Synthesis of Cascaded Lossless Commensurate Lines

Metin Şengül

***Abstract*—A scattering transfer matrix factorization based algorithm for cascaded lossless commensurate line synthesis is presented. The characteristic impedances of the extracted commensurate lines and the reflection factors of the remaining networks are formulated in terms of reflection factor coefficients of the whole circuit. There is no need to use root search routines so as to cancel common terms, to get degree reduction. The formulation of the method is explained, and an example is included, to illustrate the implementation of the synthesis algorithm.**

***Index Terms*—Lossless circuits, matrix decomposition, network synthesis, transmission line.**

I. Introduction

At microwave frequencies, because of the realization problems of the conventional lumped elements, usually distributed networks composed of transmission lines are required. Based on Richards' Theorem [1], lots of the design methods for microwave filters and matching networks incorporate finite homogenous transmission lines of commensurate lengths [2], [3]. *Richards* showed that the distributed networks composed of commensurate lengths of transmission lines could be treated as lumped element networks under the transformation

$$\lambda = \tanh(p\tau) \tag{1}$$

where $p = \sigma + j\omega$ is the complex frequency and τ is the commensurate one-way delay of the transmission line.

In [4], commensurate line synthesis based on Richards' Theorem is realized in terms of reflection factor as

$$S_R(\lambda) = \frac{S(\lambda) - S(1)}{1 - S(\lambda)S(1)} \frac{1+\lambda}{1-\lambda} \tag{2}$$

where $S(\lambda)$ is the given reflection factor, $S_R(\lambda)$ is the reflection factor of the remaining network. The characteristic impedance of the extracted commensurate line is $Z_1 = (1 + S(1))/(1 - S(1))$. It can be seen that to get a degree reduction, the denominator of (2) must has a root at $\lambda = -1$, and numerator at $\lambda = +1$.

In literature, many researches have been realized about the analysis and classification of distributed-element networks [4]–[7]. Then researchers have investigated the synthesis problem [8]–[19]. For example, in [18], a transformation is proposed to synthesize commensurate line networks. In [19], synthesis involves the extraction of commensurate lines from input impedance function. In the proposed synthesis method, the network is thought as a lossless, reciprocal two-port expressed using only three polynomials $\{g, h$ and $f\}$, in Belevitch form [20], and scattering transfer matrix factorization (SMTF) is reformulated in terms of the reflection factor coefficients, to get the polynomials of the remaining network, after a commensurate lines is extracted. There is no need to find roots at $\lambda = \pm 1$, to get a degree reduction.

In the following section, SMTF is explained briefly. Subsequently, the synthesis method is described. Finally, an example is presented, to illustrate the implementation of the proposed new formulation.

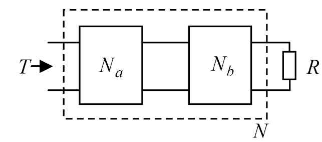

Fig. 1. Cascade decomposition of a two-port.

II. Scattering Transfer Matrix Factorization (STMF)

As it is well known, canonic form of the scattering transfer matrix $\{T\}$ of a lossless, reciprocal two-port is defined as [2]

$$T = \frac{1}{f} \begin{bmatrix} \mu g_* & h \\ \mu h_* & g \end{bmatrix} \tag{3}$$

where $\mu = f_*/f = \pm 1$ is a unimodular constant, g is a strictly Hurwitz real polynomial. These polynomials must satisfy the Feldtkeller equation $gg_* = hh_* + ff_*$ (where "$*$" denotes para-conjugation).

The problem is to decompose the lossless reciprocal two-port $\{N\}$ into two cascade connected lossless two-ports $\{N_a, N_b\}$ which are also lossless and reciprocal (Fig. 1). This amounts to factoring the scattering transfer matrix $\{T\}$ into a product of two scattering transfer matrices

$$T = T_a \cdot T_b \tag{4a}$$

where

$$T_a = \frac{1}{f_a} \begin{pmatrix} \mu_a g_{a*} & h_a \\ \mu_a h_{a*} & g_a \end{pmatrix} \text{ and } T_b = \frac{1}{f_b} \begin{pmatrix} \mu_b g_{b*} & h_b \\ \mu_b h_{b*} & g_b \end{pmatrix}. \tag{4b}$$

The polynomial sets $\{g_a, h_a, f_a\}$ and $\{g_b, h_b, f_b\}$ have the same properties as $\{g, h, f\}$, and in particular, must satify the Feldtkeller equation. Equation (4) implies the following:

$$g = g_a g_b + \mu_a h_{a*} h_b \tag{5a}$$

$$h = h_a g_b + \mu_a g_{a*} h_b \tag{5b}$$

$$f = f_a f_b \tag{5c}$$

$$\mu = \mu_a \mu_b. \tag{5d}$$

Manuscript received March 13, 2007; revised June 20, 2007. This paper was recommended by Associate Editor P. P. Sotiriadis.

The author is with Engineering Faculty, Kadir Has University, 34083 Cibali-Fatih-Istanbul, Turkey (e-mail: msengul@khas.edu.tr).

Digital Object Identifier 10.1109/TCSII.2007.904171

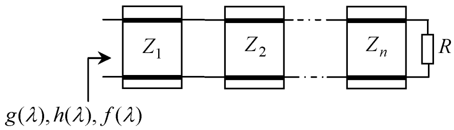

Fig. 2. Commensurate line extraction.

Under the use of these equalities, if one writes $T_b = T_a^{-1}T$, two equations can be obtained as

$$h_b = \frac{hg_a - gh_a}{\mu_a f_a f_{a*}} \tag{6a}$$

$$g_b = \frac{gg_{a*} - hh_{a*}}{f_a f_{a*}}. \tag{6b}$$

Now, for a given polynomial set $\{g, h, f\}$, the original decomposition problem (4a) is essentially reduced to solving (6) in the unknown polynomials $\{g_a, h_a, g_b, h_b\}$ subject to the Feldtkeller equation with g_a and g_b being strictly Hurwitz polynomials.

The factorization of the scattering transfer matrix of a lossless, reciprocal two-port has been treated by *Fettweis* [21]. The problem has been solved by using a modified formulation of the factorization problem [22]. In [22], instead of solving (6), a different set of equations [which can be obtained by manipulating (5a), (5b), and (6)] are choosen as the basis for the solution, and the factorization problem is reformulated. Detailed treatment of the problem stated above and all the pertinent proofs with regard to this formulation can be found in [22].

III. Reformulation of STMF for Commensurate Line Synthesis

Consider the circuit shown in Fig. 2. $g(\lambda)$, $h(\lambda)$, and $f(\lambda)$ polynomials can be described as follows:

$$g(\lambda) = g_0 + g_1\lambda + g_2\lambda^2 + \cdots + g_n\lambda^n \tag{7a}$$

$$h(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_n\lambda^n \tag{7b}$$

$$f(\lambda) = (1-\lambda^2)^{n/2}. \tag{7c}$$

Characteristic impedance Z_1 of the first commensurate line that will be extracted is calculated as

$$Z_1 = \frac{g(1) + h(1)}{g(1) - h(1)}. \tag{8}$$

Then, $g(\lambda), h(\lambda)$ and $f(\lambda)$ polynomials of the remaining network are obtained as

$$g(\lambda) = \sum_{j=1}^{n} D_j \lambda^{j-1} \tag{9a}$$

$$h(\lambda) = \sum_{j=1}^{n} N_j \lambda^{j-1} \tag{9b}$$

$$f(\lambda) = (1-\lambda^2)^{(n-1)/2} \tag{9c}$$

where

$$D_j = \sum_{i=1}^{j} (-1)^{i+j} y_i, \qquad j = 1, 2, \ldots, n \tag{10a}$$

$$N_j = \sum_{i=1}^{j} x_i, \qquad j = 1, 2, \ldots, n \tag{10b}$$

where

$$x_i = h_{i-1}g(1) - g_{i-1}h(1), \qquad i = 1, 2, \ldots, n \tag{11a}$$

$$y_i = g_{i-1}g(1) - h_{i-1}h(1), \qquad i = 1, 2, \ldots, n. \tag{11b}$$

The extraction of commensurate lines proceeds in a similar fashion until the termination resistance (R) is reached.

IV. Example

Example given in [17] is solved, to illustrate the implementation of the proposed algorithm. The given input reflection factor is

$$S(\lambda) = \frac{h(\lambda)}{g(\lambda)} = \frac{114.50\lambda^5 + 44.27\lambda^3 + 3.16\lambda}{114.50\lambda^5 + 83.21\lambda^4 + 74.48\lambda^3 + 28.89\lambda^2 + 8.53\lambda + 1}.$$

Step 1)

$$h^{(1)}(\lambda) = h(\lambda) \text{ and } g^{(1)}(\lambda) = g(\lambda).$$

Step 2)

$$Z_1 = \frac{g^{(1)}(1) + h^{(1)}(1)}{g^{(1)}(1) - h^{(1)}(1)} = 3.1782.$$

Step 3)

$$\begin{aligned}
x_1 &= h_0 g^{(1)}(1) - g_0 h^{(1)}(1) \\
&= 0 \cdot 310.61 - 1 \cdot 161.93 = -161.93 \\
x_2 &= h_1 g^{(1)}(1) - g_1 h^{(1)}(1) = 3.16 \cdot 310.61 - 8.53 \cdot 161.93 \\
&= -399.74 \\
x_3 &= h_2 g^{(1)}(1) - g_2 h^{(1)}(1) = 0 \cdot 310.61 - 28.89 \cdot 161.93 \\
&= -4678.15 \\
x_4 &= h_3 g^{(1)}(1) - g_3 h^{(1)}(1) = 44.27 \cdot 310.61 - 74.48 \cdot 161.93 \\
&= 1690.16 \\
x_5 &= h_4 g^{(1)}(1) - g_4 h^{(1)}(1) = 0 \cdot 310.61 - 83.21 \cdot 161.93 \\
&= -13474.20.
\end{aligned}$$

$$\begin{aligned}
y_1 &= g_0 g^{(1)}(1) - h_0 h^{(1)}(1) = 1 \cdot 310.61 - 0 \cdot 161.93 = 310.61 \\
y_2 &= g_1 g^{(1)}(1) - h_1 h^{(1)}(1) \\
&= 8.53 \cdot 310.61 - 3.16 \cdot 161.93 = 2137.80 \\
y_3 &= g_2 g^{(1)}(1) - h_2 h^{(1)}(1) \\
&= 28.89 \cdot 310.61 - 0 \cdot 161.93 = 8973.52 \\
y_4 &= g_3 g^{(1)}(1) - h_3 h^{(1)}(1) \\
&= 74.48 \cdot 310.61 - 44.27 \cdot 161.93 = 15965.60 \\
y_5 &= g_4 g^{(1)}(1) - h_4 h^{(1)}(1) \\
&= 83.21 \cdot 310.61 - 0 \cdot 161.93 = 25845.86.
\end{aligned}$$

Step 4)

$$\begin{aligned}N_1 &= x_1 = -161.93\\ N_2 &= x_2 + x_1 = -399.74 - 161.93 = -561.67\\ N_3 &= x_3 + x_2 + x_1 = -4678.15 - 399.74 - 161.93 = -5239.82\\ N_4 &= x_4 + x_3 + x_2 + x_1\\ &= 1690.16 - 4678.15 - 399.74 - 161.93 = -3549.66\\ N_5 &= x_5 + x_4 + x_3 + x_2 + x_1\\ &= -13474.20 + 1690.16 - 4678.15 - 399.74 - 161.93\\ &= -17023.86.\end{aligned}$$

$$\begin{aligned}D_1 &= y_1 = 310.61\\ D_2 &= y_2 - y_1 = 2137.80 - 310.61 = 1827.19\\ D_3 &= y_3 - y_2 + y_1 = 8973.52 - 2137.80 + 310.61 = 7146.33\\ D_4 &= y_4 - y_3 + y_2 - y_1\\ &= 15965.60 - 8973.52 + 2137.80 - 310.61 = 8819.27\\ D_5 &= y_5 - y_4 + y_3 - y_2 + y_1\\ &= 25845.86 - 15965.60 + 8973.52 - 2137.80 + 310.61\\ &= 17026.59.\end{aligned}$$

Step 5)

$$\begin{aligned}g^{(2)}(\lambda) &= D_1 + D_2\lambda + D_3\lambda^2 + D_4\lambda^3 + D_5\lambda^4\\ &= 310.61 + 1827.19\lambda + 7146.33\lambda^2\\ &\quad + 8819.27\lambda^3 + 17026.59\lambda^4.\end{aligned}$$

$$\begin{aligned}h^{(2)}(\lambda) &= N_1 + N_2\lambda + N_3\lambda^2 + N_4\lambda^3 + N_5\lambda^4\\ &= -161.93 - 561.67\lambda - 5239.82\lambda^2\\ &\quad - 3549.66\lambda^3 - 17023.86\lambda^4.\end{aligned}$$

After applying the algorithm until the termination resistance was reached, the following impedance values were obtained $Z_1 = 3.1782$, $Z_2 = 0.4429$, $Z_3 = 4.4505$, $Z_4 = 0.4420$, and $Z_5 = 3.1873$. Characteristic impedances found in [17] are $Z_1 = 3.18$, $Z_2 = 0.443$, $Z_3 = 4.38$, $Z_4 = 0.443$, $Z_5 = 3.18$. As mentioned in [17], the given reflection factor belongs to a symmetrical filter. So during the sysnthesis process, the symmetry property of the filter structure may be used to advantage in the computations.

Then, (2) was used to synthesize the same reflection factor without using the symmetry property, and the obtained characteristic impedances were $Z_1 = 3.1782$, $Z_2 = 0.4427$, $Z_3 = 4.4620$, $Z_4 = 0.4229$, $Z_5 = 4.8223$.

So without using the symmetry property, closer impedance values to the ones given in [17] than the ones calculated by using (2) have been obtained employing the proposed synthesis algorithm.

V. Conclusion

Richards' Theorem is a simple and powerful method, but can cause large numerical inaccuracies, as the number of commensurate lines increases. In [18], synthesis is carried out by using the reflection factor of the unterminated cascade network which is calculated by even and odd parts of the numerator and denominator terms of the input reflection factor. But in the proposed algorithm, input reflection factor is utilized directly. The method is based on the reformulation of the SMTF in terms of the input reflection factor coefficients. After a line is extracted, there is no need to employ root search routines, to get degree reduction. As a result, a very simple to implement commensurate line synthesis algorithm is presented.

Acknowledgment

The author thanks S.B. Yarman (Istanbul, Turkey) and M. Hein (Ilmenau, Germay) for the fruitful discussions.

References

[1] P. I. Richards, "Resistor-termination-line circuits," *Proc. IRE*, vol. 36, pp. 217–220, Feb. 1948.
[2] A. Aksen, "Design of lossless two-port with mixed, lumped and distributed elements for broadband matching," Ph.D. dissertatation, Ruhr University, Bochum, Germany, 1994.
[3] A. Sertbaş, "Genelleştirilmiş iki-kapılı, iki-deiķenli kayıpsız merdiven devrelerin tanımlanması," (in Turkish) Ph.D. dissertatation, İstanbul University, İstanbul, Turkey, 1997.
[4] P. S. Castro and W. W. Happ, "Distributed parameter circuits and microsystem electronics," in *Proc. NEC*, 1960, pp. 448–461.
[5] M. J. Hellstrom, "Equivalent distributed *RC* networks or transmission lines," *IRE Trans. Circuit Theory*, vol. CT-9, pp. 247–251, Sep. 1962.
[6] M. R. Wohlers, *Lumped and Distributed Passive Networks, A Generalized and Advanced Viewpoint*. New York: Academic , 1969.
[7] M. Ghausi and J. Kelly, *Introduction to Distributed-Parameter Networks*. New York: Krieger, 1977.
[8] K. Heizer, "Distributed *RC* networks with rational transfer functions," *IRE Trans. Circuit Theory*, vol. CT-9, no. 4, pp. 356–362, Dec. 1962.
[9] R. W. Wyndrum Jr., The exact synthesis of distributed *RC* networks New York University, Dept. of Elect. Eng., New York, 1963, Tech. Rept. 400-476.
[10] J. O. Scanlan and J. D. Rhodes, "Realizability and synthesis of a restricted class of distributed *RC* networks," *IEEE Trans. Circuit Theory*, vol. CT-12, no. 4, pp. 577–585, Dec. 1965.
[11] R. P. O'Shea, "Synthesis using distributed *RC* networks," *IEEE Trans. Circuit Theory*, vol. CT-12, no. 4, pp. 546–554, Dec. 1965.
[12] S. S. Penbeci and S. C. Lee, "Synthesis of distributed *RC* networks by means of orthogonal functions," *IEEE Trans. Circuit Theory*, vol. CT-16, no. 1, pp. 137–140, Feb. 1969.
[13] J. O. Scanlan and N. Ramamurty, "Cascade synthesis of distributed *RC* networks," *IEEE Trans. Circuit Theory*, vol. CT-16, no. 1, pp. 47–57, Feb. 1969.
[14] S. C. Lee and S. S. Penbeci, "Computer-aided design of distributed *RC* networks," *IEEE Trans. Circuit Theory*, vol. CT-17, no. 2, pp. 224–232, May 1970.
[15] P. Sotiriadis and Y. Tsividis, "Integrators using a single distributed *RC* elements," in *Proc. IEEE Int. Symp. Circuits Syst.*, May 2002, vol. 2, pp. 21–24.
[16] P. Sotiriadis and Y. Tsividis, "Single-URC integrators," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 2, pp. 304–307, Feb. 2003.
[17] H. J. Carlin, "Distributed circuit design with transmission line elements," *Proc. IEEE*, vol. 59, no. 7, pp. 1059–1081, Jul. 1971.
[18] J. Komiak and H. J. Carlin, "Improved accuracy for commensurate-line synthesis," *IEEE Trans. Microw. Theory Tech.*, vol. MTT-22, no. 4, pp. 212–215, Apr. 1976.
[19] M. H. Ali and R. Yarlagadda, "A note on the unit element synthesis," *IEEE Trans. Circuits Syst.*, vol. CAS-25, no. 3, pp. 172–174, Mar. 1978.
[20] V. Belevitch, *Classical Network Theory*. San Francisco, CA: Holden Day, 1968.
[21] A. Fettweis, "Factorization of transfer matrices of lossless two-ports," *IEEE Trans. Circuit Theory*, vol. CT-17, no. 1, pp. 86–94, Feb. 1970.
[22] A. Fettweis, *Cascade Synthesis of Lossless Two-Ports by Transfer Matrix Factorization*, R. Boite, Ed. New York: Gordon and Breach, 1972, Network Theory, pp. 43–103.