9$ we find that $a_{m-6} = 0$ and we show that F is polynomial in u_k and belongs to $\tilde{\mathcal{M}}_7^{(m-7)}$.

Step 5. $k = m-7$, $m = 11, 13$. At this step, F is a polynomial in the generators of $\mathcal{M}_j^{(k)}$, $j = 7, \ldots, 0$ with coefficients in $\mathcal{K}^{(m-7)}$. We omit the explicit expressions here. The conserved density conditions imply that $a_{m-6} = 0$ and F belongs to $\tilde{\mathcal{M}}_8^{(m-8)}$.

Step 6. $k = m-8, m = 11, 13$. F is now a polynomial in the generators of $\mathcal{M}_j^{(k)}$, $j = 8, \ldots, 0$. The conserved density conditions imply that for $m = 11$, a satisfies the equation above (4.11) and for $m > 11$, $a_{m-8} = 0$. We find that $m > 11$, F belongs to $\tilde{\mathcal{M}}_9^{(m-9)}$.

Step 7. $k = m-9$, $m = 13$. F is a polynomial in the generators of $\mathcal{M}_j^{(k)}$, $j = 9, \ldots, 0$. The conserved density conditions imply that $a_{m-9} = 0$ and we find that F belongs to $\tilde{\mathcal{M}}_{10}^{(m-10)}$.

Step 8. $k = m-10, m = 13$. F is a polynomial in the generators of $\mathcal{M}_j^{(k)}$, $j = 10, \ldots, 0$. For $m = 13$, a satisfies that equation above (4.11).

In sections 4.3 and 4.4 we will outline the solution procedures in some detail for $m = 7$ and $m = 9$. The final results for $m = 11$ and $m = 13$ will be given in appendix C.

4.3. Detailed computations for $m = 7$

For $m = 7$, the explicit form of the evolution equation and its canonical densities are

$$u_t = a^7 u_7 + B u_5 u_6 + C u_5^3 + E u_6 + G u_5^2 + H u_5 + K, \tag{4.14}$$

$$\begin{aligned} \rho^{(-1)} &= a^{-1} \\ \rho^{(1)} &\cong P^{(1)} u_5^2 + Q^{(1)} u_5 + R^{(1)} \\ \rho^{(3)} &\cong P^{(3)} u_6^2 + Q^{(3)} u_5^4 + R^{(3)} u_5^3 + S^{(3)} u_5^2 + T^{(3)} u_5 + V^{(3)}, \end{aligned} \tag{4.15}$$

where $B, C, \ldots K$ and $P^{(1)}, \ldots, V^{(3)}$ are functions of $x, t, u, \ldots, u_4$. According to proposition 3.4, the top level term in $D_t \rho^{(i)}$, $i = -1, 1, 3$ depends on the u_4 dependence of all coefficients. At the top level we have

$$\begin{aligned} \pi(D_t \rho^{(-1)}) &\cong (a^{-1})_4 D^4 F \\ \pi(D_t \rho^{(1)}) &\cong 2P^{(1)} u_5 D^5 F \\ \pi(D_t \rho^{(3)}) &\cong 2P^{(3)} u_6 D^6 F + 4Q^{(3)} u_5^3 D^5 F. \end{aligned} \tag{4.16}$$

We use the symbolic programming language REDUCE for our conserved density computations. For top level computations we declare dependences on u_4 only for the functions above and integrate by parts $D_t \rho^{(1)}$ and $D_t \rho^{(3)}$. Assuming that $P^{(1)} \neq 0$ and $P^{(3)} \neq 0$, the coefficients of $\{u_8^2 u_5, u_7^3, u_7^2 u_6 u_5\}$ in $D_t \rho^{(1)}$ and the coefficients of $\{u_9^2 u_5, u_8^2 u_7\}$ in $D_t \rho^{(3)}$ give

$$B = 14 a_4 a^6, \quad C = a^5 \left[\frac{7}{2} a_{4,4} a + 21 a_4^2 \right], \quad P_4^{(1)} = 7 \frac{a_4}{a} P^{(1)}, \quad P_4^{(3)} = 9 \frac{a_4}{a} P^{(3)},$$

and finally we obtain $a_4 = 0$, hence $B = C = 0$. At order 7, the computations are slow but still feasible even if we keep dependences in lower order derivatives. We compute $D_t \rho^{(-1)}$ and from the coefficients of $\{u_6^2, u_5^4, u_5^3, u_5^2\}$ we obtain

$$\frac{\partial^2 E}{\partial u_4{}^2} = 0, \quad \frac{\partial G}{\partial u_4} = 0, \quad \frac{\partial^3 H}{\partial u_4{}^3} = 0, \quad \frac{\partial^5 K}{\partial u_4{}^5} = 0,$$

hence all coefficient functions are polynomials in u_4. It follows that u_t is of the form below where $E^{(j)}$ denotes the coefficient of u_4^j in E and so on:

$$u_t = \left[a^7u_7 + E^{(1)}u_6u_4 + G^{(0)}u_5^2 + H^{(2)}u_5u_4^2 + K^{(4)}u_4^4\right] + \left[E^{(0)}u_6 + H^{(1)}u_5u_4 + K^{(3)}u_4^3\right] + \left[H^{(0)}u_5 + K^{(2)}u_4^2\right] + K^{(1)}u_4 + K^{(0)} \tag{4.17}$$

$u_t = F$ is a sum of level homogeneous terms over the base level $k = 3$. We can repeat the same type of computations to determine the dependence on u_3. Here we present only the top level part

$$u_t = a^7u_7 + 14a_3a^6u_6u_4 + \tfrac{21}{2}a_3a^6u_5^2 + a^5\left(\tfrac{35}{2}a_{3,3}a + 63a_3^2\right)u_5u_4^2 + a_3a^4\left(\tfrac{399}{8}a_{3,3}a - \tfrac{21}{4}a_3^2\right)u_4^4,$$

where a satisfies the equation (4.11).

4.4. Detailed computations for $m = 9$

We start with the form

$$u_t = a^9u_9 + Bu_7u_8 + Cu_7^3 + Eu_8 + Gu_7^2 + Hu_7 + K. \tag{4.18}$$

At the top level, the evolution equation and the canonical conserved densities are

$$\begin{aligned} &u_t = a^9u_9 + Bu_7u_8 + Cu_7^3\\ &\pi(\rho^{(1)}) \cong P^{(1)}u_7^2\\ &\pi(\rho^{(3)}) \cong P^{(3)}u_8^2 + Q^{(3)}u_7^4. \end{aligned} \tag{4.19}$$

The top level parts of the conserved density conditions are used to express B and C in terms of the derivatives of a with respect to u_6 and finally we get

$$a_6 = 0, \quad B = 0, \quad C = 0.$$

We integrate by parts $D_t\rho^{(-1)} = D_t(a^{-1})$ to get

$$\frac{\partial^2 E}{\partial {u_6}^2} = 0, \quad \frac{\partial^G}{\partial u_6} = 0, \quad \frac{\partial^3 H}{\partial {u_6}^3} = 0, \quad \frac{\partial^5 K}{\partial {u_6}^5} = 0,$$

hence these functions are polynomial in u_6.

We re-parameterize u_t as

$$u_t = a^9u_9 + Bu_8u_6 + Cu_7^2 + Eu_7u_6^2 + Gu_6^4 + Hu_8 + Iu_7u_6 + Ju_6^3 + Ku_7 + Lu_6^2 + Mu_6 + N, \tag{4.20}$$

and consider the top level parts of the equation

$$u_t = a^9u_9 + Bu_8u_6 + Cu_7^2 + Eu_7u_6^2 + Gu_6^4. \tag{4.21}$$

At this step, we also use the conserved density conditions to compute the coefficients B, C, E, G as functions of the derivatives of a with respect to u_5, and finally we obtain

$$a_5 = 0, \quad B = 0, \quad C = 0, \quad E = 0, \quad G = 0.$$

Then, we integrate by parts $D_t\rho^{(-1)}$ and use the explicit expressions of $\rho^{(1)}$ and $\rho^{(3)}$ to obtain

$$\frac{\partial^2 H}{\partial {u_5}^2} = 0, \quad \frac{\partial I}{\partial u_5} = 0, \quad \frac{\partial J}{\partial u_5} = 0, \quad \frac{\partial^3 K}{\partial {u_5}^3} = 0, \quad \frac{\partial^2 L}{\partial {u_5}^2} = 0, \quad \frac{\partial^4 M}{\partial {u_5}^4} = 0, \quad \frac{\partial^6 N}{\partial {u_5}^6} = 0.$$

Thus we prove that $u_t = F$ is a sum of level homogeneous terms of levels $5, 4, \ldots, 0$ above the base level u_4.

We collect level homogeneous terms and rename to obtain

$$\begin{aligned}u_t = {}& [a^9u_9 + Bu_8u_5 + Cu_7u_6 + Eu_7u_5^2 + Fu_6^2u_5 + Gu_6u_5^3 + Hu_5^5] \\ & + [Iu_8 + Ju_7u_5 + Ku_6^2 + Lu_6u_5^2 + Mu_5^4] + [Nu_7 + Ou_6u_5 + Pu_5^3] \\ & + [Qu_6 + Ru_5^2] + [Su_5] + T. \end{aligned} \tag{4.22}$$

Top level computations give

$$a_4 = 0,\quad B = 0,\quad C = 0,\quad E = 0,\quad F = 0,\quad G = 0,\quad H = 0$$

and lower level computations using the explicit expressions of the canonical densities $\rho^{(j)}$, $j = -1, 1, 3$ imply that

$$\frac{\partial^2 I}{\partial u_4{}^2} = \frac{\partial J}{\partial u_4} = \frac{\partial K}{\partial u_4} = L = M = 0,\qquad \frac{\partial^3 N}{\partial u_4{}^3} = \frac{\partial^2 O}{\partial u_4{}^2} = \frac{\partial P}{\partial u_4} = 0,$$

$$\frac{\partial^4 Q}{\partial u_4{}^4} = \frac{\partial^3 R}{\partial u_4{}^3} = 0,\qquad \frac{\partial^5 S}{\partial u_4{}^5} = 0,\qquad \frac{\partial^7 T}{\partial u_4{}^7} = 0.$$

At top level the explicit form of the equation is

$$\begin{aligned}u_t = {}& u_9a^9 + 27u_8u_4a_3a^8 + 57u_7u_5a_3a^8 + u_7u_4^2a^7\left(\tfrac{105}{2}a_{3,3}a + 255a_3^2\right) + \tfrac{69}{2}u_6^2a_3a^8 \\ & + 3u_6u_5u_4a^7\left(63a_{3,3}a + 284a_3^2\right) + 330u_6u_4^3a_3a^6\left(4a_{3,3}a + a_3^2\right) \\ & + u_5^3a^7\left(\tfrac{91}{2}a_{3,3}a + 199a_3^2\right) + u_5^2u_4^2a_3a^6\left(\tfrac{11187}{4}a_{3,3}a + \tfrac{1023}{2}a_3^2\right) \\ & + u_5u_4^4a^5\left(\tfrac{6699}{8}a_{3,3}^2a^2 + 11385a_{3,3}a_3^2a - \tfrac{19569}{2}a_3^4\right) \\ & + u_4^6a_3a^4\left(\tfrac{39325}{16}a_{3,3}^2a^2 + \tfrac{9295}{2}a_{3,3}a_3^2a - \tfrac{37895}{4}a_3^4\right)\end{aligned}$$

where a satisfies (4.11).

5. Results and discussion

In this paper, we introduced a new grading, which we called the 'level grading', on the algebra $\mathcal{M}^{(k)}$ of polynomials generated by the derivatives u_{k+j}, $j = 1, 2, \ldots$ over the coefficient ring $\mathcal{K}^{(k)}$ of C^∞ functions of x, t, u and u_i, $i = 1, 2, \ldots, k$. We proved that this grading has the property that the total derivatives with respect to x and t and the integrations by parts are filtered algebra maps. We also proved that if u satisfies an evolution equation $u_t = F[u]$ of order $m = k + p$ and F belongs to a submodule $\tilde{\mathcal{M}}_p^{(k)}$, then the canonical densities belong to $\tilde{\mathcal{M}}_{p+m}^{(k)}$. We applied this 'level homogeneity' property to the classification of scalar evolution equations of orders $m = 7, 9, 11, 13$ admitting non-trivial conserved densities of all orders, and we obtained their explicit expressions up to their dependences on u_2, u_1 and u. It was a remarkable fact that at all orders $a = (F_m)^{1/m}$ satisfies the same equation

$$a_{3,3,3} - 9a_{3,3}a_3a^{-1} + 12a_3^3a^{-2} = 0. \tag{5.1}$$

We note that the substitution $a = Z^{-1/2}$ leads to $Z_{3,3,3} = 0$, hence

$$a = \left(\alpha u_3^2 + \beta u_3 + \gamma\right)^{-1/2}, \tag{5.2}$$

where α, β, γ are functions of u, u_1 and u_2 in general.

The occurrence of the same form for a strongly suggests that these equations belong to a hierarchy. In fact we have shown that the same form of a has occurred in the classification of fifth order equations [12]. These equations seem to be intrinsically related to the class of fully nonlinear third order equations [13],

$$u_t = F = \left(\alpha u_3^2 + \beta u_3 + \gamma\right)^{-1/2}(2\alpha u_3 + \beta) + \delta.$$

In this equation, when we compute $\frac{\partial F}{\partial u_3}$ we find that

$$\left(\frac{\partial F}{\partial u_3}\right)^{1/3} = P\big(\alpha u_3^2 + \beta u_3 + \gamma\big)^{-1/2},$$

where $P = (2\alpha\gamma - \frac{1}{2}\beta^2)^{1/3}$. This result suggests that the equations that we have obtained possibly belong to a hierarchy starting at the fully nonlinear third order equation, and that the hierarchy is possibly generated by a second order recursion operator.

As a final remark, we emphasize that our results on the classification of evolution equations are only preliminary. We recall that in [10], we have proved that arbitrary (non-polynomial) evolution equations of orders m are of the form (4.1). Here, assuming that $E = G = H = K = 0$ and $\rho^{(-1)}$ and $\rho^{(3)}$ are non-trivial, we prove that for $m = 7, 9, 11, 13$ the expressions that we obtain are unique up to the dependence of the functions α, β and γ in x, t, u, u_1, u_2.

To obtain a complete classification, first of all we should study the cases where $\rho^{(-1)}$ and/or $\rho^{(3)}$ are trivial. The triviality of $\rho^{(3)}$ is the common property of the Sawada–Kotera/Kaup equations and of the fifth order equation [2]

$$u_t = -\frac{3}{2A}(Au_5 + B)^{-2/3} + C, \quad A_5 = B_5 = C_5 = 0.$$

Thus we expect that higher order equations characterized by the triviality of $\rho^{(3)}$ will be symmetries of these fifth order equations. The equations characterized by a nonzero but trivial $\rho^{(-1)}$ turn out to be harder to analyze. We note that the nonlinear third order equation given [8],

$$u_t = F = (Au_3 + B)^{-2} + C, \quad A_3 = B_3 = C_2 = 0,$$

has the property that $\left(\frac{\partial F}{\partial u_3}\right)^{-1/3}$ is linear in u_3. We may thus suspect that higher order equations with trivial $\rho^{(-1)}$ are related to the equation above. The second assumption, $E = G = H = K = 0$, is of a more technical nature. At order $m = 5$ we have included an almost complete classification of KdV-like equations including lower order terms [12], but this turned out to be practically impossible at higher orders. Although it would be possible to force computational power, we are hoping to prove that the top level part determines integrable evolution equations up to certain transformations to be discussed below.

The last problem with the incompleteness of our classification is related to the dependences of the functions α, β and γ on x, t, u, u_1, u_2. In preliminary computations we have seen that there seem to be identities among the derivatives of these functions, hence one has to possibly use analogues of contact or Miura transformations in order to eliminate the arbitrariness. The situation is more or less the same with the constant separant case, where at third order a large number of the equations are reduced to the KdV or the Krichever–Novikov equation.

Appendix A. Conserved densities

If the evolution equation $u_t = F[u]$ is integrable, it is known that the quantities

$$\rho^{(-1)} = F_m^{-1/m}, \quad \rho^{(0)} = F_{m-1}/F_m,$$

where

$$F_m = \frac{\partial F}{\partial u_m}, \quad F_{m-1} = \frac{\partial F}{\partial u_{m-1}}$$

are conserved densities for equations of any order [8]. Higher order conserved densities are computed in [2] as below, with the following notation

$$a = F_m^{1/m}, \quad \alpha_{(i)} = \frac{F_{m-i}}{F_m},\ i = 1, 2, 3, 4$$

$$\rho^{(1)} = a^{-1}(Da)^2 - \frac{12}{m(m+1)}Da\alpha_{(1)} + a\left[\frac{12}{m^2(m+1)}\alpha_{(1)}^2 - \frac{24}{m(m^2-1)}\alpha_{(2)}\right],$$

$$\rho^{(2)} = a(Da)\left[D\alpha_{(1)} + \frac{3}{m}\alpha_{(1)}^2 - \frac{6}{(m-1)}\alpha_{(2)}\right] + 2a^2\left[-\frac{1}{m^2}\alpha_{(1)}^3 + \frac{3}{m(m-1)}\alpha_{(1)}\alpha_{(2)} - \frac{3}{(m-1)(m-2)}\alpha_{(3)}\right],$$

$$\begin{aligned}\rho^{(3)} = {}& a(D^2a)^2 - \frac{60}{m(m+1)(m+3)}a^2D^2aD\alpha_{(1)} + \frac{1}{4}a^{-1}(Da)^4 \\ &+ 30a(Da)^2\left[\frac{(m-1)}{m(m+1)(m+3)}D\alpha_{(1)} + \frac{1}{m^2(m+1)}\alpha_{(1)}^2 - \frac{2}{m(m^2-1)}\alpha_{(2)}\right] \\ &+ \frac{120}{m(m^2-1)(m+3)}a^2Da\left[-\frac{(m-1)(m-3)}{m}\alpha_{(1)}D\alpha_{(1)} + (m-3)D\alpha_{(2)}\right. \\ &\left. - \frac{(m-1)(2m-3)}{m^2}\alpha_{(1)}^3 + \frac{6(m-2)}{m}\alpha_{(1)}\alpha_{(2)} - 6\alpha_{(3)}\right] \\ &+ \frac{60}{m(m^2-1)(m+3)}a^3\left[\frac{(m-1)}{m}(D\alpha_{(1)})^2 - \frac{4}{m}D\alpha_{(1)}\alpha_{(2)}\right. \\ &+ \frac{(m-1)(2m-3)}{m^3}\alpha_{(1)}^4 - 4\frac{(2m-3)}{m^2}\alpha_{(1)}^2\alpha_{(2)} + \frac{8}{m}\alpha_{(1)}\alpha_{(3)} \\ &\left. + \frac{4}{m}\alpha_{(2)}^2 - \frac{8}{(m-3)}\alpha_{(4)}\right].\end{aligned}$$

Appendix B. Submodules and their generators

The submodules $\mathcal{M}_i^{(k)}$ and their generating monomials where: $i = 1, 2, 3, \ldots, 13$ and the base $k = m-3, m-4, \ldots, 3$. The generating monomials of the quotient submodules (the monomials that are not total derivatives) are shown in bold. The generators of a submodule $\mathcal{M}_p^{(k)}$ can be obtained by using the partitions of the integer p into a sum of integers, as for example in $4 = 4+0 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$ or $5 = 5+0 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$.

$$\mathcal{M}_1^{(k)} = \langle u_{k+1}\rangle$$
$$\mathcal{M}_2^{(k)} = \langle u_{k+2}, \mathbf{u_{k+1}^2}\rangle$$
$$\mathcal{M}_3^{(k)} = \langle u_{k+3}, u_{k+2}u_{k+1}, \mathbf{u_{k+1}^3}\rangle$$
$$\mathcal{M}_4^{(k)} = \langle u_{k+4}, u_{k+3}u_{k+1}, u_{k+2}u_{k+1}^2, \mathbf{u_{k+2}^2}, \mathbf{u_{k+1}^4}\rangle$$
$$\mathcal{M}_5^{(k)} = \langle u_{k+5}, u_{k+4}u_{k+1}, u_{k+3}u_{k+2}, u_{k+3}u_{k+1}^2, u_{k+2}u_{k+1}^3, \mathbf{u_{k+2}^2u_{k+1}}, \mathbf{u_{k+1}^5}\rangle$$
$$\begin{aligned}\mathcal{M}_6^{(k)} = {}&\langle u_{k+6}, u_{k+5}u_{k+1}, u_{k+4}u_{k+2}, u_{k+4}u_{k+1}^2, u_{k+3}u_{k+2}u_{k+1}, u_{k+3}u_{k+1}^3, u_{k+2}u_{k+1}^4, \\ &\mathbf{u_{k+3}^2}, \mathbf{u_{k+2}^3}, \mathbf{u_{k+2}^2u_{k+1}^2}, \mathbf{u_{k+1}^6}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{M}_7^{(k)} = \langle & u_{k+7}, u_{k+6}u_{k+1}, u_{k+5}u_{k+2}, u_{k+5}u_{k+1}^2, u_{k+4}u_{k+3}, u_{k+4}u_{k+2}u_{k+1}, u_{k+4}u_{k+1}^3,\\ & u_{k+3}u_{k+2}^2, u_{k+3}u_{k+2}u_{k+1}^2, u_{k+3}u_{k+1}^4, u_{k+2}u_{k+1}^5, \mathbf{u_{k+3}^2u_{k+1}}, \mathbf{u_{k+2}^3u_{k+1}},\\ & \mathbf{u_{k+2}^2u_{k+1}^3}, \mathbf{u_{k+1}^7}\rangle\\ \mathcal{M}_8^{(k)} = \langle & u_{k+8}, u_{k+7}u_{k+1}, u_{k+6}u_{k+2}, u_{k+6}u_{k+1}^2, u_{k+5}u_{k+3}, u_{k+5}u_{k+2}u_{k+1}, u_{k+5}u_{k+1}^3,\\ & u_{k+4}u_{k+3}u_{k+1}, u_{k+4}u_{k+2}^2, u_{k+4}u_{k+2}u_{k+1}^2, u_{k+4}u_{k+1}^4, u_{k+3}u_{k+2}^2u_{k+1}, u_{k+3}u_{k+2}u_{k+1}^3,\\ & u_{k+3}u_{k+1}^5, u_{k+2}u_{k+1}^6, \mathbf{u_{k+4}^2}, \mathbf{u_{k+3}^2u_{k+2}}, \mathbf{u_{k+3}^2u_{k+1}^2}, \mathbf{u_{k+2}^4}, \mathbf{u_{k+2}^3u_{k+1}^2}, \mathbf{u_{k+2}^2u_{k+1}^4}, \mathbf{u_{k+1}^8}\rangle\\ \mathcal{M}_9^{(k)} = \langle & u_{k+9}, u_{k+8}u_{k+1}, u_{k+7}u_{k+2}, u_{k+7}u_{k+1}^2, u_{k+6}u_{k+3}, u_{k+6}u_{k+2}u_{k+1}, u_{k+6}u_{k+1}^3, u_{k+5}u_{k+4},\\ & u_{k+5}u_{k+3}u_{k+1}, u_{k+5}u_{k+2}^2, u_{k+5}u_{k+2}u_{k+1}^2, u_{k+5}u_{k+1}^4, u_{k+4}u_{k+3}u_{k+2}, u_{k+4}u_{k+3}u_{k+1}^2,\\ & u_{k+4}u_{k+2}^2u_{k+1}, u_{k+4}u_{k+2}u_{k+1}^3, u_{k+4}u_{k+1}^5, u_{k+3}u_{k+2}^3, u_{k+3}u_{k+2}^2u_{k+1}^2, u_{k+3}u_{k+2}u_{k+1}^4,\\ & u_{k+3}u_{k+1}^6, u_{k+2}u_{k+1}^7, \mathbf{u_{k+4}^2u_{k+1}}, \mathbf{u_{k+3}^3}, \mathbf{u_{k+3}^2u_{k+2}u_{k+1}}, \mathbf{u_{k+3}^2u_{k+1}^3}, \mathbf{u_{k+2}^4u_{k+1}}, \mathbf{u_{k+2}^3u_{k+1}^3}\\ & \mathbf{u_{k+2}^2u_{k+1}^5}, \mathbf{u_{k+1}^9}\rangle\\ \mathcal{M}_{10}^{(k)} = \langle & u_{k+10}, u_{k+9}u_{k+1}, u_{k+8}u_{k+2}, u_{k+8}u_{k+1}^2, u_{k+7}u_{k+3}, u_{k+7}u_{k+2}u_{k+1}, u_{k+7}u_{k+1}^3, u_{k+6}u_{k+4},\\ & u_{k+6}u_{k+3}u_{k+1}, u_{k+6}u_{k+2}^2, u_{k+6}u_{k+2}u_{k+1}^2, u_{k+6}u_{k+1}^4, u_{k+5}u_{k+4}u_{k+1}, u_{k+5}u_{k+3}u_{k+2},\\ & u_{k+5}u_{k+3}u_{k+1}^2, u_{k+5}u_{k+2}^2u_{k+1}, u_{k+5}u_{k+2}u_{k+1}^3, u_{k+5}u_{k+1}^5, u_{k+4}u_{k+3}^2, u_{k+4}u_{k+3}u_{k+2}u_{k+1},\\ & u_{k+4}u_{k+3}u_{k+1}^3, u_{k+4}u_{k+2}^3, u_{k+4}u_{k+2}^2u_{k+1}^2, u_{k+4}u_{k+2}u_{k+1}^4, u_{k+4}u_{k+1}^6, u_{k+3}u_{k+2}^3u_{k+1},\\ & u_{k+3}u_{k+2}^2u_{k+1}^3, u_{k+3}u_{k+2}u_{k+1}^5, u_{k+3}u_{k+1}^7, u_{k+2}u_{k+1}^8, \mathbf{u_{k+5}^2}, \mathbf{u_{k+4}^2u_{k+2}}, \mathbf{u_{k+4}^2u_{k+1}^2}, \mathbf{u_{k+3}^3u_{k+1}},\\ & \mathbf{u_{k+3}^2u_{k+2}^2}, \mathbf{u_{k+3}^2u_{k+2}u_{k+1}^2}, \mathbf{u_{k+3}^2u_{k+1}^4}, \mathbf{u_{k+2}^5}, \mathbf{u_{k+2}^4u_{k+1}^2}, \mathbf{u_{k+2}^3u_{k+1}^4}, \mathbf{u_{k+2}^2u_{k+1}^6}, \mathbf{u_{k+1}^{10}}\rangle.\end{aligned}$$

Appendix C. Results for $m = 11, m = 13$

We present the top level parts of F computed as described in section 4.2, Steps 1–8, for $m = 11$ and $m = 13$.

Scalar integrable evolution equations of order $m = 11$ have the form:

$$\begin{aligned}u_t = a^{11}u_{11} &+ B_0u_{10}u_4 + B_1u_9u_5 + B_2u_9u_4^2 + B_3u_8u_6 + B_4u_8u_5u_4 + B_5u_8u_4^3\\ &+ B_6u_7^2 + B_7u_7u_6u_4 + B_8u_7u_5^2 + B_9u_7u_5u_4^2 + B_{10}u_7u_4^4 + B_{11}u_6^2u_5\\ &+ B_{12}u_6^2u_4^2 + B_{13}u_6u_5^2u_4 + B_{14}u_6u_5u_4^3 + B_{15}u_6u_4^5 + B_{16}u_5^4 + B_{17}u_5^3u_4^2\\ &+ B_{18}u_5^2u_4^4 + B_{19}u_5u_4^6 + B_{20}u_4^8\end{aligned}$$

where

$$\begin{aligned}&B_0 = 44a_3a^{10},\ B_1 = 121a_3a^{10},\ B_2 = a^9\left(\tfrac{231}{2}a_{3,3}a + 726a_3^2\right),\ B_3 = 209a_3a^{10},\\ &B_4 = 198a^9\big(3a_{3,3}a + 17a_3^2\big),\ B_5 = a_3a^8\left(\tfrac{9867}{2}a_{3,3}a + 3432a_3^2\right),\ B_6 = \tfrac{253}{2}a_3a^{10},\\ &B_7 = 66a^9\big(14a_{3,3}a + 75a_3^2\big),\ B_8 = a^9\left(\tfrac{1353}{2}a_{3,3}a + 3333a_3^2\right),\\ &B_9 = a_3a^8\left(\tfrac{63\,921}{2}a_{3,3}a + 17\,160a_3^2\right),\\ &B_{10} = a^7\left(\tfrac{39\,039}{8}a_{3,3}^2a^2 + 78\,936a_{3,3}a_3^2a - \tfrac{96\,525}{2}a_3^4\right),\\ &B_{11} = a^9\left(\tfrac{1749}{2}a_{3,3}a + 4026a_3^2\right),\ B_{12} = a_3a^8\left(\tfrac{81\,081}{4}a_{3,3}a + 9867a_3^2\right),\\ &B_{13} = 858a_3a^8\big(67a_{3,3}a + 26a_3^2\big),\ B_{14} = a^7\left(\tfrac{70\,785}{2}a_{3,3}^2a^2 + 552\,981a_{3,3}a_3^2a - 376\,662a_3^4\right),\\ &B_{15} = a_3a^6\left(\tfrac{708\,279}{4}a_{3,3}^2a^2 + 465\,465a_{3,3}a_3^2a - 738\,309a_3^4\right),\\ &B_{16} = a_3a^8\left(\tfrac{54\,483}{8}a_{3,3}a + \tfrac{8437}{4}a_3^2\right),\\ &B_{17} = a^7\left(\tfrac{100\,815}{4}a_{3,3}^2a^2 + 385\,242a_{3,3}a_3^2a - 280\,709a_3^4\right),\end{aligned}$$

$B_{18} = a_3a^6\left(\frac{9988\,407}{16}a_{3,3}^2a^2 + 1571\,856a_{3,3}a_3^2a - \frac{10\,401\,105}{4}a_3^4\right),$
$B_{19} = a^5\left(\frac{1210\,495}{16}a_{3,3}^3a^3 + \frac{18\,214\,911}{8}a_{3,3}^2a_3^2a^2 - \frac{4236\,375}{4}a_{3,3}a_3^4a - \frac{6180\,031}{2}a_3^6\right),$
$B_{20} = a_3a^4\left(\frac{28\,717\,403}{128}a_{3,3}^3a^3 + \frac{86\,866\,923}{64}a_{3,3}^2a_3^2a^2 - \frac{89\,492\,403}{32}a_{3,3}a_3^4a + \frac{4538\,677}{16}a_3^6\right).$

Scalar integrable evolution equations of order $m = 13$ have the form:

$$\begin{aligned}u_t = a^{13}u_{13} &+ B_0u_{12}u_4 + B_1u_{11}u_5 + B_2u_{11}u_4^2 + B_3u_{10}u_6 + B_4u_{10}u_5u_4 + B_5u_{10}u_4^3\\ &+ B_6u_9u_7 + B_7u_9u_6u_4 + B_8u_9u_5^2 + B_9u_9u_5u_4^2 + B_{10}u_9u_4^4 + B_{11}u_8^2 + B_{12}u_8u_7u_4\\ &+ B_{13}u_8u_6u_5 + B_{14}u_8u_6u_4^2 + B_{15}u_8u_5^2u_4 + B_{16}u_8u_5u_4^3 + B_{17}u_8u_4^5 + B_{18}u_7^2u_5\\ &+ B_{19}u_7^2u_4^2 + B_{20}u_7u_6^2 + B_{21}u_7u_6u_5u_4 + B_{22}u_7u_6u_4^3 + B_{23}u_7u_5^3 + B_{24}u_7u_5^2u_4^2\\ &+ B_{25}u_7u_5u_4^4 + B_{26}u_7u_4^6 + B_{27}u_6^3u_4 + B_{28}u_6^2u_5^2 + B_{29}u_6^2u_5u_4^2 + B_{30}u_6^2u_4^4\\ &+ B_{31}u_6u_5^3u_4 + B_{32}u_6u_5^2u_4^3 + B_{33}u_6u_5u_4^5 + B_{34}u_6u_4^7 + B_{35}u_5^5 + B_{36}u_5^4u_4^2\\ &+ B_{37}u_5^3u_4^4 + B_{38}u_5^2u_4^6 + B_{39}u_5u_4^8 + B_{40}u_4^{10}\end{aligned}$$

where

$B_0 = 65a_3a^{12},\ B_1 = 221a_3a^{12},\ B_2 = a^{11}\left(\frac{429}{2}a_{3,3}a + 1664a_3^2\right),\ B_3 = 494a_3a^{12},$
$B_4 = 130a^{11}(11a_{3,3}a + 76a_3^2),\ B_5 = 260a_3a^{10}(53a_{3,3}a + 62a_3^2),\ B_6 = 793a_3a^{12},$
$B_7 = 273a^{11}(11a_{3,3}a + 71a_3^2),\ B_8 = a^{11}\left(\frac{4433}{2}a_{3,3}a + 12\,857a_3^2\right),$
$B_9 = a_3a^{10}\left(\frac{239\,473}{2}a_{3,3}a + 114\,062a_3^2\right),$
$B_{10} = a^9\left(\frac{147\,147}{8}a_{3,3}^2a^2 + \frac{699\,855}{2}a_{3,3}a_3^2a - \frac{241\,917}{2}a_3^4\right),$
$B_{11} = \frac{923}{2}a_3a^{12},\ B_{12} = 13a^{11}(330a_{3,3}a + 2077a_3^2),\ B_{13} = 13a^{11}(638a_{3,3}a + 3351a_3^2),$
$B_{14} = a_3a^{10}\left(\frac{433\,043}{2}a_{3,3}a + 183\,937a_3^2\right),\ B_{15} = a_3a^{10}\left(\frac{614\,601}{2}a_{3,3}a + 220\,779a_3^2\right),$
$B_{16} = 39a^9(4895a_{3,3}^2a^2 + 88\,882a_{3,3}a_3^2a - 39\,704a_3^4),$
$B_{17} = a_3a^8\left(\frac{8620\,989}{8}a_{3,3}^2a^2 + \frac{7607\,925}{2}a_{3,3}a_3^2a - \frac{9264\,099}{2}a_3^4\right),$
$B_{18} = a^{11}\left(\frac{10\,153}{2}a_{3,3}a + 26\,065a_3^2\right),$
$B_{19} = a_3a^{10}\left(\frac{525\,395}{4}a_{3,3}a + 107\,900a_3^2\right),\ B_{20} = a^{11}\left(\frac{13\,299}{2}a_{3,3}a + 31\,655a_3^2\right),$
$B_{21} = 13a_3a^{10}\big(73\,438a_{3,3}a + 45\,379a_3^2\big),$
$B_{22} = 195a^9\big(1529a_{33}^2a^2 + 27\,189a_{3,3}a_3^2a - 13\,286a_3^4\big),$
$B_{23} = a_3a^{10}\left(\frac{452\,751}{2}a_{3,3}a + 113\,269a_3^2\right),$
$B_{24} = a^9\left(\frac{2563\,275}{4}a_{33}^2a^2 + 11\,068\,278a_{3,3}a_3^2a - 6059\,391a_3^4\right),$
$B_{25} = a_3a^8\left(\frac{94\,273\,959}{8}a_{3,3}^2a^2 + \frac{77\,468\,235}{2}a_{3,3}a_3^2a - \frac{102\,276\,369}{2}a_3^4\right),$
$B_{26} = a^7\left(\frac{11\,510\,785}{16}a_{3,3}^3a^3 + \frac{195\,575\,055}{8}a_{3,3}^2a_3^2a^2 - \frac{765\,765}{4}a_{3,3}a_3^4a - \frac{82\,683\,835}{2}a_3^6\right),$
$B_{27} = 91a_3a^{10}(2239a_{3,3}a + 1222a_3^2),\ B_{28} = a_3a^{10}\left(\frac{1739\,647}{4}a_{3,3}a + 184\,639a_3^2\right),$
$B_{29} = a^9\left(\frac{3302\,871}{4}a_{3,3}^2a^2 + 14\,017\,107a_{3,3}a_3^2a - 8104785a_3^4\right),$
$B_{30} = a_3a^8\left(\frac{120\,547\,323}{16}a_{3,3}^2a^2 + \frac{96\,860\,985}{4}a_{3,3}a_3^2a - \frac{130\,949\,793}{4}a_3^4\right),$
$B_{31} = a^7\left(\frac{62\,945\,883}{8}a_{3,3}^3a^3 + \frac{522\,998\,931}{2}a_{3,3}^2a_3^2a^2 - \frac{59\,372\,313}{2}a_{3,3}a_3^4a - 427\,213\,332a_3^6\right),$
$B_{32} = a^9\left(\frac{1584\,297}{2}a_{3,3}^2a^2 + 13\,011\,531a_{3,3}a_3^2a - 8261\,656a_3^4\right),$
$B_{33} = a_3a^8\left(\frac{85\,420\,257}{2}a_{3,3}^2a^2 + 130\,920\,621a_{3,3}a_3^2a - 185\,780\,556a_3^4\right),$
$B_{34} = a_3a^6\left(\frac{136\,530\,485}{4}a_{3,3}^3a^3 + \frac{978\,346\,005}{4}a_{3,3}^2a_3^2a^2 - 409\,150\,560a_{3,3}a_3^4a - 36\,300\,355a_3^6\right),$
$B_{35} = a^9\left(\frac{457\,743}{8}a_{3,3}^2a^2 + \frac{1813\,227}{2}a_{3,3}a_3^2a - \frac{1244\,997}{2}a_3^4\right),$
$B_{36} = a_3a^8\left(\frac{242\,491\,587}{16}a_{3,3}^2a^2 + \frac{176\,827\,183}{4}a_{3,3}a_3^2a - \frac{263\,231\,553}{4}a_3^4\right),$

$$B_{37} = a^7 \left(\tfrac{149\,426\,277}{16} a_{3,3}^3 a_3^3 + \tfrac{2445\,811\,641}{8} a_{3,3}^2 a_3^2 a^2 - \tfrac{227\,840\,613}{4} a_{3,3} a_3^4 a - \tfrac{972\,312\,705}{2} a_3^6 \right),$$
$$B_{38} = a_3 a^6 \left(\tfrac{5405\,460\,879}{32} a_{3,3}^3 a_3^3 + \tfrac{18\,948\,697\,131}{16} a_{3,3}^2 a_3^2 a^2 - \tfrac{16\,438\,417\,359}{8} a_{3,3} a_3^4 a - \tfrac{455\,797\,251}{4} a_3^6 \right),$$
$$B_{39} = a^5 \left(\tfrac{1410\,011\,603}{128} a_{3,3}^4 a^4 + \tfrac{2339\,829\,765}{4} a_{3,3}^3 a_3^2 a^3 + \tfrac{9914\,229\,507}{16} a_{3,3}^2 a_3^4 a^2 \right.$$
$$\left. - \tfrac{7090\,620\,355}{2} a_{3,3} a_3^6 a + \tfrac{13\,127\,824\,983}{8} a_3^8 \right),$$
$$B_{40} = a_3 a^4 \left(\tfrac{8400\,372\,435}{256} a_{3,3}^4 a^4 + \tfrac{3249\,837\,045}{8} a_{3,3}^3 a_3^2 a^3 - \tfrac{19\,344\,268\,125}{32} a_{3,3}^2 a_3^4 a^2 \right.$$
$$\left. - \tfrac{3278\,936\,115}{4} a_{3,3} a_3^6 a + \tfrac{14\,521\,758\,615}{16} a_3^8 \right).$$

It is a remarkable fact that at all orders $m \geqslant 7$, the separant a satisfies the same equation (4.11).

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