Available online at www.sciencedirect.com

SCIENCE DIRECT®

Topology and its Applications 153 (2005) 350–358

Topology and its Applications

www.elsevier.com/locate/topol

Chebyshev nets formed by Ricci curves in a 3-dimensional Weyl space

Gülçin Çivi Yıldırım [a,*], Abdülkadir Özdeğer [b]

[a] *Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469 Maslak-Istanbul, Turkey*

[b] *Kadir Has University, Faculty of Arts and Sciences, 34230 Cibali-Istanbul, Turkey*

Received 28 October 2002; accepted 16 May 2003

Abstract

In this paper, Ricci curves in a 3-dimensional Weyl space $W_3(g, T)$ are defined and it is shown that any 3-dimensional Chebyshev net formed by the three families of Ricci curves in a $W_3(g, T)$ having a definite metric and Ricci tensors is either a geodesic net or it consists of a geodesic subnet the members of which have vanishing second curvatures. In the case of an indefinite Ricci tensor, only one of the members of the geodesic subnet under consideration has a vanishing second curvature.

MSC: primary 53A30; secondary 53A40

Keywords: Weyl space; Ricci curve; Chebyshev net; Geodesic net

1. Introduction

A manifold of dimension n with a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition

$$\nabla g - 2g \otimes T = 0$$

* Corresponding author.
E-mail addresses: civi@itu.edu.tr (G.Ç. Yıldırım), aozdeger@khas.edu.tr (A. Özdeğer).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2003.05.011

or, in local coordinates,

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

where T is a 1-form (covariant vector field) is called a Weyl space which will be denoted by $W_n(g,T)$ [1]. Under the renormalization

$$\tilde{g} = \lambda^2 g \tag{1.2}$$

of the metric tensor g, T is transformed by the rule

$$\widetilde{T}_k = T_k + \partial_k(\ln\lambda)$$

where λ is a scalar function defined on $W_3(g,T)$ [1].

If, under the renormalization (1.2) of the metric tensor g the quantity A is changed according to the rule

$$\tilde{A} = \lambda^p A,$$

then A is a called a satellite of g of weight $\{p\}$.

The prolonged covariant derivative of A with respect to ∇ is defined by

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.3}$$

Let R_{ij} be the components of the Ricci tensor of the 3-dimensional Weyl space $W_3(g,T)$ and let $R_{(ij)}$ the symmetric part of R_{ij}. Let the principal directions and the corresponding principal values of $R_{(ij)}$ be denoted, respectively, by $\underset{1}{v}, \underset{2}{v}, \underset{3}{v}$ and $\underset{1}{M}, \underset{2}{M}, \underset{3}{M}$. We then have

$$(R_{(ij)} + \underset{r}{M} g_{ij})\underset{r}{v}^i = 0 \quad (i,j,r = 1,2,3). \tag{1.4}$$

It is clear that R_{ij} and $\underset{r}{M}$ are satellites of g of weights $\{0\}$ and $\{-2\}$, respectively. We call $\underset{1}{v}, \underset{2}{v}$ and $\underset{3}{v}$ the Ricci's principal directions. The integral curves of these vector fields will be named as the Ricci curves of $W_3(g,T)$. These curves may be considered as the generalization of Ricci curves in a Riemannian space [2–4] to a Weyl space. Since g is assumed to be definite, the Ricci curves are all real.

Suppose that the vector fields $\underset{1}{v}, \underset{2}{v}$ and $\underset{3}{v}$ are normalized by the conditions

$$g_{ij}\underset{r}{v}^i\underset{r}{v}^j = 1 \quad (r = 1,2,3).$$

Accordingly, from (1.4) it follows that

$$\underset{r}{M} = -R_{(ij)}\underset{r}{v}^i\underset{r}{v}^j = -R_{ij}\underset{r}{v}^i\underset{r}{v}^j, \tag{1.5}$$

$$R_{(ij)}\underset{r}{v}^i\underset{s}{v}^j = 0 \quad (r \neq s). \tag{1.6}$$

We note that $\underset{r}{M}$ is the mean curvature of $W_3(g,T)$ in the direction of $\underset{r}{v}$.

2. Chebyshev nets formed by Ricci curves in a $W_3(g,T)$

Let $\delta \equiv (\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ be the 3-dimensional net formed by the tangent vector fields $\underset{1}{v}, \underset{2}{v}$ and $\underset{3}{v}$ of the three families of Ricci curves in $W_3(g,T)$.

If any vector field belonging to δ undergoes a parallel displacement along the integral curves of the remaining two vector fields in δ, then δ is said to be a Chebyshev net of the first kind or, simply, a Chebyshev net [5].

Suppose that δ is a Chebyshev net. This will be the case if and only if the conditions

$$\underset{r}{v}^k \dot{\nabla}_k \underset{s}{v}^i = 0 \quad (r \neq s;\ r,s = 1,2,3). \tag{2.1}$$

Let $\underset{r}{n}$ and $\underset{r}{b}$ be, respectively, the principal normal and binormal vector fields of the integral curve $\underset{r}{C}$ of the vector field $\underset{r}{v}$ which are normalized by the conditions $g_{ij}\underset{r}{n}^i\underset{r}{n}^j = 1$, $g_{ij}\underset{r}{b}^i\underset{r}{b}^j = 1$. In this case, the Frenet formulas

$$\underset{r}{v}^k \dot{\nabla}_k \underset{r}{v}^j = \underset{r}{\rho}\underset{r}{n}^j, \qquad \underset{r}{v}^k \dot{\nabla}_k \underset{r}{n}^j = -\underset{r}{\rho}\underset{r}{v}^j + \underset{r}{\tau}\underset{r}{b}^j, \qquad \underset{r}{v}^k \dot{\nabla}_k \underset{r}{b}^j = -\underset{r}{\tau}\underset{r}{n}^j \tag{2.2}$$

hold [1].

Taking the absolute derivative of (1.4) in the direction of $\underset{s}{v}^l$ and transvecting the so-obtained equation by $\underset{r}{n}^j$ and remembering that $g_{ij}\underset{r}{v}^i\underset{r}{n}^j = 0$, we get

$$\underset{s}{v}^l (\dot{\nabla}_l R_{(ij)}) \underset{r}{v}^i \underset{r}{n}^j + (R_{(ij)} + \underset{r}{M} g_{ij}) \underset{r}{n}^j \underset{s}{v}^l \dot{\nabla}_l \underset{r}{v}^i = 0. \tag{2.3}$$

Since δ is assumed to be a Chebyshev net, according to (2.1), (2.3) becomes

$$\underset{s}{v}^l \dot{\nabla}_l R_{(ij)} \underset{r}{v}^i \underset{r}{n}^j = 0 \quad (r \neq s). \tag{2.4}$$

On the other hand, we have the relations

$$\begin{aligned}
&\underset{1}{n}^j = \cos\theta_1 \underset{2}{v}^j + \sin\theta_1 \underset{3}{v}^j, \qquad \underset{2}{n}^j = \cos\theta_2 \underset{3}{v}^j + \sin\theta_2 \underset{1}{v}^j,\\
&\underset{3}{n}^j = \cos\theta_3 \underset{1}{v}^j + \sin\theta_3 \underset{2}{v}^j\\
&\underset{1}{b}^j = \cos\theta_1 \underset{3}{v}^j - \sin\theta_1 \underset{2}{v}^j, \qquad \underset{2}{b}^j = \cos\theta_2 \underset{1}{v}^j - \sin\theta_2 \underset{3}{v}^j,\\
&\underset{3}{b}^j = \cos\theta_3 \underset{2}{v}^j - \sin\theta_3 \underset{1}{v}^j
\end{aligned} \tag{2.5}$$

where

$$\theta_1 = \angle(\underset{2}{v}, \underset{1}{n}), \qquad \theta_2 = \angle(\underset{3}{v}, \underset{2}{n}), \qquad \theta_3 = \angle(\underset{1}{v}, \underset{3}{n}).$$

Choosing $r = 1$, $s = 2$ in (2.4) and using the relations (2.5) we find that

$$\underset{2}{v}^l (\dot{\nabla}_l R_{(ij)}) \underset{1}{v}^i \underset{2}{v}^j \cos\theta_1 + \underset{2}{v}^l (\dot{\nabla}_l R_{(ij)}) \underset{1}{v}^i \underset{3}{v}^j \sin\theta_1 = 0. \tag{2.6}$$

The absolute derivative of (1.6) in the directions of $\underset{p}{v}^l$ is

$$\underset{p}{v}^l (\dot{\nabla}_l R_{(ij)}) \underset{r}{v}^i \underset{s}{v}^j + \underset{p}{v}^l (\dot{\nabla}_l \underset{r}{v}^i) R_{(ij)} \underset{s}{v}^j + \underset{p}{v}^l (\dot{\nabla}_l \underset{s}{v}^j) R_{(ij)} \underset{r}{v}^i = 0. \tag{2.7}$$

Taking first $r=1$, $p=s=2$ and then $p=2$, $r=1$, $s=3$ in (2.7) we, respectively, obtain

$$\underset{2}{v}^l(\dot\nabla_l R_{(ij)})\underset{1}{v}^i\underset{2}{v}^j=-\underset{2}{v}^l(\dot\nabla_l\underset{2}{v}^j)R_{(ij)}\underset{1}{v}^i \tag{2.8}$$

and

$$\underset{2}{v}^l(\dot\nabla_l R_{(ij)})\underset{1}{v}^i\underset{3}{v}^j=0. \tag{2.9}$$

By (2.8) and (2.9), (2.6) reduces to

$$\underset{2}{v}^l(\dot\nabla_l\underset{2}{v}^i)\underset{1}{v}^j R_{(ij)}\cos\theta_1=0. \tag{2.10}$$

On the other hand, since the vector $\underset{2}{v}^l(\dot\nabla_l\underset{2}{v}^j)$ is perpendicular to $\underset{2}{v}^j$, we can write

$$\underset{2}{v}^l(\dot\nabla_l\underset{2}{v}^j)=\underset{2}{\rho}\underset{2}{n}^j=\lambda_2\underset{1}{v}^j+\mu_2\underset{3}{v}^j \tag{2.11}$$

so that, by (1.5) and (1.6), (2.10) transforms into

$$\lambda_2\underset{1}{M}\cos\theta_1=0. \tag{2.12}$$

Similarly, choosing $r=2$, $s=1$; $r=3$, $s=1$; $r=3$, $s=2$; $r=1$, $s=3$ and $r=2$, $s=3$ in (2.4) and making use of (1.5), (1.6), (2.1), (2.5) and (2.7) we, respectively, obtain

$$\lambda_1\underset{2}{M}\sin\theta_2=0, \tag{2.13}$$

$$\mu_1\underset{3}{M}\cos\theta_3=0, \tag{2.14}$$

$$\mu_2\underset{3}{M}\sin\theta_3=0, \tag{2.15}$$

$$\lambda_3\underset{1}{M}\sin\theta_1=0, \tag{2.16}$$

$$\mu_3\underset{2}{M}\cos\theta_2=0 \tag{2.17}$$

in which the functions λ_1, μ_1, λ_3 and μ_3 are defined by

$$\underset{1}{v}^l(\dot\nabla_l\underset{1}{v}^j)=\underset{1}{\rho}\underset{1}{n}^j=\lambda_1\underset{2}{v}^j+\mu_1\underset{3}{v}^j, \tag{2.18}$$

$$\underset{3}{v}^l(\dot\nabla_l\underset{3}{v}^j)=\underset{3}{\rho}\underset{3}{n}^j=\lambda_3\underset{1}{v}^j+\mu_3\underset{2}{v}^j. \tag{2.19}$$

Case I. Let the Ricci tensor R_{ij} be definite. Since, by (1.5),

$$\underset{r}{M}=-R_{(ij)}\underset{r}{v}^i\underset{r}{v}^j=-R_{ij}\underset{r}{v}^i\underset{r}{v}^j\quad(r=1,2,3)$$

the conditions (2.12)–(2.17) are, respectively, reduced to

$$\lambda_2\cos\theta_1=0, \tag{2.20}$$

$$\lambda_1\sin\theta_2=0, \tag{2.21}$$

$$\mu_1\cos\theta_3=0, \tag{2.22}$$

$$\mu_2\sin\theta_3=0, \tag{2.23}$$

$$\lambda_3\sin\theta_1=0, \tag{2.24}$$

$$\mu_3\cos\theta_2=0. \tag{2.25}$$

Case I-a. $\lambda_1 \neq 0$. Under this condition, (2.21) and (2.25) give

$$\theta_2 = 0, \qquad \mu_3 = 0. \tag{2.26}$$

Since $\theta_2 = 0$, from (2.5) it follows that

$$\underset{2}{b} = \underset{1}{v}, \qquad \underset{2}{n} = \underset{3}{v}. \tag{2.27}$$

Then, by (2.2), we have

$$0 = \underset{2}{v}^j \dot{\nabla}_j \underset{1}{v}^i = \underset{2}{v}^j \dot{\nabla}_j \underset{2}{b}^i = -\underset{2}{\tau}\underset{2}{n}^i, \tag{2.28}$$

$$0 = \underset{2}{v}^j \dot{\nabla}_j \underset{3}{v}^i = \underset{2}{v}^j \dot{\nabla}_j \underset{2}{n}^i = -\underset{2}{\rho}\underset{2}{v}^i + \underset{2}{\tau}\underset{2}{b}^i \tag{2.29}$$

from which it follows that

$$\underset{2}{\rho} = \underset{2}{\tau} = 0. \tag{2.30}$$

By (2.11) we have

$$\lambda_2 = \mu_2 = 0, \tag{2.31}$$

showing that Eqs. (2.20) and (2.23) are automatically satisfied. Moreover, by (2.2), (2.18) and (2.19) we obtain

$$\underset{3}{v}^l \dot{\nabla}_l \underset{3}{v}^i = \underset{3}{\rho}\underset{3}{n}^i = \lambda_3 \underset{1}{v}^i \tag{2.32}$$

from which we have either

(a) $\underset{3}{\rho} = \lambda_3 \neq 0$ $(\underset{3}{n} = \underset{1}{v})$ or
(b) $\underset{3}{\rho} = \lambda_3 = 0$.

In the case (a), by (2.5), $\underset{3}{n} = \underset{1}{v}$ implies $\theta_3 = 0$ and $\underset{3}{b} = \underset{2}{v}$. So, we must have

$$0 = \underset{3}{v}^l \dot{\nabla}_l \underset{1}{v}^i = \underset{3}{v}^l \dot{\nabla}_l \underset{3}{n}^i = -\underset{3}{\rho}\underset{3}{v}^i + \underset{3}{\tau}\underset{3}{b}^i = -\underset{3}{\rho}\underset{3}{v}^i + \underset{3}{\tau}\underset{2}{v}^i \quad \Rightarrow \quad \underset{3}{\rho} = \underset{3}{\tau} = 0$$

contradicting the condition $\underset{3}{\rho} \neq 0$. Consequently, only the case (b), i.e.,

$$\underset{3}{\rho} = \lambda_3 = 0 \tag{2.33}$$

can occur.

Under these conditions, Eqs. (2.20)–(2.25) are reduced to the single equation

$$\mu_1 \cos\theta_3 = 0. \tag{2.34}$$

In (2.34) μ_1 can not vanish since, otherwise, by (2.2) and (2.18) we must have

$$\underset{1}{v}^l \dot{\nabla}_l \underset{1}{v}^i = \underset{1}{\rho}\underset{1}{n}^i = \lambda_1 \underset{2}{v}^i$$

from which we obtain

$$\underset{1}{n} = \underset{2}{v} \quad (\underset{1}{\rho} = \lambda_1 \neq 0) \tag{2.35}$$

and, consequently, by (2.5)

$$\theta_1 = 0, \qquad \underset{1}{b} = \underset{3}{v}. \tag{2.36}$$

But (2.35), (2.36), (2.1) and (2.2) imply that

$$0 = \underset{1}{v}^l \dot{\nabla}_l \underset{2}{v}^i = \underset{1}{v}^l \dot{\nabla}_l \underset{1}{n}^i = -\underset{1}{\rho} \underset{1}{v}^i + \underset{1}{\tau} \underset{1}{b}^i = -\underset{1}{\rho} \underset{1}{v}^i + \underset{1}{\tau} \underset{3}{v}^i \quad \Rightarrow \quad \underset{1}{\rho} = \underset{1}{\tau} = 0$$

contradicting the condition $\underset{1}{\rho} = \lambda_1 \neq 0$.

So, in (2.34) $\mu_1 \neq 0$ so that $\theta_3 = \pi/2$. From (2.5), we get

$$\underset{3}{n} = \underset{2}{v}, \qquad \underset{3}{b} = -\underset{1}{v}$$

by means of which we obtain

$$0 = \underset{3}{v}^l \dot{\nabla}_l \underset{2}{v}^i = \underset{3}{v}^l \dot{\nabla}_l \underset{3}{n}^i = -\underset{3}{\rho} \underset{3}{v}^i + \underset{3}{\tau} \underset{3}{b}^i = -\underset{3}{\rho} \underset{3}{v}^i - \underset{3}{\tau} \underset{1}{v}^i$$

or equivalently

$$\underset{3}{\rho} = \underset{3}{\tau} = 0, \tag{2.37}$$

where we have used (2.1) and (2.2).

(2.30) and (2.37) show that the two families of Ricci curves which are the integral curves of the vector fields $\underset{2}{v}$ and $\underset{3}{v}$ are geodesics with vanishing torsion (second curvature).

In a very similar way, it can be shown that

$$\lambda_i \neq 0 \text{ implies } \mu_i \neq 0,\ \lambda_j = \mu_j = 0\ (i \neq j;\ i, j = 1, 2, 3), \tag{2.38}$$

$$\mu_i \neq 0 \text{ implies } \lambda_i \neq 0,\ \mu_j = \lambda_j = 0\ (j \neq i). \tag{2.39}$$

But these conditions say that the two families of Ricci curves are geodesics with vanishing second curvatures in Case I-a.

Case I-b. $\lambda_1 = 0$. We first note that $\mu_1 = 0$ since according to (2.39) $\mu_1 \neq 0$ would imply $\lambda_1 \neq 0$.

Case I-b$_1$. $\lambda_1 = \mu_1 = 0$, $\mu_2 \neq 0$ (or $\mu_3 \neq 0$). In this case, according to (2.39) $\lambda_2 \neq 0$ and $\lambda_3 = \mu_3 = 0$ so that

$$\underset{1}{\rho} = 0, \qquad \underset{3}{\rho} = 0.$$

Under these conditions, from (2.20) and (2.23) we find that $\theta_1 = \pi/2$ and $\theta_3 = 0$, respectively. Then, by (2.5) we obtain

$$\underset{1}{n} = \underset{3}{v} \quad \text{and} \quad \underset{3}{n} = \underset{1}{v}$$

by means of which we get

$$0 = \underset{1}{v}^l \dot{\nabla}_l \underset{3}{v}^i = \underset{1}{v}^l \dot{\nabla}_l \underset{1}{n}^i = -\underset{1}{\rho} \underset{1}{v}^i + \underset{1}{\tau} \underset{1}{b}^i \quad \Rightarrow \quad \underset{1}{\rho} = \underset{1}{\tau} = 0,$$

$$0 = \underset{3}{v}^l \dot{\nabla}_l \underset{1}{v}^i = \underset{3}{v}^l \dot{\nabla}_l \underset{3}{n}^i = -\underset{3}{\rho} \underset{3}{v}^i + \underset{3}{\tau} \underset{3}{b}^i \quad \Rightarrow \quad \underset{3}{\rho} = \underset{3}{\tau} = 0.$$

Accordingly, the two families of Ricci curves are geodesics with vanishing second curvatures (torsions).

It is easy to see that a similar conclusion may be drawn for the

Case I-b$_2$. $\lambda_1 = \mu_1 = 0,\ \lambda_2 \neq 0$ (or $\lambda_3 \neq 0$).

Case I-c. $\lambda_i = \mu_i = 0$ $(i = 1, 2, 3)$. In this case, Eqs. (2.20)–(2.25) are automatically satisfied. By (2.11), (2.18) and (2.19) we have

$$\underset{1}{\rho} = \underset{2}{\rho} = \underset{3}{\rho} = 0$$

showing that the three families of Ricci curves are geodesics.

According to the above considerations these are the main cases that have to be considered.

It is to be noted that in the Case I-c, $W_3(g, T)$ becomes an affine space since it has one Chebyshevian and geodesic net.

We are now able to state

Theorem 2.1. *In a $W_3(g, T)$ having a definite metric and a definite Ricci tensor any* 3*-dimensional Chebyshev net formed by the three families of Ricci curves is either a geodesic net or it consists of a geodesic sub-net whose members have vanishing second curvatures* (*torsions*).

Case II. We now consider the case for which $W_3(g, T)$ has an indefinite Ricci tensors and assume that the Ricci's principal values $\underset{1}{M}, \underset{2}{M}$ and $\underset{3}{M}$ are distinct. Then, only one of $\underset{1}{M}, \underset{2}{M}$ and $\underset{3}{M}$ may be zero. So, if we take $\underset{1}{M} = 0$ $(\underset{2}{M} \neq \underset{3}{M} \neq 0)$ in (2.12) and (2.16) then Eqs. (2.13)–(2.17) are respectively transformed into Eqs. (2.21)–(2.23) and (2.25). Namely,

$$\lambda_1 \sin\theta_2 = 0, \qquad \mu_1 \cos\theta_3 = 0,$$
$$\mu_2 \sin\theta_3 = 0, \qquad \mu_3 \cos\theta_2 = 0. \tag{2.40}$$

Let $\lambda_1 = 0$ in (2.40). Then according to (2.2) and (2.18) we get

$$\underset{1}{v}^l \dot{\nabla}_l \underset{1}{v}^i = \mu_1 \underset{3}{v}^i = \underset{1}{\rho} \underset{1}{n}^i$$

from which we have either

(a) $\mu_1 = \underset{1}{\rho} \neq 0$ implying $\underset{3}{v} = \underset{1}{n}$ $(\theta_1 = \pi/2)$, $\underset{1}{b} = -\underset{2}{v}$ so that according to (2.1) and (2.2) we obtain

$$0 = \underset{1}{v}^l \dot{\nabla}_l \underset{3}{v}^i = \underset{1}{v}^l \dot{\nabla}_l \underset{1}{n}^i = -\underset{1}{\rho} \underset{1}{v}^i + \underset{1}{\tau} \underset{1}{b}^i = -\underset{1}{\rho} \underset{1}{v}^i - \underset{1}{\tau} \underset{2}{v}^i$$

which is impossible, or

(b) $$\mu_1 = \underset{1}{\rho} = 0. \tag{2.41}$$

In this case, Eqs. (2.40) reduce to

$$\mu_2 \sin\theta_3 = 0, \tag{2.42}$$

$$\mu_3 \cos\theta_2 = 0. \tag{2.43}$$

These equations will be satisfied if

(i) $\mu_2 = \mu_3 = 0$ or
(ii) $\mu_2 \neq 0$ (or $\mu_3 \neq 0$).

In the case of (i), with the help of (2.2), (2.11) and (2.19) we respectively have

$$\underset{2}{v}^l \dot{\nabla}_l \underset{2}{v}^i = \underset{2}{\rho}\underset{2}{n}^i = \lambda_2 \underset{1}{v}^i, \tag{2.44}$$

$$\underset{3}{v}^l \dot{\nabla}_l \underset{3}{v}^i = \underset{3}{\rho}\underset{3}{n}^i = \lambda_3 \underset{1}{v}^i. \tag{2.45}$$

The respective solutions of (2.44) and (2.45) are

$$\underset{2}{\rho} = \lambda_2 = 0 \quad \text{and} \quad \underset{3}{\rho} = \lambda_3 = 0. \tag{2.46}$$

Note that the cases $\underset{2}{\rho} = \lambda_2 \neq 0$ and $\underset{3}{\rho} = \lambda_3 \neq 0$ in (2.44) and (2.45) can not occur.

Combining (2.41) and (2.46) we conclude that in the case of (i) three families of Ricci curves become geodesics.

In the case of (ii), from (2.42), we obtain $\theta_3 = 0$. Then, by means of (2.5), (2.1) and (2.2) we find that

$$\underset{3}{n} = \underset{1}{v}, \qquad \underset{3}{b} = \underset{2}{v},$$

$$0 = \underset{3}{v}^l \dot{\nabla}_l \underset{1}{v}^i = \underset{3}{v}^l \dot{\nabla}_l \underset{3}{n}^i = -\underset{3}{\rho}\underset{3}{v}^i + \underset{3}{\tau}\underset{3}{b}^i$$

or, equivalently

$$\underset{3}{\rho} = \underset{3}{\tau} = 0. \tag{2.47}$$

On the other hand, by (2.2), (2.27) and (2.47) we get

$$\underset{3}{v}^l \dot{\nabla}_l \underset{3}{v}^i = \underset{3}{\rho}\underset{3}{n}^i = 0 = \lambda_3 \underset{1}{v}^i + \mu_3 \underset{2}{v}^i$$

from which it follows that $\lambda_3 = \mu_3 = 0$. So, Eq. (2.43) is also satisfied.

Eqs. (2.41) and (2.47) say that in case of (ii) the two families of Ricci curves are geodesics one family of which has vanishing second curvature.

Hence we may state

Theorem 2.2. *In a $W_3(g, T)$ having a definite metric tensor and an indefinite Ricci tensor whose principal directions are distinct, any 3-dimensional Chebyshev net formed by the three families of Ricci curves is either a geodesic net or it consists of a geodesic subnet a member of which has a vanishing second curvature.*

Acknowledgement

The authors would like to thank the referee for useful remarks.

References

[1] V. Hlavaty, Theorie d'immersion d'une W_m dans W_n, Ann. Soc. Polon. Math. 21 (1949) 196–206.

[2] P. Walberer, Riemannsche Raume mit vorgeschriebenen Riccikurven, Hamburg Abhandlungen 10 (1934) 136–151.

[3] L. Eisenhart, The Ricci principal direction vectors, Proc. Nat. Acad. Sci. USA 45 (1959) 1028–1031.

[4] C.E. Weatherburn, An Introduction to Riemannian Geometry and the Tensor Calculus, Cambridge University Press, Cambridge, UK, 1966.

[5] G. Zlatanov, B. Tsareva, On the geometry of the nets in the n-dimensional space of Weyl, J. Geometry 38 (1990) 182–197.