



## Three-fast-searchable graphs



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### ABSTRACT

In the edge searching problem, searchers move from vertex to vertex in a graph to capture an invisible, fast intruder that may occupy either vertices or edges. Fast searching is a monotonic internal model in which, at every move, a new edge of the graph  $G$  must be guaranteed to be free of the intruder. That is, once all searchers are placed the graph  $G$  is cleared in exactly  $|E(G)|$  moves. Such a restriction obviously necessitates a larger number of searchers. We examine this model, and characterize graphs for which 2 or 3 searchers are sufficient. We prove that the corresponding decision problem is NP-complete.

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### 1. Introduction

Graph searching was first introduced by Breisch in [6], as a problem in spelunking to find an individual lost in a cave. Obviously, in such a situation, the person who is lost may be moving (though not actively avoiding the searchers), and due to the darkness it may be hard to see. Such a person could easily be in the middle of a cave corridor, as opposed to a junction. This problem was first developed mathematically by Parsons in [15], who examined it particularly for trees.

The graph searching (or edge searching) problem is to move agents called *searchers* in such a way as to guarantee the capture of an intruder. Capture occurs when an intruder and a searcher both occupy the same vertex at the same time. Initially, an intruder may be located on any edge or vertex of the graph. Intruders are invisible to the searcher, and as such all edges that may contain an intruder are said to be *contaminated*, or *dirty*. A path that does not contain any searcher is called an *unguarded path*. The intruder can move at any time, and can move from its present location along any unguarded path to any other vertex or edge in the graph. The intruder has full knowledge of the graph and the location of the searchers.

On the other hand, the searchers only stop on vertices. They have full knowledge of the graph, and each others' locations, but not that of any intruders. They move one at a time. In Parsons' original model, only three moves were allowed: a searcher may be placed on any vertex  $u$ ; a searcher may be removed from any vertex  $u$  and a searcher on  $u$  may slide along any edge  $uv$ . An edge  $uv$  becomes *clear* by a sliding move in two ways: There may be (at least) two searchers on  $u$ , and one traverses the edge  $uv$  while the other remains on  $u$ . The second way is when  $u$  contains one searcher, denoted by  $\sigma$ , and all edges incident to  $u$ , other than  $uv$ , are clear. Then that searcher  $\sigma$  traverses  $uv$ .

Given a graph  $G$ , a sequence of moves using  $k$  searchers that ends with all edges of  $G$  being clear is called a  $k$ -search strategy for  $G$ ; in that case  $G$  is called  $k$ -searchable. The minimum number of searchers needed to clear the edges of  $G$  is called the search number of  $G$ , denoted by  $s(G)$ .

In a search strategy, if, after removing a searcher, a clear edge is incident with a vertex that is connected by an unguarded path to a contaminated edge, we say that the clear edge becomes *recontaminated*. A search strategy is said to be *monotonic* if,

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after an edge is cleared, searchers must move in a way to never allow recontamination. A search strategy is said to be *internal* if all searchers are first placed in the graph, and then are allowed only to slide along edges, and may never be removed. That is, they may not “jump” from a vertex to a non-adjacent vertex. Using these properties, we analogously define the *monotonic search number*,  $ms(G)$ , the *internal search number*,  $is(G)$ , and the *monotonic internal search number*,  $mis(G)$ . It is straightforward to see that for a connected graph  $G$ ,  $s(G) = is(G)$ . It was shown independently in [5,13] that  $s(G) = ms(G)$ . It is clear that  $s(G) \leq mis(G)$ , and many examples of graphs exist where the ratio of these parameters may be arbitrarily large [3,18].

In [14], it is shown that computing the search number of trees is polynomial, graphs with search number at most 3 were characterized, and, when combined with the results of [5,13], that the decision problem for search number is NP-complete.

Here we discuss the *fast search* model introduced in [7]. In this model, the intruder behaves as in the standard search model. The searchers have full knowledge of the graph and each others’ positions, and move from vertex to vertex. However, after placing all the searchers in the graph, we require that every subsequent move clears an edge and no edge gets recontaminated. In addition the searchers cannot jump from vertex to vertex. Thus, any such strategy must be both monotonic and internal. A strategy that corresponds to such a search is called a *fast search strategy*. The *fast search number*, denoted by  $s_f(G)$ , is the minimum number of searchers needed for any fast search strategy. If  $k$  searchers are enough to fast search  $G$ , then we say that  $G$  is *k-fast searchable*. The complexity result for fast searching of trees is shown in [7]. It was observed by Yang [17] that the fast searching problem is related with the graph brushing problem [1] and the balanced vertex ordering problem [4]. Other time constrained models are discussed in [2], and a general survey of graph searching is available [9].

This paper develops the fast search problem further. We first characterize those graphs which have fast search number at most 3, and then show that the fast search decision problem is NP-complete through a reduction from a new search model called the *weak searching* problem.

## 2. Preliminaries

Let  $G$  be a connected graph which may have loops or parallel edges. Let  $S$  denote a fast search strategy for  $G$ . Assume that  $\sigma$  denotes one of the searchers used in  $S$ . We say that  $v$  is the *start vertex* for  $\sigma$  if  $\sigma$  is initially placed on  $v$  according to the strategy  $S$ . We say that  $u$  is the *end vertex* for  $\sigma$  if  $\sigma$  stops at  $u$  (and never moves again). The start and end vertices for  $\sigma$  are denoted by  $b(\sigma)$  and  $t(\sigma)$ , respectively. Moreover,  $W(\sigma)$  is the walk traversed by  $\sigma$  during the search of  $G$ . We say that the *start vertices* of a search strategy are the vertices of the set  $\cup_{\sigma} b(\sigma)$ . Similarly, the *end vertices* of a search strategy are the vertices of the set  $\cup_{\sigma} t(\sigma)$ . The *degree* of a vertex  $u$  in  $G$ , denoted by  $\deg_G(u)$ , is the number of the endpoints of the edges incident to  $u$  in  $G$ . This in particular implies that adding a loop to a vertex increases its degree by 2. We call a vertex  $v$  even (resp. odd) if it has even (resp. odd) degree. Let  $V_o$  be the set of odd vertices in  $G$ . Since an odd vertex is a start or an end vertex in a fast search strategy, we have the following lemma.

**Lemma 1** ([7]). *If  $V_o$  is the set of odd degree vertices in a graph  $G$ , then  $s_f(G) \geq \frac{|V_o|}{2}$ .*

Following the development of edge searching we are interested in characterizing graphs where  $s_f(G)$  is small.

We define *reduction* as reducing any path with consecutive vertices of degree two to a single edge. Note that reduction does not change the fast search number, and that  $s_f(G) = 1$  whenever  $G$  can be reduced to a single edge. If a graph  $G$  does not contain any vertices of degree 2 then  $G$  is a *reduced graph*.

The characterization of 3-searchable graphs is given in [14]. For the characterization of 2-fast searchable graphs  $G$  we have the following result.

**Theorem 2.** *For any reduced graph  $G$ ,  $s_f(G) \leq 2$  if and only if  $G$  consists of a path with vertex set  $\{v_0, v_1, \dots, v_n\}$  together with the following conditions:*

1. *For every  $i = 0, 1, \dots, n - 1$  there are exactly two parallel edges between each pair of consecutive vertices  $v_i$  and  $v_{i+1}$ .*
2. *For every  $i = 0, 1, \dots, n$  there may be an arbitrary number of loops attached to each  $v_i$ .*
3. *There may be at most two pendant edges attached to  $v_0$  or to  $v_n$ .*

**Proof.** Necessity is obvious. In order to show sufficiency we consider the graphs  $G$  for which  $s(G) \leq 2$  since  $s(G) \leq s_f(G)$  for every graph. It is known [14] that the graphs for which  $s(G) \leq 2$  are those that are paths  $P$  together with possible pendant edges and loops attached to each vertex. By Lemma 1, we have  $|V_o| \leq 4$ . Let  $e = uv_i$  be a pendant edge attached to  $v_i$  where  $v_i$  is an internal vertex of the path  $P$ ; hence  $i \in \{1, 2, \dots, n - 1\}$ . Let  $\deg(u) = 1$ . Thus  $u$  is either a start or an end vertex for a searcher. Assume that  $u$  is a start vertex for a searcher. The other case can be shown similarly. This implies that the vertices  $v_0, v_1, \dots, v_{i-1}$  must be cleared by a single searcher. Similarly  $v_{i+1}, v_{i+2}, \dots, v_n$  must be cleared by a single searcher. Thus the graph induced by  $v_0, v_1, \dots, v_{i-1}$  must be a path. Similarly  $v_{i+1}, v_{i+2}, \dots, v_n$  also induce a path. Since  $G$  is a reduced graph,  $G = K_{1,3}$  or  $K_{1,4}$  and  $v_0 = v_n = v_i$ . This contradicts the assumption that  $v_i$  is an internal vertex. Thus no pendant edge is attached to an internal vertex. Hence we have proved Condition 3. Using a similar discussion we conclude that all internal vertices must have even degree, thus Condition 1 and Condition 2 hold.  $\square$

### 3. Characterization of biconnected 3-fast searchable graphs

Here we give the structure of the biconnected graphs that can be cleared with 3 searchers using a fast search strategy. These three searchers are denoted by  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$ . First, we give the necessary definitions.

Recall that a biconnected graph is a graph which remains connected when we remove any single vertex. For a graph  $G = (V, E)$  and  $X \subseteq V$ ,  $G[X]$  denotes the *induced subgraph* that is the subgraph of  $G$  with the vertex set  $X$  and the edge set  $E(G[X]) = \{uv \in E(G) : u, v \in X\}$ . For a given pair of vertices  $u, v$ ,  $\mu(u, v)$  is the number of edges between  $u$  and  $v$ . In this paper, we consider multigraphs and in that case we assume that  $E$  is a multiset, i.e. multiple occurrences of an edge with the same endpoints are allowed. Then,  $E \setminus \{e\}$  means removing only one occurrence of  $e$  from  $E$ .

The *contraction* of an edge  $uv \in E$  of  $G$  is an operation of removing all the edges between  $u$  and  $v$ , and unifying the vertices  $u$  and  $v$ , i.e. these vertices are replaced with a new vertex that is adjacent to all the vertices adjacent to  $u$  or  $v$  in the initial graph preserving the multiplicities of the edges. The *deletion* of an edge  $e \in E$  is the operation of removing the edge  $e$ . If  $H$  is the graph obtained from  $G$  by sequence of edge deletions and contractions, then  $H$  is called a *minor* of  $G$ . The following lemma says that edge searching is closed under taking minors.

**Lemma 3** ([15]). *If  $H$  is a minor of  $G$ , then  $s(H) \leq s(G)$ .*

However, a similar lemma does not hold for fast searching [7] and this forces us to establish different ground rules of characterizational discussions.

A biconnected graph  $G$  is *outerplanar* if it has a planar embedding in which the infinite face includes all of its vertices. Consider an outerplanar embedding of a biconnected planar graph  $G$ . We say that  $uv \in E$  is a *boundary edge* if it is incident to the infinite face. All the remaining edges of  $G$  are called *internal edges*. We fix an outerplanar embedding of  $G$ , in order boundary edges and internal edges to be well defined.

A *boundary path* in  $G$  is a path containing only boundary edges. Note that a boundary path is not necessarily an induced path. Note that the biconnectedness of  $G$  implies that the boundary edges form a cycle. If  $uv$  and  $u'v'$  are different boundary edges then a boundary path  $P$  is *connecting* them if it is one of the two boundary paths formed from removing  $uv$  and  $u'v'$  from the set of all boundary edges. Here the term path means a simple graph, i.e. a path does not contain parallel edges.

A *chord* of a path  $P$  in a graph  $G$  is an edge  $e \in E(G) \setminus E(P)$  between two vertices of  $P$ . Two chords  $e, e'$  of  $P$  are *nested* if the graph  $(V(P), E(P) \cup \{e, e'\})$  is contractible to a multigraph with two vertices and three parallel edges between them.

Two edges  $e, e' \in E$  are called *opposing poles* of  $G$  if the two boundary paths connecting  $e$  and  $e'$  have no nested chords. An edge  $e \in E$  is a *pole* if there exists  $e' \in E$  such that  $e$  and  $e'$  are opposing poles. If such two edges exist or  $G$  consists of a single edge, then we say that  $G$  is *bipolar*.

**Theorem 4** ([14]). *If  $G$  is a reduced biconnected multigraph then the search number of  $G$  is at most 3 if and only if  $G$  is outerplanar and bipolar.*

Since every fast search strategy is also a search strategy, by **Theorem 4** we have the following result.

**Corollary 5.** *If a reduced biconnected graph  $G$  is 3-fast searchable then  $G$  is bipolar and outerplanar.  $\square$*

To be able to conclude that a property shown for the set of start vertices is also true for the set of end vertices in some search strategy, we use the concept of reversibility.

**Definition 1** ([16]). Given a search strategy  $S$ , define the *reverse* of a step  $a \in S$ , denoted by  $a^{-1}$ , as follows:

- If  $a$  is 'slide  $\sigma_i$  from  $v$  to  $u$ ' then  $a^{-1}$  is 'slide  $\sigma_i$  from  $u$  to  $v$ '.
- If  $a$  is 'remove  $\sigma_i$  from  $v$ ', then  $a^{-1}$  is 'place  $\sigma_i$  on  $v$ '.
- If  $a$  is 'place  $\sigma_i$  on  $v$ ', then  $a^{-1}$  is 'remove  $\sigma_i$  from  $v$ '.

Given a search strategy  $S = (a_1, a_2, \dots, a_l)$  that uses  $l$  steps, define the *inverse* of  $S$ , denoted by  $S^{-1}$  to be  $S^{-1} = (a_l^{-1}, a_{l-1}^{-1}, \dots, a_1^{-1})$ .

We say that a search strategy  $S$  is *reversible* if  $S^{-1}$  is a search strategy. Lemma 3 in [16] proves that monotonic search strategies are reversible. However, their proof can be easily generalized to general search strategies on general graphs. Moreover, if  $S$  is a fast search strategy, then an operation of sliding a searcher along an edge  $e$  appears in  $S^{-1}$  exactly once for each edge  $e$ . This gives us the following lemma.

**Lemma 6.** *If  $S$  is a fast search strategy that uses  $k$  searchers for a multigraph  $G$ , then  $S^{-1}$  is a fast search strategy that uses  $k$  searchers for  $G$ .  $\square$*

**Lemma 7.** *If  $G$  is a reduced biconnected 3-fast searchable multigraph then for any fast search strategy with 3 searchers, the set  $X$  of start (end) vertices is such that  $|X| = 1$ , or  $|X| = 2$  and the two vertices in  $X$  are adjacent.*

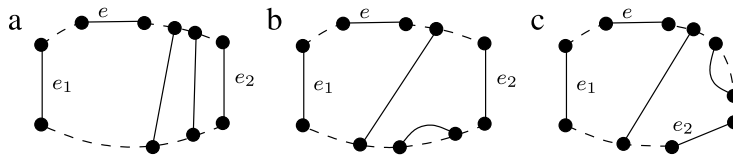


Fig. 1. Nested chords of a boundary path between  $e$  and  $e_1$ .

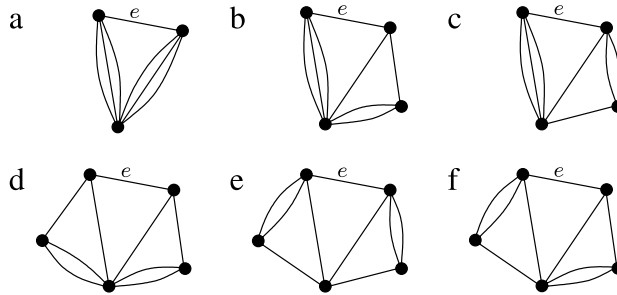


Fig. 2. The graphs  $H$  such that  $G$  is contractible to  $H$ .

**Proof.** Assume that  $S = (s_1, \dots, s_l)$  is a search strategy for  $G$ . Assume for a contradiction that the set of start vertices  $X$  is of size 3. When the first move of sliding a searcher  $\sigma_i$  from  $u$  to  $v$  occurs, then  $\deg_G(u) = 1$ , because no other searcher is at  $u$ . This contradicts the fact that  $G$  is biconnected. Hence  $|X| \leq 2$ .

Let the initial placement of the searchers be  $u = b(\sigma_1) = b(\sigma_2) \neq b(\sigma_3) = v$  and suppose that there is no edge between  $u$  and  $v$ . As before,  $\sigma_3$  cannot change his position until another searcher reaches  $v$ . Then, after either of  $\sigma_1$  or  $\sigma_2$  move, they are on vertices of degree at least three since  $G$  is reduced. Each of those two vertices must be incident to at least two contaminated edges, and hence unable to move. Thus, the edges incident to  $v$  can never be clear, a contradiction. Therefore there is an edge between  $u$  and  $v$ . By Lemma 6, the property also holds for the set of end vertices.  $\square$

**Lemma 8.** *If  $G$  is a reduced biconnected outerplanar bipolar 3-searchable multigraph then for each monotonic search strategy using 3 searchers we have  $b(\sigma_i) \in e$  for each  $i = 1, 2, 3$  and  $t(\sigma_i) \in e'$  for each  $i = 1, 2, 3$ , where  $e, e'$  are some opposing poles of  $G$ . Furthermore,  $e$  is the first cleared edge and  $e'$  is the last.*

**Proof.** By Lemma 7 the set  $X$  of start vertices has size at most 2. If all the searchers start at the same vertex  $u$  then the first move of a searcher results in a situation when one edge  $e$  incident to  $u$  is clear and both end vertices of  $e = uv$  are occupied by the searchers. If the set of starting vertices contains more than one vertex then by Lemma 7,  $u = b(\sigma_1) = b(\sigma_2) \neq b(\sigma_3) = v$  and  $uv \in E$ . Then, if  $e = uv$  is not the first cleared edge, then all searchers get stuck after the first move (as  $G$  is a reduced graph).

Thus, we let  $e = uv$  be the first cleared edge. We prove that in both cases  $e$  is a pole of  $G$ .

Fix an outerplanar embedding of  $G$ . Assume that  $e$  is not a pole. Let  $e_1, e_2$  be two poles of  $G$ . We consider two cases and in both of them we obtain a contradiction by proving that 3 searchers are not sufficient to clear  $G$ .

Case 1:  $e$  is a boundary edge. The edges  $e$  and  $e_1$  are not opposing poles. Let  $P_1, P_2$  be the two boundary paths connecting  $e$  and  $e_1$  in  $G$ . One of these paths contains  $e_2$ . Assume without loss of generality that  $P_2$  contains  $e_2$ . Note that  $P_1$  cannot have nested chords because then a boundary path connecting  $e_1$  and  $e_2$  would have nested chords. Thus,  $P_2$  has two nested chords and one of these chords has a subpath that contains  $e_2$  between its end vertices.

All possible configurations of chords of  $P_2$  are depicted in Fig. 1.

Similarly, since  $e$  and  $e_2$  are not opposing poles and since we consider a planar embedding of  $G$ , the pairs of chords related to the pairs  $e, e_1$  and  $e, e_2$  are edge-disjoint. Then every such graph contains one of the graphs shown in Fig. 2 as a minor. Call this minor  $H$ .

It is easy to see that it is impossible to search  $H$  if the three searchers are initially placed on the vertices incident to  $e$  and  $e$  is the first cleared edge. By Lemma 3, there exists no 3-search strategy that clears  $G$ , such that the first edge cleared is  $e$ . Thus, no 3-fast search strategy for  $G$  starting by clearing  $e$  exists. This leads to a contradiction.

Case 2:  $e$  is an internal edge. Let  $P_1$  and  $P_2$  be the two different boundary paths connecting the end vertices of  $e$ . Assume, without loss of generality, that  $e_i \in E(P_i)$  for  $i = 1, 2$ . For each  $i = 1, 2$ ,  $P_i$  has a chord, because otherwise either  $G$  is not a reduced graph or  $e$  is a boundary edge. There are two possibilities when we take into consideration the position of the chord of  $P_i$  with respect to the pole  $e_i$ . Both possibilities in the case of  $P_2$  are depicted in Fig. 3(a). As before, each such graph contains one of the graphs in Fig. 3(b) as a minor  $H$ . Then it is easy to see that, by Lemma 3, it is not possible to clear  $G$  when all the searchers are initially placed at the vertices of  $e$ . The situation for  $P_1$  is analogous.

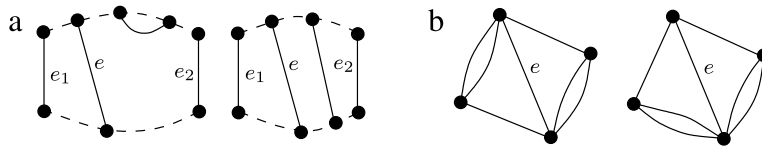


Fig. 3. (a) A graph  $G'$  with two possible positions of the chord of  $P_2$ ; (b) possible graphs  $H$  such that  $G'$  is contractible to  $H$ .

In this way we showed that in each 3-fast search strategy  $b(\sigma_i) \in e$  for some pole  $e$  of  $G$ . Since each search strategy is reversible, we have  $t(\sigma_i) \in e'$  for a pole  $e'$  of  $G$ . It is easy to see that  $e$  and  $e'$  must be opposing poles.  $\square$

We define *starting* and *ending poles* to be a pair of opposing poles in a biconnected bipolar 3-fast searchable graph from which the searchers in a 3-fast search strategy start and end. Thus  $e$  and  $e'$  from the statement of Lemma 8 is a pair of starting and ending poles.

**Lemma 9.** *If  $G$  is a reduced biconnected 3-fast searchable graph then*

1. *if all vertices of  $G$  are even, then  $G$  contains a pair of adjacent starting and ending poles;*
2. *if  $G$  has an odd vertex, then there are exactly two odd vertices, one of which is a vertex of a starting pole and the other is a vertex of a corresponding ending pole.*

**Proof.** Let  $G$  be a reduced biconnected 3-fast searchable graph, and  $e$  and  $e'$  be the starting and ending poles of  $G$ . By Lemma 1,  $|V_o| \leq 6$ . On the other hand by Lemma 8, the searchers start and end at the poles, thus  $V_o \subseteq e \cup e'$ , and thus  $|V_o| \leq 4$ . Since the number of odd vertices must be even, we must have 0, 2 or 4 odd vertices. If the poles are incident, then  $|V_o| \leq 3$ , and hence  $|V_o| \leq 2$ . Assume that the poles are not incident and  $|V_o| = 4$ . Then, each vertex of each pole must be odd. Lemma 8 implies that each pole contains a vertex that is initially occupied by even number of searchers. This is a contradiction. Thus, when the poles are not incident  $|V_o| \leq 2$ .

Assume that all the vertices of  $G$  have even degree. There exists a vertex  $u \in e$  such that 1 or 3 searchers start at  $u$ . This vertex has even degree only if  $t(\sigma_i) = u$  for some  $i \in \{1, 2, 3\}$ . Thus  $e' = uv'$  for some  $v' \in V$ . Therefore  $e$  and  $e'$  are adjacent and we have the first part.

Finally assume that there are two vertices of odd degree in  $G$ . Since an odd degree vertex is a start or an end vertex in each fast search strategy, by Lemma 8 we have that both of these vertices belong to  $e \cup e'$ . Let  $e = uv$ . If only one of  $u$  or  $v$  is odd, then  $e'$  contains the second odd vertex and the statement follows. Otherwise, either  $e$  or  $e'$  contains both of the odd vertices. Without loss of generality we can consider the case when both  $u$  and  $v$  are of odd degree. We have that the number of searchers that start at one of the vertices in  $e$ , say  $u$ , is even. Then,  $\deg_G(u)$  is odd if and only if  $t(\sigma_i) = u$  for some  $i \in \{1, 2, 3\}$ . This means that  $u \in e'$ . So, each pole has at least one vertex of odd degree and the lemma follows.  $\square$

The following theorem completes the classification of biconnected 3-fast searchable graphs. Together with Corollary 11 they give the characterization of start and end vertices in a 3-fast search.

**Theorem 10.** *Let  $G$  be a reduced biconnected graph. Then  $G$  is 3-fast searchable if and only if  $G$  is outerplanar, bipolar and satisfies*

$$(V_o = \emptyset \text{ and } e \cap e' \neq \emptyset) \text{ or } (|V_o| = 2, V_o \subseteq e \cup e', e \cap V_o \neq \emptyset \text{ and } e' \cap V_o \neq \emptyset) \tag{1}$$

for some opposing poles  $e$  and  $e'$ . Moreover, for each 3-fast search strategy for  $G$ ,  $\exists e, e' \in E$  such that the following holds:

$$\text{for } v \in e \setminus e', \quad \deg_G(v) \Leftrightarrow |\{\sigma_i : b(\sigma_i) = v, i = 1, 2, 3\}| \text{ is even, and} \tag{2}$$

$$\text{for } u \in e' \setminus e, \quad \deg_G(u) \Leftrightarrow |\{\sigma_i : t(\sigma_i) = u, i = 1, 2, 3\}| \text{ is even.} \tag{3}$$

**Proof.** If a reduced biconnected graph  $G$  is 3-fast searchable then, by Corollary 5 and by Lemma 9,  $G$  is outerplanar, bipolar and Condition (1) holds.

Let us prove the reverse implication by constructing a 3-fast search strategy for  $G$ . First, fix an outerplanar embedding of  $G$ . Since  $G$  is a reduced biconnected outerplanar bipolar graph, [14] tell us that  $G$  is 3-searchable. Then by Lemma 8, there exist two opposing poles  $e = u_1 v_1$  and  $e' = u_{n_1} v_{n_2}$  such that  $b(\sigma_i) \in e$  and  $t(\sigma_i) \in e'$  for each  $i = 1, 2, 3$ . Let  $P_1$  and  $P_2$  be the two disjoint boundary paths connecting  $e$  and  $e'$ . Let  $P_1$  and  $P_2$  contain the vertices  $u_1, \dots, u_{n_1}$  and  $v_1, \dots, v_{n_2}$ , respectively. Assume without loss of generality that  $u_1 \notin \{u_{n_1}, v_{n_2}\}$ , i.e. if the poles share a vertex then this vertex is  $v_1$ .

All possible configurations of the poles (and initial and final placements of searchers) are summarized in Fig. 4. The vectors  $(m; n)$  and  $(k; l)$  at the vertices  $u_1$  and  $v_1$ , respectively, characterize the possible initial configurations:  $m$  searchers start at  $u_1$  while  $k$  searchers start at  $v_1$  or  $n$  searchers start at  $u_1$  while  $l$  searchers start at  $v_1$ . Note that it must hold that  $m + k = n + l = 3$ .

We will restrict our discussion to Case (a), the other cases follow in a similar fashion. Note that the parity of the vertices in edge  $e$  in Fig. 4(d) is independent of the fact which vertex in the pole  $u_{n_1} v_{n_2}$  is of odd degree.

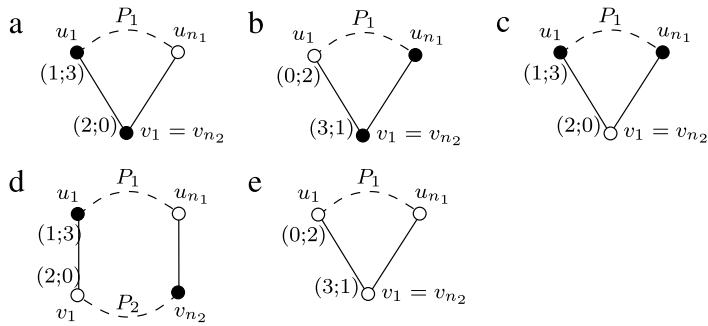


Fig. 4. All possible configurations of the poles (filled vertices are of odd degree).

Now we prove that there exists a 3-fast search strategy for  $G$ . We use induction on  $i + j$  to show that if all the edges of the graph

$$G' = G[\{u_1, \dots, u_i, v_1, \dots, v_j\}] \tag{4}$$

have been cleared, then we can extend the cleared graph by adding one of the vertices  $u_{i+1}, v_{j+1}$  (and its corresponding edges) to the clear part of  $G$ . First we show that  $G[\{u_1, v_1\}]$  can be cleared. We first clear the edge  $e$ . If  $b(\sigma_1) = u_1$  and  $b(\sigma_2) = b(\sigma_3) = v_1$ , then move  $\sigma_3$  along  $e$  to  $u_1$ , clearing  $e$ . If  $b(\sigma_1) = b(\sigma_2) = b(\sigma_3) = u_1$ , move  $\sigma_2$  along  $e$  to  $v_1$ , clearing  $e$ . In either case,  $\sigma_3$  next clears all the edges between  $u_1$  and  $v_1$  parallel to  $e$ . In this way  $G[\{u_1, v_1\}]$  has been cleared.

Moreover,  $\sigma_1$  occupies  $u_1$  while  $\sigma_2$  occupies  $v_1$ , and those searchers will follow the paths  $P_1$  and  $P_2$ , respectively, during the search.

Assume that  $G'$  in Eq. (4) is clear. Since  $G$  is outerplanar and bipolar, we have the possibilities:

- (1) All the contaminated edges incident to  $u_i$  are incident to  $u_{i+1}$ . Assume first that  $\sigma_3$  is at  $u_i$ . We have  $\mu(u_i, u_{i+1}) \leq 2$ , because otherwise the graph has a pair of nested chords. First we show that there are exactly two edges between  $u_i$  and  $u_{i+1}$ .  
 If  $i = 1$  then it follows from how we clear  $G[\{u_1, v_1\}]$  that there are exactly two searchers at  $u_1$  if and only if  $\mu(u_1, v_1)$  is odd. Since we consider the case where  $u_1$  has odd degree and since the only neighbors of  $u_1$  are  $v_1$  and  $u_2$ , we must have that  $\mu(u_i, u_{i+1})$  is even. Since nested chords are not allowed,  $\mu(u_i, u_{i+1}) = 2$ .  
 If  $i > 1$  then, by assumption,  $\deg_G(u_i)$  is even. Moreover,  $\deg_{G'}(u_i)$  is also even, because each time a searcher visited and left the vertex  $u_i$ , two edges incident to  $u_i$  were cleared, and two searchers occupy  $u_i$ , which means that each of them cleared one edge incident to  $u_i$  when reaching  $u_i$  last time. So, an even number of cleared edges is incident to  $u_i$ , and consequently, an even number of contaminated edges is incident to  $u_i$ . This gives that there are exactly two contaminated edges and, by assumption, those edges are between  $u_i$  and  $u_{i+1}$ . Thus, both of the searchers at  $u_i$  can proceed to  $u_{i+1}$ . Then,  $\sigma_1$  remains at  $u_{i+1}$  and  $\sigma_3$  clears all edges, if any, between  $u_{i+1}$  and  $v_j$ .  
 We can prove similarly that if  $\sigma_3$  is at  $v_j$  then  $\mu(u_i, u_{i+1}) = 1$ , and  $\sigma_2$  can slide to  $u_{i+1}$ . Then  $\sigma_3$  clears all edges between  $v_j$  and  $u_{i+1}$ .
- (2) There exist contaminated edges connecting  $u_i$  and  $v_{j'}$  for some  $j' > j$ . Since  $G$  is outerplanar and bipolar no edge  $v_j u_{i'}$ ,  $i' > i$ , is possible and the situation is analogous to (1).

At this point, we have shown that a 3-fast search strategy exists for all graphs corresponding to (a) in Fig. 4. Now we show that statements (2) and (3) are true, again for case (a). Assume, by way of contradiction, that an even number of searchers begin at  $u_1$ . That is, either two or zero searchers begin there. Then, every time a searcher enters and leaves  $u_1$ , he clears an even number of edges. Since the  $\deg_G(u_1)$  is odd, to clear all incident edges, at least one searcher must finish the search at  $u_1$ . However, by Lemma 8, no searcher may end at  $u_1$ , a contradiction. Again, similar arguments suffice for the other cases of Fig. 4.  $\square$

Since the search strategies described in Theorem 10 differ only by their first move, we have the following result.

**Corollary 11.** *If  $G$  is a reduced biconnected 3-fast searchable graph with a pair of starting and ending poles that are not adjacent, then the initial placement of the searchers is independent of the final configuration of the searchers.*  $\square$

Since any 3-fast searchable graph is 3-searchable, a complete characterization follows in the same manner as in Theorem 6 of [14], though in this case, the statement of such a theorem becomes even less pleasant. In Theorem 10 we have characterized the 3-fast searchable biconnected components of any 3-fast searchable graph.

#### 4. NP-completeness of fast searching

In this section, we show that the fast searching problem is NP-complete. This result has been independently obtained by Yang in [17] by a direct reduction from node searching.



We introduce a new model called *weak searching* and prove that it is NP-complete, by reducing the node searching problem to weak searching. Then, we give a reduction from the weak searching problem to the fast searching problem. We consider only monotonic search strategies, so the following definitions do not allow any recontamination.

**Definition 2.** Given a graph  $G$ , a search strategy  $S$  is called a *monotonic  $k$ -node search strategy* if it satisfies the following:

- In the initial state all the edges of  $G$  are contaminated while in the final state all the edges of  $G$  are clear.
- Both in initial and final states no vertex of  $G$  is occupied by a searcher.
- Each step of  $S$  is one of the following.
  - (i) (*placing a searcher*) Place a searcher at an arbitrary vertex of  $G$ . No more than  $k$  searchers are occupying the vertices of  $G$  at each step.
  - (ii) (*removing a searcher*) Remove a searcher from a vertex  $v$  of  $G$ . This operation is allowed only if all the edges incident to  $v$  are clear or if there is another searcher located at  $v$ .
- An edge of  $G$  becomes clear whenever both of its end vertices are occupied by a searcher.

Notice that searching a multigraph does not differ from searching an underlying simple graph where parallel edges are replaced with a single edge.

**Definition 3.** Given a graph  $G$  a *monotonic  $k$ -weak search strategy*  $S$  is defined as follows: The initial and final states are as in a node search strategy. Each step of  $S$  consists of one of the following:

- (i) (*placing a searcher*) Place a searcher at an arbitrary vertex of  $G$ . No more than  $k$  searchers are occupying the vertices of  $G$  at the end of this step.
- (ii) (*removing a searcher*) Remove a searcher from a vertex  $v$  of  $G$ . This operation is allowed only if all the edges incident to  $v$  are clear or if there is another searcher located at  $v$ .
- (iii) (*clearing an edge*) If at least two searchers are at  $u$  and at least one searcher is at  $v$  at the beginning of this step then clear an edge  $uv$  by sliding one of the searchers occupying  $u$  along the edge  $uv$ . The searcher moved along  $uv$  stays at  $v$  at the end of the step.

Both models of searching we introduced above are monotonic, so we may omit the term “monotonic” when we mention these models. Note that in both the node and weak search strategy we consider the graph as searched once all the edges have been cleared and all searchers have been removed from the graph.

Using the equivalence of the pathwidth and node searching problems [12] we have the following.

**Theorem 12 ([10]).** *Given an integer  $k$  and a graph  $G$ , the problem of deciding whether a monotonic  $k$ -node search for  $G$  exists is NP-complete.*

**Theorem 13.** *Given a graph  $G$  and an integer  $k$ , there exists a monotonic  $k$ -node search for  $G$  if and only if there exists a monotonic  $(k + 1)$ -weak search for  $G$ .*

**Proof.** Let  $S$  be a  $k$ -node search for  $G$  using searchers  $\sigma_1, \dots, \sigma_k$ . Define a  $(k + 1)$ -weak search  $S'$  for  $G$  as follows. Initial (empty) states of  $S$  and  $S'$  are identical. If in the  $j$ th step of  $S$  a searcher  $\sigma_i$ ,  $i \in \{1, \dots, k\}$ , is placed on (removed from)  $v \in V(G)$ , then we extend  $S'$  by placing  $\sigma_i$  on  $v$  (removing  $\sigma_i$  from  $v$ , respectively). Moreover, if a set  $X = \{vv_1, vv_2, \dots, vv_l\}$  of edges gets clear in  $S$  as a result of placing a searcher at  $v$ , then each  $v_i$ ,  $i = 1, 2, \dots, l$ , must contain a searcher. In  $S'$  we clear the edges in  $X$  by adding at most  $3|X|$  moves: select an edge  $vv_i \in X$ , place the searcher  $\sigma_{k+1}$  on  $v$ , slide it from  $v$  to  $v_i$ , remove it from  $v_i$ . Then remove  $uv_i$  from  $X$ . Continue repeating this procedure until all edges in  $X$  are cleared.

Assume now that  $S'$  is a  $(k + 1)$ -weak search for  $G$ . Observe, that if in the  $j$ th step of  $S'$   $\sigma_i$  is placed on  $v$  and this results in a situation when the  $k + 1$  searchers are occupying  $k + 1$  pairwise different vertices then in the  $(j + 1)$ th step the only allowed operation is removing a searcher from a vertex. So, we can exchange these operations. After a finite number of such modifications we obtain a  $(k + 1)$ -weak search  $S''$  for  $G$ . Then  $S''$  has the property that in each step at most  $k$  different vertices are occupied by searchers. If we remove from  $S''$  all the operations of clearing edges and the operations of placing a searcher on a vertex occupied by another searcher, then we get the desired  $k$ -node search for  $G$ .  $\square$

Now we describe a polynomial-time reduction from the weak searching problem to the fast searching problem. Let  $k$  be an integer and  $G = (V, E)$  be a (simple) graph such that  $G$  and  $k$  constitute an instance of the monotonic  $k$ -weak search problem. Define  $G^{(t)}$  as a multigraph with  $t$  parallel edges between each pair of vertices that are adjacent in  $G$  where  $t = 2(|V(G)| + 1) \cdot |E(G)|$ . The input to the fast searching problem is a graph  $\tilde{G}$  constructed as follows. We take  $2k$  vertex disjoint copies of  $G^{(t)}$ , denoted by  $G_1^{(t)}, \dots, G_{2k}^{(t)}$ , and we introduce another vertex  $v_0$ . For each  $v \in V(G_1^{(t)}) \cup \dots \cup V(G_{2k}^{(t)})$  there are  $t$  parallel edges between  $v_0$  and  $v$ . Note here that the size of  $\tilde{G}$  is polynomial in the size of  $G$ , since we assume without loss of generality that  $k \leq |V(G)| + 1$ .

**Theorem 14.** *Let  $G$  be a connected graph and let  $k \geq 2$ . There exists a  $k$ -weak search for a  $G$  containing at least one edge if and only if there exists a  $(k + 1)$ -fast search for  $\tilde{G}$ .*

**Proof.** Let  $S$  be a  $k$ -weak search strategy for  $G$  with searchers  $\sigma_1, \sigma_2, \dots, \sigma_k$ . We make five assumptions concerning  $S$ . Note that none of the assumptions on  $S$  made below increases the number of searchers used.

1. We do not move a searcher along a clear edge  $e = uv$ . Such an operation can be replaced by two steps: removing the searcher from  $u$  and placing it on the adjacent vertex  $v$ .
2. We do not place a searcher on a clear vertex (a vertex with no dirty edges incident to it). Such a move is not necessary, since we consider only monotone strategies, and moving a searcher along a clear edge is forbidden. This means that when all the edges adjacent to a vertex  $v$  are cleared we place no more searchers at  $v$  and we remove (not necessarily immediately) the searchers at  $v$  in the forthcoming steps.
3. All edges are cleared by the searcher  $\sigma_k$ . As in the proof of [Theorem 13](#), we know that no more than  $k - 1$  vertices ever contain searchers at any time. Thus, there is always one searcher “free”.
4. No vertex ever contains more than 2 searchers. (Prior to such an event, surplus searchers may be removed.)
5. When  $\sigma_k$  clears an edge  $uv$ , the move immediate before was placing  $\sigma_k$  at  $u$ , and the move immediately following is to remove  $\sigma_k$  from  $v$ .

We will construct a fast search strategy  $S'$  for  $\tilde{G}$  using  $k + 1$  searchers. The initial placement of the searchers in  $S'$  is  $b(\sigma_i) = v_0$  for each  $i = 1, \dots, k + 1$ .  $S'$  proceeds in such a way that  $v_0$  is occupied by  $\sigma_{k+1}$  during the entire search strategy and the subgraphs  $\tilde{G}[\{v_0\} \cup V(G_i^{(t)})]$  are cleared one by one. So, we only show how to clear  $\tilde{G}[\{v_0\} \cup V(G_1^{(t)})]$ . The strategy  $S'$  for this subgraph reflects the moves done in  $S$  as follows:

1. If in  $S$  a searcher  $\sigma_i$  has been placed on a vertex  $v \in V(G)$ , then in  $S'$  we slide  $\sigma_i$  from  $v_0$  to  $v \in V(G_1^{(t)})$ .
2. If in  $S$  a searcher  $\sigma_i$  ( $i \neq k$ ) has been removed from  $v \in V(G)$ , then in  $S'$  we slide  $\sigma_i$  from  $v \in V(G_1^{(t)})$  to  $v_0$ .
3. Assume that in  $S$  a searcher  $\sigma_k$  slides along an edge  $uv \in V(G)$  from  $u$  to  $v$ . In the search strategy  $S'$ , the searcher  $\sigma_k$  slides from  $v_0$  to  $u$  (and in this way clears one of the parallel edges between  $v_0$  and  $u$ ). Then,  $\sigma_k$  clears all the edges between  $u$  and  $v$  in  $\tilde{G}$ . As a result,  $\sigma_k$  ends at  $u$ , because there are  $t$  edges between  $u$  and  $v$  in  $G_1^{(t)}$ , and  $t$  is even. Then  $\sigma_k$  slides to  $v_0$ . Moreover, if all the edges adjacent to  $u$  (respectively  $v$ ) in  $G$  are clear then we also use  $\sigma_k$  to clear all the edges between  $u$  ( $v$ , resp.) and  $v_0$  of  $G_1^{(t)}$ . Note that after performing all above moves the searcher  $\sigma_k$  occupies  $v_0$ , because  $\mu(v_0, x)$  is even for each  $x \in V(G_1^{(t)})$ .

Due to the choice of  $t$  the construction of  $S'$  is correct. In particular, note that after clearing  $G_1^{(t)}$ , the searchers end at  $v_0$ , ready to clear  $G_2^{(t)}$ , and so on.

Assume now that  $S'$  is a  $(k + 1)$ -fast search strategy for  $\tilde{G}$ . We will construct a  $k$ -weak search strategy for  $G$ . Since there are  $2k$  subgraphs  $G_j^{(t)}$ , initially at least  $k$  of them have all the vertices unoccupied by the searchers in the strategy  $S'$ . (Otherwise, no edge could be cleared in  $S'$ .) For any such unoccupied subgraph, we know that when a searcher first reaches a vertex of  $G_j^{(t)}$  then another searcher, say  $\sigma_{k+1}$ , is at  $v_0$ , because no recontamination is allowed. Without loss of generality, assume that the unoccupied subgraphs are  $G_1^{(t)}$  to  $G_k^{(t)}$ .

Let  $\sigma_{k+1}$  be the last searcher leaving  $v_0$  (if this ever happens in  $S'$ ). Suppose that none of  $G_1^{(t)}$  through  $G_k^{(t)}$  have been completely cleared at this point. Then, since there is at least one contaminated edge in each of these copies of  $G$ , there must be at least two searchers present in each of the  $k$  subgraphs, accounting for  $2k > k + 1$  searchers, a contradiction. Thus, one of these subgraphs must have been completely cleared while the searcher  $\sigma_{k+1}$  remained at  $v_0$ . In the following we assume that this distinguished subgraph is  $G_1^{(t)}$ .

We will construct a  $k$ -weak search strategy  $S$  for  $G$ . Remember that, according to the definition, initially no vertices of  $G$  are occupied by the searchers in  $S$ . Since  $\tilde{G}$  is a multigraph with exactly  $t$  edges between each pair of adjacent vertices  $u, v$  we know that there exists a step in  $S'$  when a searcher slides along an edge between  $u$  and  $v$ ,  $u, v \in V(G_1^{(t)})$ , and two other searchers occupy  $u$  and  $v$ . We build  $S$  based on  $S'$  as follows.

1. Assume that  $\sigma_i$  slides from  $u$  to  $v$ ,  $u, v \in V(G_1^{(t)})$ , and after this move  $\sigma_i$  is the only searcher at  $v$ . This means that all the remaining edges between  $u$  and  $v$  in  $G_1^{(t)}$  are still dirty. In  $S$  we add two steps in which  $\sigma_i$  is first removed from  $u$  and then placed at  $v$ .
2. Assume that  $\sigma_i$  slides from  $u$  to  $v$ ,  $u, v \in V(G_1^{(t)})$ , and there is a searcher at each of  $u$  and  $v$ . If this is the first time this has occurred, then in  $S$  we slide  $\sigma_i$  from  $u$  to  $v$ , clearing this edge. Otherwise, we remove  $\sigma_i$  from  $u$  and subsequently place it on  $v$ .
3. Assume that  $\sigma_i$  slides from  $u$  to  $v$ ,  $u, v \in V(G_1^{(t)})$ , and after this move no searcher is at  $u$ . This can only occur if all the other edges between  $u$  and  $v$  in  $G_1^{(t)}$  are clear; thus one of the previous moves has cleared  $uv$  in  $G$ . We add two steps to  $S$ : remove  $\sigma_i$  from  $u$ ; place  $\sigma_i$  at  $v$ .

The above rules imply that the configuration of searchers  $\sigma_1, \dots, \sigma_k$  is identical in  $G_1^{(t)}$  and in  $G$  after the corresponding moves in  $S'$  and  $S$ , respectively. Moreover, after each pair of corresponding steps in the search strategies, when a set of all parallel edges in  $G_1^{(t)}$  is cleared, the corresponding edge in  $G$  is cleared.  $\square$



**Theorem 15.** *The fast searching problem is NP-complete for multigraphs and for simple graphs.*

**Proof.** Theorems 13 and 14 imply the NP-hardness of the decision version of fast searching. Due to monotonicity, the problem is clearly in NP. In order to get the result for simple graphs one can observe that there exists a  $k$ -fast search for a multigraph if and only if there exists a  $k$ -fast search for a simple graph obtained by “dividing” each edge of the multigraph into a path with two edges.  $\square$

## 5. Conclusion and future directions

We investigate some of the classical areas of graph searching as they correspond to the fast search problem; namely, we have considered graphs with small fast search number, and the complexity of the decision problem. The major area that remains unconsidered is the fast search number’s relation to any of the width family of graph parameters. It is well known from [8,11] that the search number of a graph is, essentially, its pathwidth, denoted by  $pw(G)$ . In fact,  $pw(G) \leq s(G) \leq pw(G) + 2$ . As a result, we know that  $pw(G) \leq s(G) \leq s_f(G)$ ; however, this is a terrible lower bound for fast search number, as the ratio  $\frac{s_f(G)}{s(G)}$  is shown to be arbitrarily large [7]. The same example shows that treewidth is equally bad. The “right” width parameter remains to be seen.

Another restriction that is commonly placed on a search strategy is that at every step the set of clear edges induces a connected subgraph; such a strategy is called *connected*. We could apply the same restriction to the fast searching problem, as was briefly discussed in [7]. Investigation of the connected fast search number  $cs_f(G)$  remains wide open.

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