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E. Fatih Yetkin, and Hasan Dağ



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# Sparsity Preserving Computation for Spectral Projectors

E. Fatih Yetkin\* and Hasan Dağ†

\**Istanbul Technical University, Informatics Institute, Istanbul/TURKEY*

†*Kadir Has University, Information Technologies Department, Istanbul/TURKEY*

**Abstract.** Several areas of applications, such as model order reduction, preconditioner design, and eigenvalue problems for spectral projectors can be found in the literature. In this paper, a fast and sparsity preserving approach for computing the spectral projectors is proposed. The suggested approach can be used in both Newton iteration and integral representation based methods. A comparison of the original and the suggested approaches in terms of computation time, sparsity preservation and accuracy is presented in this paper.

**Keywords:** Model Order Reduction, Spectral Projectors, Sparse Approximate Inverse

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## INTRODUCTION

The simulation of large scale dynamical systems requires huge amount of computational power. Thus, reducing the computational cost of these types of simulations deserves attention in the applied mathematics community. A well-known solution method for these types of problems is the model order reduction (MOR), which aims to build an approximate dynamical model of the system under study with an order significantly smaller than that of the original system. There are several approaches for MOR [1]. An important application area of spectral projectors is the eigenvalue based model order reduction. These types of methods are based on determination of the dominant poles of dynamical system for approximation and named as modal approximation generally [2, 3]. To find the dominant eigentriplet of the system matrices, one can use Matrix Sign Function (MSF) iteratively. MSF behaves like its scalar equivalent, which extracts the sign of a real number. Similarly, MSF detects the signs of eigenvalues of a matrix and its output is two blocks of identity matrix. The sizes of the first and the second block matrices correspond to the number of positive and negative eigenvalues of the matrix respectively.

Another possible application area of the spectral projectors is the preconditioner design for iterative solution of a linear equation systems. One can use the spectral projectors to vanish the extremal eigenvalues of an ill-conditioned coefficient matrix to accelerate the solution with Krylov subspace methods like GMRES [4].

In the literature one can find several methods for computation of MSF [5]. The inverse of a square matrix has to be taken explicitly several times in all of these spectral projector computation algorithms. Therefore, the computational cost of these types of algorithms is very high. Most of the time, the system matrices of the large scale dynamical systems are very large and sparse. Thus, the inverse becomes full. Hence, it is a challenging problem to find an efficient and sparsity preserving method for the computation of the MSF of the system matrices. In the present work, the MSF required by the model reduction algorithm proposed in [3] is computed by both the Newton-type methods and the integral form methods with sparse approximate inversion [6]. The results are compared in terms of accuracy, sparsity preservation capability and the computational efficiency.

The rest of the paper is as follows. In the second chapter, some definitions and well known techniques for computation of MSF and the proposed method are given. In the third chapter, some numerical tests are shown. Finally in the last chapter, the concluding remarks and the planned future work are given.

## COMPUTATION OF SPECTRAL PROJECTORS

Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be given and its spectrum be composed of two different parts as  $\Lambda(\mathbf{A}) = \Lambda_1 \cup \Lambda_2$  and  $\Lambda_1 \cap \Lambda_2 = \emptyset$ . Then one can define an  $\mathbf{A}$ -invariant subspace  $\mathbf{Z}$ , which can be called as a spectral projector corresponding to  $\Lambda_1$ . A well-known spectral projector is matrix sign function (MSF). MSF can be defined as  $sign(\mathbf{A}) = 2P(\mathbf{A}) - I$  where the

function  $P$  is defined as a contour integral,

$$P(\mathbf{A}) = \frac{1}{2\pi i} \oint_{\Gamma} (zI - \mathbf{A})^{-1} dz. \quad (1)$$

In (1),  $\Gamma$  can be defined as any closed curve enclosing the desired eigenvalues of matrix  $\mathbf{A}$ . But for MSF,  $\Gamma$  has to be selected to enclose the negative eigenvalues of matrix  $\mathbf{A}$  in complex plane.

There are several methods for computing the MSF [5]. The simplest iterative method is based on the Newton's iteration, which uses the equation  $((\text{sign}(\mathbf{A}))^2 - I) = 0$ . If Newton's method is applied to this equation, one can obtain an iteration, which converges to the  $\text{sign}(\mathbf{A})$ .

$$\mathbf{A}_{k+1} = \frac{1}{2}(\mathbf{A}_k + \mathbf{A}_k^{-1}), \quad \mathbf{A}_0 = \mathbf{A} \quad (2)$$

The iteration is quadratically convergent [5]. To accelerate this iteration, some scaling procedures can be applied. To do this, in each step,  $\mathbf{A}_k$  is replaced by  $\frac{1}{\alpha_k} \mathbf{A}_k$ . Some popular selections for  $\alpha_k$  can be given as determinantal scaling, norm scaling, and Robert's scaling [7].

More general iterative schemes based on Pade approximation can be also found in literature [5]. Halley iteration can be given as an example for Pade type iterations. The iterative scheme of the Halley iteration is given below.

$$\mathbf{A}_{k+1} = \mathbf{A}_k(3I + \mathbf{A}_k^2)(I + 3\mathbf{A}_k^2)^{-1} \quad (3)$$

Once the MSF of matrix  $\mathbf{A}$  is computed, it can be used for the spectral block decomposition of the matrix as shown in Alg. 1.

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**Algorithm 1** BLOCK DECOMPOSITION

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**Require:** Spectral Projector  $\mathbf{S}$

**Ensure:** Diagonal Blocks of  $\mathbf{A}$  matrix

- 1: Compute rank revealing decomposition of the projector,  $S = QR\Pi$
- 2: Build  $A$ -invariant  $\mathcal{S}_1$  subspace from the first  $k$  column of the orthogonal matrix  $Q$
- 3: Compute the below transformation

$$\hat{A} = Q^T A Q = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (4)$$

- 4:  $\Lambda(A_{11}) = \Lambda_1$  and  $\Lambda(A_{22}) = \Lambda_2$
- 

Another way of computing the spectral projector is to compute explicitly the contour integral given in 1. The most appropriate integration method will be the Gauss quadrature [8]. Let us a polygon contour with  $m$  edges as  $\Gamma$ . Then, the integral can be written for each edge with appropriate boundaries and variable changes as,

$$\mathbf{P}(\mathbf{A}) = \frac{1}{2\pi i} \sum_{j=1}^m \int_{z_j}^{z_{j+1}} (zI - \mathbf{A})^{-1} dz \quad (5)$$

After the evaluation of this integral with Gaussian quadrature in a few points, the spectral projector can be used to decompose the matrix spectrum in a similar way with the Alg. 1. It is known that the trace of the  $\mathbf{P}$  is equal to the number of the eigenvalues of the matrix  $\mathbf{A}$  in the preselected contour  $\Gamma$ . Implementation of the spectral projector is shown in the Alg. 2. All of these methods need matrix inversion. To overcome this problem, one can use some special

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**Algorithm 2** BLOCK DECOMPOSITION WITH INTEGRATION

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**Require:** Spectral Projector  $\mathbf{P}$

**Ensure:** Diagonal Blocks of matrix  $\mathbf{A}$

- 1: Compute the trace of  $\mathbf{P}$  and assign it to  $r$ .
  - 2: Compute the SVD decomposition of the projector,  $P = USV^T$
  - 3: Let  $U_r = U(:, 1 : r)$ , and  $U_p = U(:, r + 1 : n)$
  - 4: Compute the transformations  $A_{11} = U_r^H A U_r$  and  $A_{22} = U_p^H A U_p$ .
  - 5:  $\Lambda(A_{11}) = \Lambda_1$  and  $\Lambda(A_{22}) = \Lambda_2$
-

sparse matrix tools like Sparse Approximate Inverse (SPAI) [6]. The SPAI algorithm computes a sparse approximate inverse  $\mathbf{M}$  of a given sparse matrix  $\mathbf{A}$  by minimising  $\|I - \mathbf{AM}\|_F$  provided that the sparsity structure of  $\mathbf{M}$  is the same as the original matrix. To compute MSF of  $\mathbf{A}$ , one can employ the SPAI to produce the approximate inverse in every step of the Newton iteration given in (1) or in every integration point of Gauss quadrature given in the integral (5). Because of the use of approximate inverse instead of exact inverse, while the number of Newton iterations in (1) increases, the accuracy of the integration in (5) decreases. But the computation time in both cases decreases.

Another important advantage of using SPAI to obtain a spectral projector is the preservation of sparsity. Because of the matrix inversion, sparsity structure of matrix  $\mathbf{A}$  is lost. Hence, the computation of the orthogonal bases of the spectral projector needs more computational effort. But if SPAI is employed, the sparsity structure of matrix  $\mathbf{A}$  is completely preserved.

In this work, the effect of the usage of sparse approximate inversion instead of the exact inverse methods is investigated. To do this, matrix inversions in both type of methods (direct integration and iteration schemes) for spectral projector computation is replaced with the sparse approximate inverses produced by the SPAI algorithm.

## NUMERICAL EXAMPLES

To illustrate the effect of SPAI usage on spectral projector computation, some well-known electrical power network data [4] are selected. The dimensions of the matrices are changed from 106 to 530. In the first experiment, the computational time of the algorithms for building spectral projectors are compared. The comparison of the sparsity structures is also given. These comparisons are made for Newton iteration and explicit integration with Gauss quadratures. In the second test, the effect of the SPAI is discussed for each method in terms of the accuracy of the results.

In the first experiment,  $\varepsilon$  parameter for the SPAI method is selected as 0.1.  $\varepsilon$  parameter controls the quality of the approximation of  $\mathbf{M}$  to the inverse of  $\mathbf{A}$ . It has to be between in 0 and 1. Smaller  $\varepsilon$  values corresponds to usually better results [6].

Sparsity structure of the methods are compared according to a measure which can be defined as,

$$sparsity = \frac{nnz(A)}{n^2} \quad (6)$$

where  $nnz$  stands for the number of non-zeros of the  $n \times n$  dimensional matrix  $A$ .

**TABLE 1.** Comparison of the computational time and sparsity structures for various IEEE test matrices

	Newton *	Newton SPAI	Gauss	Gauss SPAI		Newton †	Newton SPAI	Gauss	Gauss SPAI
IEEE 57	1.10	1.86	1.80	0.37	IEEE 57	1	0.075	1	0.076
IEEE 118	3.81	1.40	5.19	0.87	IEEE 118	1	0.043	1	0.055
IEEE 300	65.40	5.04	44.50	3.05	IEEE 300	1	0.014	1	0.006

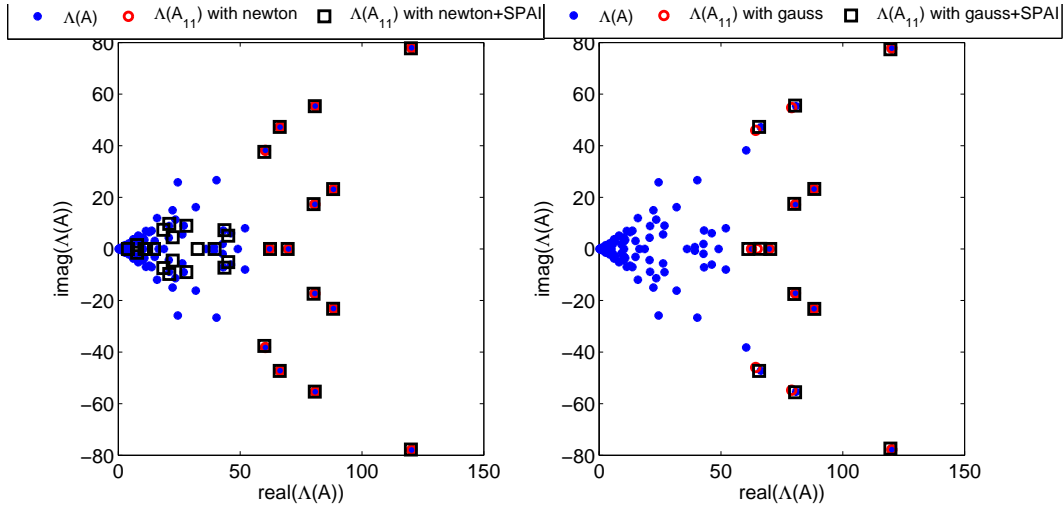
\* computational times in seconds

† sparsity measures

In the second experiment, the methods are compared to each other with respect to their accuracy. To do this, the well known property of the spectral projectors is employed. The trace of a spectral projector gives the number of the eigenvalues of the matrix  $A$  in the predefined domain in complex plane. The eigenvalues of the matrix  $A$  and the eigenvalues computed via spectral projectors with the suggested methods are given in Fig. 1. The traces of the spectral projectors and the exact number of the eigenvalues in selected domains for various examples are given in Table 2. In the experiments, the number of the integration points with Gaussian quadrature based methods is selected as relatively small. So the results are not exactly true as in the Newton iteration based methods. On the other hand, the computed eigenvalues are found as close enough to the original eigenvalues of matrix  $A$  in selected domain of complex plane. The SPAI gives more accurate results with the Gauss quadrature approach. These results can be seen in Fig.1.

**TABLE 2.** Number of the eigenvalues in the selected domain in complex plain with several examples

	Number of eigenvalues	Newton	Newton SPAI	Gauss	Gauss SPAI
IEEE 57	14	14	14	13.14	12.7
IEEE 118	53	53	53.28	53.37	53.35
IEEE 300	23	23	23	24	23.75



**FIGURE 1.** Eigenvalues computed via Newton iteration, Newton iteration with SPAI , Gauss quadrature and Gauss quadrature with SPAI methods.

## CONCLUSIONS AND FUTURE WORK

In this work, a faster and sparsity preserving method for the spectral projector computation is proposed. To realize this aim, sparse approximate inversion algorithm is used with the well-known computational methods for spectral projectors. In the experiments it is shown that the sparse approximate inversion can be an efficient tool to accelerate the computational time. Another important advantage of the SPAI tool is the sparsity preservation. Main reason of using SPAI is to avoid matrix inversion.

The experimental test matrices were selected from the power system matrices. In future work, the proposed method will be tested on different matrices from different disciplines. We are also planning to focus on the optimal selection of the  $\varepsilon$  value for SPAI algorithm and the realization of the algorithm on parallel environments.

## REFERENCES

1. A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, Philadelphia, USA, (2006).
2. J. Rommes, N. Martins, *IEEE Trans. on Power Systems*, **21**, 1218–1226, (2006).
3. P. Benner, E. S. Quintana-Orti, *Dimension Reduction of Large-Scale Systems*, edited by P. Benner, V. Mehrmann, D. C. Sorensen, Springer, 5–48, (2005).
4. H. Dağ, E. F. Yetkin, M. Manguoglu, *IREE Int. Review of Electrical Engineering*, **6**, 1339–1348, (2011).
5. J. , H. Higham, *Functions of Matrices Theory and Computations*, SIAM, Philadelphia, USA, (2008).
6. M. J. Grote and T. Huckle, *SIAM Journal of Scientific Computing*, **18**, 838–853, (1997).
7. Z. Bai, J. Demmel, *Design of a Parallel Nonsymmetric Eigenroutine Toolbox Part I*, Technical Report, UCB/CSD-92-718, UC Berkeley, (1992).
8. W. W. Hager, *Applied Numerical Linear Algebra*, Prentice-Hall, New Jersey, USA, (1988).