Cascaded Lossless Commensurate Line Synthesis

Metin Şengül
Kadir Has University
Electronics Engineering Department
Cibali-Fatih, Istanbul, Turkey
msengul@khas.edu.tr

***Abstract*— In this paper, a synthesis algorithm for cascaded lossless commensurate lines is summarized. The algorithm is based on transfer matrix factorization. Firstly the characteristic impedance of the extracted element is calculated and then after extracting the element, the reflection factor of the remaining network is calculated. This process is repeated until the termination resistance is reached. An example is included, to illustrate the implementation of the algorithm.**

Keywords-synthesis; commensurate lines; lossless networks

I. INTRODUCTION

Especially at microwave frequencies, distributed element networks are preferred. In lots of microwave filter and matching network designs, finite homogenous transmission lines of commensurate lengths have been used [1, 2]. This approach is based on that the distributed networks composed of commensurate lengths of transmission lines could be regarded as lumped element networks under the transformation [3],

$$\lambda = \tanh(p\tau) \tag{1}$$

where $p = \sigma + j\omega$ is the complex frequency and τ is the commensurate delay of the transmission line.

In literature, characteristic impedance of the extracted commensurate line is calculated by using the following formula [4], if $S(\lambda)$ is the given reflection factor,

$$Z_1 = \frac{1+S(1)}{1-S(1)} \tag{2}$$

Then, the reflection factor of the remaining network is [4]

$$S_R(\lambda) = \frac{S(\lambda)-S(1)}{1-S(\lambda)S(1)}\frac{1+\lambda}{1-\lambda}. \tag{3}$$

As can be seen from Eq. (3), to get a degree reduction, the denominator must has a root at $\lambda = -1$, and numerator at $\lambda = +1$. But in the new method, there is no need to find roots at $\lambda = \pm 1$, to get a degree reduction.

In literature, many researches have been worked on the analysis [5-8] and synthesis problem [9-17] of distributed-element networks. For instance in [18], a transformation is proposed, in [19], input impedance function is utilized in synthesis process. But in the synthesis method presented here, the network is thought as a lossless, reciprocal two-port expressed in Belevitch form [20], and directly the reflection factor is used.

In [21], the synthesis algorithm presented here has been proposed. But when the number of transmission lines is increased, the coefficients used in the algorithm get larger. So a coefficient normalization step must be inserted to get more precise element values. In the updated algorithm presented in this paper, this coefficient normalization step has been included.

In the following section, transfer matrix factorization is explained briefly. Then, the modified synthesis algorithm is given. Finally, an example is presented, to illustrate the implementation of the algorithm.

II. TRANSFER MATRIX FACTORIZATION

Canonic form of the scattering transfer matrix $\{T\}$ of a lossless, reciprocal two-port is defined as [1],

$$T = \frac{1}{f}\begin{bmatrix} \mu\, g_* & h \\ \mu\, h_* & g \end{bmatrix}, \tag{4}$$

where $\mu = \frac{f_*}{f} = \pm 1$ is a unimodular constant, g is a strictly Hurwitz real polynomial. These polynomials satisfy the Feldtkeller equation, $gg_* = hh_* + ff_*$ (where "*" represents paraconjugation).

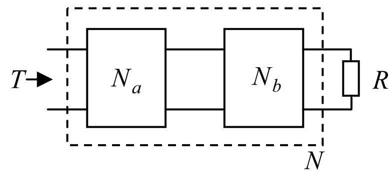

Fig. 1. Cascade decomposition of a two-port.

It is desired to decompose the lossless, reciprocal two-port $\{N\}$ into two cascade connected lossless two-ports $\{N_a, N_b\}$ which are also lossless and reciprocal (Fig. 1). This means to factor the scattering transfer matrix $\{T\}$ into a product of two scattering transfer matrices,

$$T = T_a \cdot T_b \,, \quad (5a)$$

where

$$T_a = \frac{1}{f_a}\begin{pmatrix} \mu_a g_{a*} & h_a \\ \mu_a h_{a*} & g_a \end{pmatrix} \text{ and } T_b = \frac{1}{f_b}\begin{pmatrix} \mu_b g_{b*} & h_b \\ \mu_b h_{b*} & g_b \end{pmatrix}. \quad (5b)$$

The polynomials $\{g_a, h_a, f_a\}$ and $\{g_b, h_b, f_b\}$ have the same properties as $\{g, h, f\}$, and must satisfy the Feldtkeller equation. Namely,

$$g = g_a g_b + \mu_a \, h_{a*} h_b \,, \quad (6a)$$

$$h = h_a g_b + \mu_a \, g_{a*} h_b \,, \quad (6b)$$

$$f = f_a f_b \,, \quad (6c)$$

$$\mu = \mu_a \mu_b \,. \quad (6d)$$

So if one writes $T_b = T_a^{-1} T$, two equations can be reached as,

$$h_b = \frac{h g_a - g h_a}{\mu_a f_a f_{a*}} \,, \quad (7a)$$

$$g_b = \frac{g g_{a*} - h h_{a*}}{f_a f_{a*}} \,. \quad (7b)$$

Now, the problem is to solve (7) in the unknown polynomials $\{g_a, h_a, g_b, h_b\}$ subject to the Feldtkeller equation with g_a and g_b being strictly Hurwitz polynomials.

The problem has been solved by *Fettweis* by using a modified formulation of the factorization problem [22, 23]. In [23], a different set of equations are chosen as the basis for the solution instead of solving (7).

III. Algorithm

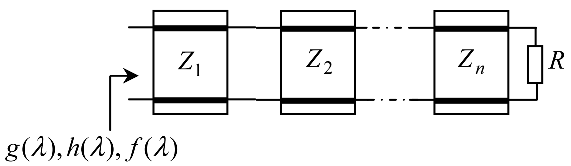

Fig. 2 Commensurate line extraction.

Consider the circuit shown in Fig. 2. $g(\lambda)$, $h(\lambda)$ and $f(\lambda)$ polynomials are given as follows,

$$g(\lambda) = g_0 + g_1\lambda + g_2\lambda^2 + \ldots + g_n\lambda^n \,, \quad (8a)$$

$$h(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \ldots + h_n\lambda^n \,, \quad (8b)$$

$$f(\lambda) = (1-\lambda^2)^{n/2} \,. \quad (8c)$$

Characteristic impedance Z_1 of the first commensurate line can be calculated as,

$$Z_1 = \frac{g(1) + h(1)}{g(1) - h(1)} \,. \quad (9)$$

Then, $g(\lambda)$, $h(\lambda)$ and $f(\lambda)$ polynomials of the remaining network are obtained as,

$$g(\lambda) = \sum_{j=1}^{n} D_j \lambda^{j-1} \,, \quad (10a)$$

$$h(\lambda) = \sum_{j=1}^{n} N_j \lambda^{j-1} \,, \quad (10b)$$

$$f(\lambda) = (1-\lambda^2)^{(n-1)/2} \,, \quad (10c)$$

where

$$D_j = \sum_{i=1}^{j} (-1)^{i+j} y_i \,, \; j = 1,2,\ldots,n \,, \quad (11a)$$

$$N_j = \sum_{i=1}^{j} x_i \,, \; j = 1,2,\ldots,n \,, \quad (11b)$$

where

$$x_i = h_{i-1} g(1) - g_{i-1} h(1) \,, \; i = 1,2,\ldots,n \,, \quad (12a)$$

$$y_i = g_{i-1} g(1) - h_{i-1} h(1) \,, \; i = 1,2,\ldots,n \,. \quad (12b)$$

The coefficients of the polynomials found in (10) may be very large after a few elements are extracted. In the next steps this cause to work with large numbers. As a result, element values may not be calculated. So at this step coefficients must be normalized.

The extraction of commensurate lines is implemented in a similar fashion until the termination resistance (R) is reached.

IV. Example

Example given in [18] is solved, to illustrate the implementation of the proposed algorithm. The given input reflection factor is

$$S(\lambda) = \frac{h(\lambda)}{g(\lambda)}$$

where

$$h(\lambda) = 121.7\lambda^{10} - 136.2\lambda^9 + 209.2\lambda^8 - 151.8\lambda^7 + 111.8\lambda^6 - 52.45\lambda^5 + 21.75\lambda^4 - 6.072\lambda^3 + 1.298\lambda^2 - 0.165\lambda + 0.0105$$

$$g(\lambda) = 121.7\lambda^{10} + 167.9\lambda^9 + 248.8\lambda^8 + 206.7\lambda^7 + 152.6\lambda^6 + 79.44\lambda^5 + 33.76\lambda^4 + 10.31\lambda^3 + 2.29\lambda^2 + 0.316\lambda + 0.0211$$

Step 1 :

$h^{(1)}(\lambda) = h(\lambda)$ and $g^{(1)}(\lambda) = g(\lambda)$.

Step 2 :

$$Z_1 = \frac{g^{(1)}(1) + h^{(1)}(1)}{g^{(1)}(1) - h^{(1)}(1)} = 1.2632$$

Step 3 :

$x_1 = h_0 g^{(1)}(1) - g_0 h^{(1)}(1)$
$= 0.0105 \cdot 1023.8 - 0.0211 \cdot 119.0715 = 8.2379$

$x_2 = h_1 g^{(1)}(1) - g_1 h^{(1)}(1)$
$= -0.165 \cdot 1023.8 - 0.316 \cdot 119.0715 = -206.5597$

$x_3 = h_2 g^{(1)}(1) - g_2 h^{(1)}(1)$
$= 1.298 \cdot 1023.8 - 2.29 \cdot 119.0715 = 1056.3$

$x_4 = h_3 g^{(1)}(1) - g_3 h^{(1)}(1)$
$= -6.072 \cdot 1023.8 - 10.31 \cdot 119.0715 = -7444.4$

$x_5 = h_4 g^{(1)}(1) - g_4 h^{(1)}(1)$
$= 21.75 \cdot 1023.8 - 33.76 \cdot 119.0715 = 18249$

$x_6 = h_5 g^{(1)}(1) - g_5 h^{(1)}(1)$
$= -52.45 \cdot 1023.8 - 79.44 \cdot 119.0715 = -63159$

$x_7 = h_6 g^{(1)}(1) - g_6 h^{(1)}(1)$
$= 111.8 \cdot 1023.8 - 152.6 \cdot 119.0715 = 96295$

$x_8 = h_7 g^{(1)}(1) - g_7 h^{(1)}(1)$
$= -151.8 \cdot 1023.8 - 206.7 \cdot 119.0715 = -180030$

$x_9 = h_8 g^{(1)}(1) - g_8 h^{(1)}(1)$
$= 209.2 \cdot 1023.8 - 248.8 \cdot 119.0715 = 184560$

$x_{10} = h_9 g^{(1)}(1) - g_9 h^{(1)}(1)$
$= -136.2 \cdot 1023.8 - 167.9 \cdot 119.0715 = -159440$

$y_1 = g_0 g^{(1)}(1) - h_0 h^{(1)}(1)$
$= 0.0211 \cdot 1023.8 - 0.0105 \cdot 119.0715 = 20.3527$

$y_2 = g_1 g^{(1)}(1) - h_1 h^{(1)}(1)$
$= 0.316 \cdot 1023.8 + 0.165 \cdot 119.0715 = 343.1793$

$y_3 = g_2 g^{(1)}(1) - h_2 h^{(1)}(1)$
$= 2.29 \cdot 1023.8 - 1.298 \cdot 119.0715 = 2190$

$y_4 = g_3 g^{(1)}(1) - h_3 h^{(1)}(1)$
$= 10.31 \cdot 1023.8 + 6.072 \cdot 119.0715 = 11279$

$y_5 = g_4 g^{(1)}(1) - h_4 h^{(1)}(1)$
$= 33.76 \cdot 1023.8 - 21.75 \cdot 119.0715 = 31975$

$y_6 = g_5 g^{(1)}(1) - h_5 h^{(1)}(1)$
$= 79.44 \cdot 1023.8 + 52.45 \cdot 119.0715 = 87579$

$y_7 = g_6 g^{(1)}(1) - h_6 h^{(1)}(1)$
$= 152.6 \cdot 1023.8 - 111.8 \cdot 119.0715 = 142930$

$y_8 = g_7 g^{(1)}(1) - h_7 h^{(1)}(1)$
$= 206.7 \cdot 1023.8 + 151.8 \cdot 119.0715 = 229700$

$y_9 = g_8 g^{(1)}(1) - h_8 h^{(1)}(1)$
$= 248.8 \cdot 1023.8 - 209.2 \cdot 119.0715 = 229820$

$y_{10} = g_9 g^{(1)}(1) - h_9 h^{(1)}(1)$
$= 167.9 \cdot 1023.8 + 136.2 \cdot 119.0715 = 188120$

Step 4 :

$N_1 = x_1 = 8.2379$

$N_2 = x_2 + x_1 = -206.5597 + 8.2379 = -198.3218$

$N_3 = x_3 + x_2 + x_1 = 1056.3 - 206.5597 + 8.2379 = 857.9450$

$N_4 = x_4 + x_3 + x_2 + x_1 = -7444.4 + 1056.3 - 206.5597$
$+ 8.2379 = -6586.4$

$N_5 = x_5 + x_4 + x_3 + x_2 + x_1 = 18249 - 7444.4 + 1056.3$
$- 206.5597 + 8.2379 = 11662$

$N_6 = x_6 + x_5 + x_4 + x_3 + x_2 + x_1$
$= -63159 + 18249 - 7444.4 + 1056.3 - 206.5597$
$+ 8.2379 = -51497$

$N_7 = x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1$
$= 96295 - 63159 + 18249 - 7444.4 + 1056.3$
$- 206.5597 + 8.2379 = 44798$

$N_8 = x_8 + x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1$
$= -180030 + 96295 - 63159 + 18249 - 7444.4 + 1056.3$
$- 206.5597 + 8.2379 = -135230$

$N_9 = x_9 + x_8 + x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1$
$= 184560 - 180030 + 96295 - 63159 + 18249 - 7444.4 + 1056.3$
$- 206.5597 + 8.2379 = 49329$

$N_{10} = x_{10} + x_9 + x_8 + x_7 + x_6 + x_5 + x_4 + x_3 + x_2 + x_1$
$= -159440 + 184560 - 180030 + 96295 - 63159 + 18249$
$- 7444.4 + 1056.3 - 206.5597 + 8.2379 = -110110$

$D_1 = y_1 = 20.3527$

$D_2 = y_2 - y_1 = 343.1793 - 20.3527 = 322.8266$

$D_3 = y_3 - y_2 + y_1 = 2190 - 343.1793 + 20.3527 = 1867.2$

$D_4 = y_4 - y_3 + y_2 - y_1 = 11279 - 2190 + 343.1793$
$- 20.3527 = 9411.6$

$D_5 = y_5 - y_4 + y_3 - y_2 + y_1 = 31975 - 11279 + 2190$
$- 343.1793 + 20.3527 = 22563$

$D_6 = y_6 - y_5 + y_4 - y_3 + y_2 - y_1$
$= 87579 - 31975 + 11279 - 2190 + 343.1793$
$- 20.3527 = 65016$

$D_7 = y_7 - y_6 + y_5 - y_4 + y_3 - y_2 + y_1$
$= 142930 - 87579 + 31975 - 11279 + 2190$
$- 343.1793 + 20.3527 = 77910$

$D_8 = y_8 - y_7 + y_6 - y_5 + y_4 - y_3 + y_2 - y_1$
$= 229700 - 142930 + 87579 - 31975 + 11279 - 2190$
$+ 343.1793 - 20.3527 = 151790$

$D_9 = y_9 - y_8 + y_7 - y_6 + y_5 - y_4 + y_3 - y_2 + y_1$
$= 229820 - 229700 + 142930 - 87579 + 31975 - 11279$
$+ 2190 - 343.1793 + 20.3527 = 78029$

$D_{10} = y_{10} - y_9 + y_8 - y_7 + y_6 - y_5 + y_4 - y_3 + y_2 - y_1$
$= 188120 - 229820 + 229700 - 142930 + 87579 - 31975$
$+ 11279 - 2190 + 343.1793 - 20.3527 = 110090$

Step 5 :

$$g^{(2)}(\lambda) = D_1 + D_2\lambda + D_3\lambda^2 + D_4\lambda^3 + D_5\lambda^4 + D_6\lambda^5 + D_7\lambda^6 + D_8\lambda^7 + D_9\lambda^8 + D_{10}\lambda^9 = 20.2327 + 322.8266\lambda + 1867.2\lambda^2 + 9411.6\lambda^3 + 22563\lambda^4 + 65016\lambda^5 + 77910\lambda^6 + 151790\lambda^7 + 78029\lambda^8 + 110090\lambda^9$$

$$h^{(2)}(\lambda) = N_1 + N_2\lambda + N_3\lambda^2 + N_4\lambda^3 + N_5\lambda^4 + N_6\lambda^5 + N_7\lambda^6 + N_8\lambda^7 + N_9\lambda^8 + N_{10}\lambda^9 = 8.2379 - 198.3218\lambda + 857.9450\lambda^2 - 6586.4\lambda^3 + 11662\lambda^4 - 51497\lambda^5 + 44798\lambda^6 - 135230\lambda^7 + 49329\lambda^8 - 110110\lambda^9$$

As can be seen from the expressions of $g^{(2)}(\lambda)$ and $h^{(2)}(\lambda)$, the coefficients are very large even at the first step. If the process is repeated, the coefficients will be much larger and it will be impossible to work with these larger numbers. Only seven elements can be extracted in this manner. So at this step, the coefficients must be normalized. After normalizing and applying the algorithm until the termination resistance (R) is reached, the following impedance values are obtained

$Z_1 = 1.2632, Z_2 = 0.5662, Z_3 = 2.3295, Z_4 = 0.3876, Z_5 = 2.7783$
$Z_6 = 0.3564, Z_7 = 2.9046, Z_8 = 0.3453, Z_9 = 2.9743, Z_{10} = 0.3431$
$R = 2.9811.$

Characteristic impedances found in [18] are
$Z_1 = 1.26, Z_2 = 0.566, Z_3 = 2.33, Z_4 = 0.387, Z_5 = 2.79$
$Z_6 = 0.355, Z_7 = 2.92, Z_8 = 0.346, Z_9 = 2.97, Z_{10} = 0.343$
$R = 2.9779.$

V. CONCLUSION

The proposed algorithm in [21] is a simple and powerful method, but can cause large polynomial coefficients, as the number of commensurate lines increases. So it needs a modification; at each step the coefficients must be normalized to be able to calculate all characteristic impedances. As a result, a very simple to implement commensurate line synthesis algorithm is obtained.

ACKNOWLEDGMENT

Fruitful discussions with S.B. Yarman (Istanbul) are gratefully acknowledged.

REFERENCES

[1] A. Aksen. "Design of lossless two-port with mixed, lumped and distributed elements for broadband matching," Ph.D. dissertation, Bochum, Ruhr University, 1994.

[2] A. Sertbaş. Genelleştirilmiş iki-kapılı, iki-değişkenli kayıpsız merdiven devrelerin tanımlanması, Ph.D. dissertation (in Turkish), İstanbul, İstanbul University, 1997.

[3] P. I. Richards, "Resistor-termination-line circuits," *Proc. IRE*, vol. 36, pp. 217-220, Feb. 1948.

[4] H. J. Carlin, "Distributed circuit design with transmission line elements," *Proc. IEEE*. vol. 59, no. 7, pp. 1059-1081, July 1971.

[5] P. S. Castro and W. W. Happ, *Distributed parameter circuits and microsystem electronics*, *Proc. NEC*, pp. 448-461, 1960.

[6] M. J. Hellstrom, "Equivalent distributed RC networks or transmission lines," *IRE Trans. Circuit Theory*, vol. CT-9, pp. 247-251, Sept. 1962.

[7] M. R. Wohlers, *Lumped and distributed passive networks, a generalized and advanced viewpoint*. New York and London: Academic Press, 1969.

[8] M. Ghausi and J. Kelly, *Introduction to distributed-parameter networks.* New York: Krieger, 1977.

[9] K. Heizer, "Distributed RC networks with rational transfer functions," *IRE Trans. Circuit Theory*, vol. CT-9, pp. 356-362, Dec. 1962.

[10] R. W. Wyndrum, Jr., *The exact synthesis of distributed RC networks*, Dept. of Elec. Eng., New York University, N. Y., Tech. Rept. 400-476, May 1963.

[11] J. O. Scanlan and J. D. Rhodes, "Realizability and synthesis of a restricted class of distributed RC networks," *IEEE Trans. Circuit Theory*, vol. CT-12, no. 4, pp. 577-585, Dec. 1965.

[12] R. P. O'Shea, "Synthesis using distributed RC networks," *IEEE Trans. Circuit Theory*, vol. CT-12, no. 4, pp. 546-554, Dec. 1965.

[13] S. S. Penbeci and S. C. Lee, "Synthesis of distributed RC networks by means of orthogonal functions," *IEEE Trans. Circuit Theory*, pp. 137-140, Feb. 1969.

[14] J. O. Scanlan and N. Ramamurty, "Cascade synthesis of distributed RC networks," *IEEE Trans. Circuit Theory*, vol. CT-16, no. 1, pp. 47-57, Feb. 1969.

[15] S. C. Lee and S. S. Penbeci, "Computer-aided design of distributed RC networks," *IEEE Trans. Circuit Theory*, vol. CT-17, no. 2, pp. 224-232, May 1970.

[16] P. Sotiriadis and Y. Tsividis, "Integrators using a single distributed RC elements," *IEEE International Symp. on Circuit and Systems 2002*, vol. 2, pp. 21-24, May. 2002.

[17] P. Sotiriadis and Y. Tsividis, "Single-URC integrators," *IEEE Trans. Circuit and System-I*, vol. 50, no. 2, pp. 304-307, Feb. 2003.

[18] J. Komiak and H. J. Carlin, "Improved accuracy for commensurate-line synthesis," *IEEE Trans. Microwave Theory and Techniques*, pp. 212-215, Apr. 1976.

[19] M. Hyder Ali and R. Yarlagadda, "A note on the unit element synthesis," *IEEE Trans. Circuit and Systems*, vol. cas-25, no. 3, pp. 172-174, March 1978.

[20] V. Belevitch. *Classical network theory*. San Francisco, CA: Holden Day; 1968.

[21] M. Şengül, "Synthesis of cascaded lossless commensurate lines," IEEE Trans. Circuit and Systems: Express Briefs, vol: 55, no: 1, pp: 89-91, Jan. 2008.

[22] A. Fettweis, "Factorization of transfer matrices of lossless two-ports," *IEEE Trans. Circuit Theory*, vol. 17, pp. 86-94, Feb. 1970.

[23] A. Fettweis, *Cascade synthesis of lossless two-ports by transfer matrix factorization*, in R.Boite, ed., Network Theory, Gordon & Breach, 1972, pp. 43-103.