Coulomb corrections in the lepton-pair production in ultrarelativistic nuclear collisions

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We solve the perturbative electron-positron pair production exactly by calculating the second-order Feynman diagrams. We compare our result with Born methods that include Coulomb corrections. We find that a small-momentum approximation is not adequate to obtain exact Coulomb corrections and higher-order terms should also be included. We also compare the impact parameter dependence cross sections.

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I. INTRODUCTION

The process of electron-positron pair production in ultrarelativistic heavy-ion collisions has received renewed interest in the past several years. Because of coherence, large electromagnetic fields are generated in a short period of time. Therefore, photon-photon and nucleus-photon interactions play an important role in these collisions and this could be the source of the production of exotic particles.

There have been many attempts to calculate electromagnetic lepton-pair production cross section. The two-photon process [1–3] has been modeled through the equivalent-photon approximation. In this model, the equivalent-photon flux associated with a relativistic charged particle is obtained via a Fourier decomposition of the electromagnetic interaction. Cross sections are obtained by folding the elementary, real two-photon cross section for the pair production process with the equivalent-photon flux produced by each ion. Although the results for the total cross sections are reasonably accurate, however, the details of the differential cross sections, spectra, and impact-parameter dependence differ. The method loses applicability at impact parameters less than the Compton wavelength of the lepton, which is the region of greatest interest for the study of nonperturbative effects.

Baltz [4] obtained an exact solution to the time-dependent Dirac equation by calculating all orders of pair production. Segev and Wells [5] solved the gauge-transformed Dirac equation using light-front variables and a light-front representation, and obtained nonperturbative results for the free pair-creation amplitudes in the collider frame. Their result reproduces the result of second-order perturbation theory in the small charge limit while nonperturbative effects arise for heavy ions. Similar results are also obtained in other papers [6,7]. All these results coincide with the lowest-order perturbative result without any Coulomb corrections. On the other hand, Ivanov *et al.* [8] and Lee and Milstein [9,10] have argued that a correct regularization of the exact Dirac equation amplitude should give the Coulomb corrections. They also further argue that the Coulomb corrections are universal function of $Z\alpha$ and $f(\alpha Z)$, where

$$f(Z\alpha) = Z^2\alpha^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+Z^2\alpha^2)} \tag{1}$$

is the same function of Bethe and Maximon obtained for Coulomb corrections to electron-positron pair production [11]. In a recent article [12], Baltz has agreed that the Coulomb corrections indeed exist. However, it must be not only the function of $Z\alpha$ and $f(\alpha Z)$ but also γ and ω, the frequency of the virtual photons.

In previous works [15–17], we have calculated the electron-positron pair production cross section by using second-order Feynman diagrams. We have employed Monte Carlo methods and solved it exactly. We have generalized this calculation for all energies and charges of the heavy ions. This gives us a semianalytic cross section and impact parameter dependence of cross-section expressions. We then compare our results with the results obtained by Lee and Milstein. In the following sections, we present our results and argue that the Coulomb correction terms are not exact and these terms need to be improved.

II. FORMALISM

Previously, we obtained the impact parameter dependence cross section of the electron-positron pair production cross section by calculating the second-order Feynman diagrams. Including the direct and exchange terms of Feynman diagrams, the total cross section can be written as

$$\sigma = \int d^2\rho \sum_{\sigma_k}\sum_{\sigma_q}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3} \times |\langle\chi_k^+|S_{direct}|\chi_q^-\rangle + \langle\chi_k^+|S_{exchange}|\chi_q^-\rangle|^2, \tag{2}$$

where

$$\langle\chi_k^+|S_{direct}|\chi_q^-\rangle = \frac{1}{4\beta^2}\int\frac{d^2p_\perp}{(2\pi)^2}e^{i\{\mathbf{p}_\perp-[(\mathbf{k}_\perp+\mathbf{q}_\perp)/2]\}\cdot\boldsymbol{\rho}}\mathcal{A}^+(k,q;\mathbf{p}_\perp) \tag{3}$$

is the explicit form of the direct term and

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$$\langle \chi_k^+ | S_{exchange} | \chi_q^- \rangle = \frac{1}{4\beta^2} \int \frac{d^2 p_\perp}{(2\pi)^2} e^{-i\{\mathbf{p}_\perp - [(\mathbf{k}_\perp + \mathbf{q}_\perp)/2]\}\cdot \boldsymbol{\rho}} \mathcal{A}^-(k,q;\mathbf{p}_\perp) \tag{4}$$

is the exchange term. When we write these explicit forms of the amplitudes in the cross-section expression, we can obtain the total cross-section expression.

$$\sigma = \frac{1}{4\beta^2} \int d^2\rho \sum_{\sigma_k} \sum_{\sigma_q} \int \frac{dk_z dq_z d^2k_\perp d^2p_\perp d^2p'_\perp}{(2\pi)^{10}} e^{i\boldsymbol{\rho}\cdot[\mathbf{p}_\perp - \mathbf{p}'_\perp]} \times [\mathcal{A}^+(k,q;\mathbf{p}_\perp) + \mathcal{A}^-(k,q;\mathbf{k}_\perp + \mathbf{q}_\perp - \mathbf{p}_\perp)] \times [\mathcal{A}^+(k,q;\mathbf{p}'_\perp) + \mathcal{A}^-(k,q;\mathbf{k}_\perp + \mathbf{q}_\perp - \mathbf{p}'_\perp)]^\dagger. \tag{5}$$

We have calculated this cross section numerically by employing Monte Carlo techniques and obtained the following expression as a function of energy for colliding beams of heavy ions:

$$\sigma = C_0 \lambda_C^2 Z_A^2 Z_B^2 \alpha^4 \ln^3(\gamma), \tag{6}$$

where C_0 is the fitted parameter and is equal to 2.19. $\lambda_C = \hbar/mc$ is the reduced Compton wavelength of the electron, and Z_A and Z_B are the charges of the colliding ions.

The impact parameter dependence cross section can be also obtained as

$$\frac{d\sigma}{d\rho} = C_0 \lambda_C^2 Z_A^2 Z_B^2 \alpha^4 \ln^3(\gamma) \frac{\rho_0 \rho}{(\rho_0^2 + \rho^2)^{3/2}}, \tag{7}$$

where ρ_0 is a constant and determined computationally as $1.35\lambda_C$. When we integrate this impact parameter dependence cross section over the impact parameter, we obtain the total cross section, Eq. (1), and the parameter ρ_0 disappears in the total cross section. Details of the calculation of this parameter are explained in Ref. [17].

Lee and Milstein tried to obtain an analytic expression for the total cross section. First, they assume that for all terms in the total cross section, the main contribution to the integrals comes from the region of integration,

$$|\mathbf{k}|, |\mathbf{k}'| \ll m, \quad |p_z|, |q_z| \ll m\gamma, \quad |\mathbf{p}_\perp - \mathbf{q}_\perp| \sim m,$$

and expand $\mathcal{M}$ around $\mathbf{k}=0$—i.e., $\mathcal{M}(\mathbf{k}) \simeq \mathbf{k}\cdot\mathbf{L}$, where

$$\mathcal{M} = \bar{u}(p) \left\{ \frac{\alpha(\mathbf{k} - \mathbf{p}_\perp) + \gamma_0 m}{-p_+ q_- - (\mathbf{k} - \mathbf{p}_\perp)^2 - m^2} \gamma_- + \frac{-\alpha(\mathbf{k} - \mathbf{q}_\perp) + \gamma_0 m}{-p_- q_+ - (\mathbf{k} - \mathbf{q}_\perp)^2 - m^2} \gamma_+ \right\} u(-q) \tag{8}$$

is the matrix element in the amplitude for electron-positron pair production. After this approximation, they calculate the integral representing the difference between the exact and perturbative solution,

$$G = \int \frac{d^2k}{(2\pi)^2} k^2 [|F(\mathbf{k})|^2 - |F^0(\mathbf{k})|^2], \tag{9}$$

where

$$F(\mathbf{k}) = \int d^2\rho \exp[-i\mathbf{k}\cdot\boldsymbol{\rho}]\{\exp[-i\chi(\rho)] - 1\}, \tag{10}$$

$$\chi(\rho) = \int_{-\infty}^{\infty} dz V(z,\rho) \tag{11}$$

and

$$F^0(\mathbf{k}) = -i \int d^2\boldsymbol{\rho} \exp[-i\mathbf{k}\cdot\boldsymbol{\rho}]\chi(\rho) \tag{12}$$

is the perturbative limit of $F(\mathbf{k})$. With the help of G, the authors derive the following terms in the total cross section of electron-positron pair production:

$$\sigma_{total} = \sigma^b + \sigma_A^c + \sigma_B^c + \sigma_{AB}^c, \tag{13}$$

where

$$\sigma^b = \frac{28(Z_A\alpha)^2(Z_B\alpha)^2}{27\pi m^2} \ln^3(\gamma^2) \tag{14}$$

is the Born cross section,

$$\sigma_A^c = -\frac{28(Z_A\alpha)^2(Z_B\alpha)^2}{9\pi m^2} f(Z_A\alpha)\ln^2(\gamma^2) \tag{15}$$

is the Coulomb correction obtained from nucleus A by taking nucleus B to lowest order in $Z\alpha$, and

$$\sigma_B^c = -\frac{28(Z_A\alpha)^2(Z_B\alpha)^2}{9\pi m^2} f(Z_B\alpha)\ln^2(\gamma^2) \tag{16}$$

is the Coulomb correction obtained from nucleus B by taking nucleus A to lowest order in $Z\alpha$. The last term in the total cross-section expression

$$\sigma_{AB}^c = \frac{56(Z_A\alpha)^2(Z_B\alpha)^2}{9\pi m^2} f(Z_A\alpha) f(Z_B\alpha)\ln(\gamma^2) \tag{17}$$

is also obtained with the same approach. In the above equations,

$$f(Z\alpha) = Z^2\alpha^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + Z^2\alpha^2)} \tag{18}$$

was also derived exactly by Bethe and Maximon for Coulomb corrections to lepton-pair production in the collisions of heavy ions. Neglecting the last term σ_{AB}^c in the total cross section, the result is in complete agreement with the Coulomb corrections obtained by Ivanov *et al.* [8].

In a recent paper [13], the Racah formula for the total electron-positron pair production cross section in perturbative theory [14] is given as

$$\sigma_R = \frac{(Z_A\alpha)^2(Z_B\alpha)^2}{\pi m^2} \left\{ \frac{28}{27}\mathcal{L}^3 - \frac{178}{27}\mathcal{L}^2 + \frac{370 + 7\pi^2}{27}\mathcal{L} - \frac{116}{9} - \frac{13\pi^2}{54} + \frac{7}{9} 1.202 \right\}, \tag{19}$$

where

$$\mathcal{L} = \log 2(2\gamma^2 - 1). \tag{20}$$

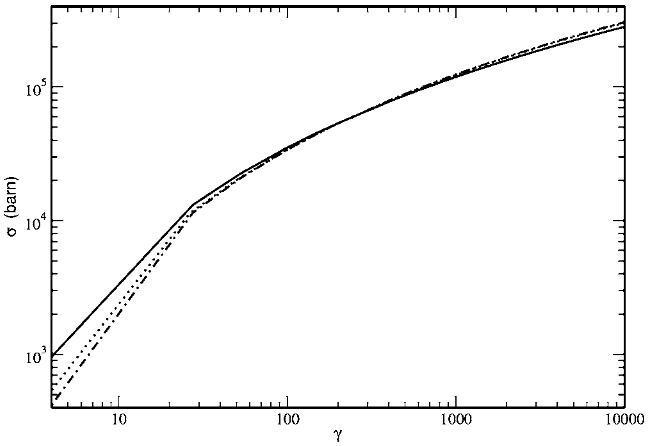

FIG. 1. Total cross section of electron-positron pair production as a function of energy. The solid line is the Monte Carlo calculation, the dashed-dot line is the Born approximation with Coulomb corrections, and the dotted line is the Racah equation.

In Fig. 1, we compare the total e^+e^- pair production cross sections for the Monte Carlo calculations, Born approximation with Coulomb corrections, and Racah formula. We have also tabulated the total cross sections obtained from these methods for the energies $\gamma=10$, 100, and 3400 and for Au+Au collisions in Table I. These calculations show that the agreement of the Monte Carlo calculation, Born approximation with Coulomb correction, and Racah equation is very good especially for $\gamma \geqslant 20$.

III. RESULTS

In this section, we are going to present the numerical calculation of the differential cross sections of momenta of produced pairs. These calculations are shown in Figs 2 and 3. The ratio of the small-momentum region to all longitudinal momenta can be written as

$$\frac{\int_0^{k_z} dk_z' \frac{d\sigma}{dk_z'}}{\int_0^{\infty} dk_z' \frac{d\sigma}{dk_z'}} \tag{21}$$

and similarly the ratio of the small-transverse-momentum $k_\perp$ region to all transverse momenta can be written as

TABLE I. Total pair production cross sections for Au+Au collisions.

	$\gamma=10$	$\gamma=100$	$\gamma=3400$
Monte Carlo calculation	4349	35148	193499
Born approximation	5298	42383	233329
Born approximation with c.c.	3284	34035	206903
Racah equation	3622	34087	205073

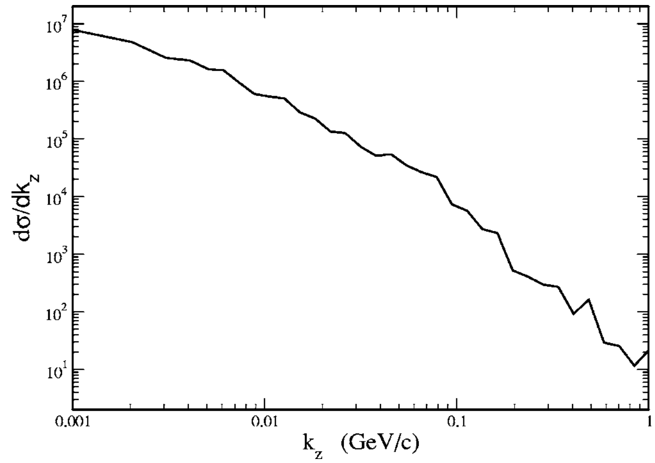

FIG. 2. Monte Carlo calculation of the differential cross section of pair production as a function of the longitudinal momentum of produced pairs.

$$\frac{\int_0^{k_\perp} dk_\perp' \frac{d\sigma}{dk_\perp'}}{\int_0^{\infty} dk_\perp' \frac{d\sigma}{dk_\perp'}}. \tag{22}$$

In this calculation, the energy of the colliding heavy ions is $\gamma=100$, and we used the fully stripped gold ions, $Z=79$. In Figs. 2 and 3, we show the differential cross sections of the transverse and longitudinal momentum spectrum. In the calculation, it is very clear that the longitudinal momentum of the produced leptons is much higher than the transverse momentum, and the produced leptons move along the heavy ions, which is observed also experimentally. Although in the small-momentum approximation the longitudinal momentum is much smaller than γm ($k_z \ll \gamma m$), in our calculation, Eq. (21), we have taken the upper limit as $k_z \sim 0.1\gamma m$, about 10% of γm, which is at the order of or higher than the small-momentum approximation. We find that the ratio is about

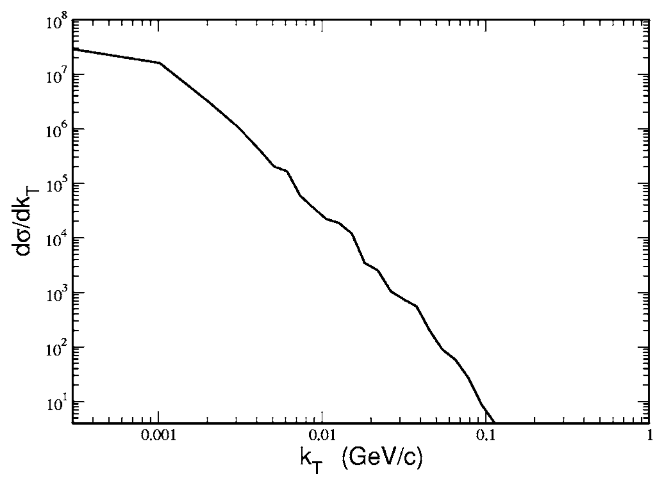

FIG. 3. Monte Carlo calculation of the differential cross section of pair production as a function of the transverse momentum of produced pairs.

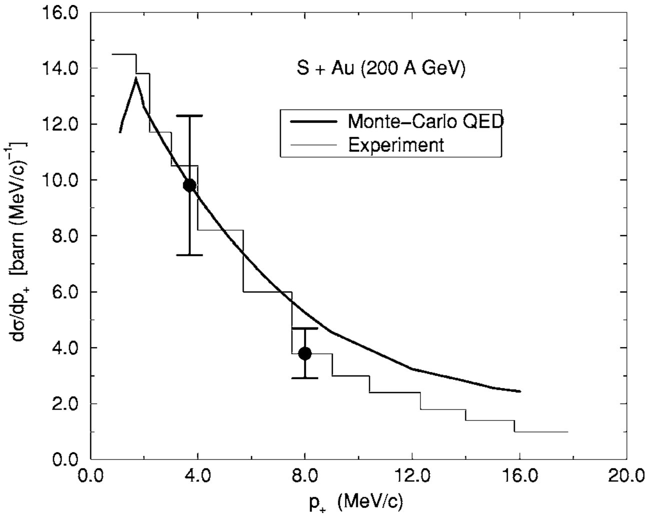

FIG. 4. Experimental and Monte Carlo calculations of the differential cross section as a function of positron momentum.

0.4–0.5. For the smaller upper limit the ratio becomes even smaller. In Eq. (22), we have taken the upper limit of the transverse momentum as $k_\perp \sim 0.5m$ and $k_\perp \sim m$. In these limits the ratio is about between 0.1 and 0.4 depending on the upper limits of the transverse momentum. It is clear that the small-transverse-momentum approximation cannot describe the Coulomb corrections, since 60% and higher of the contribution to the integral comes from $k_\perp \sim$ the $0.5m \sim m$ region. This clearly shows that first-order terms in k are not sufficient to calculate Coulomb corrections. One should include the higher-order terms in k to obtain more accurate expressions. Baltz has evaluated Eq. (9), the integral representation of the difference between the exact solution and the perturbative solution numerically. In this calculation, there is strong evidence that the Coulomb correction terms should be a function of γ, ω, and $Z\alpha$.

We have used the Monte Carlo method to calculate these differential cross sections. This calculation was done by Bottcher and Strayer previously [15]. We have generalized this calculation and also compared with CERN data. The convergences are obtained for 100×10^4 sets of random numbers for the variables in the equation. We also monitor the error in the calculation, and it is within 1% of the total values.

In Fig. 4 we present the differential cross section for the magnitude of the positron momentum, $d\sigma/dp^+$, for 200A GeV (at fixed target) sulfur on gold. The experiment measured positrons with momenta in the range $1-17$ MeV/c. In order to compare our calculation with the experiment, we have applied a similar cut to the calculation. We see that the theory does quite well with the experiment, particularly at smaller momenta below 6 MeV/c. Integrating over the range of the data (from 1 to 17 MeV/c), our calculation give 98 barns while the data give 85 ∓ 12 mb. From this calculation and experimental data, it is quite clear that most of the momentum of the produced leptons is higher than γm. In Ref. [18] we have shown our calculation of angular width $1/e$ of the positron spectrum as a function of the magnitude of the positron momentum. This calculation also agrees with the experimental observation.

Another shortcoming of the Born approximation is the small-impact-parameter region for pair production. In Ref. [19], Lee, Milstein, and Serbo have plotted the impact parameter dependence probability of the electron-positron production. They have improved the previously derived equivalent photon approximation

$$W_0(\rho) = \frac{14}{9\pi^2}\frac{Z_A^2 Z_B^2\alpha^4}{\rho^2}\left(\ln\frac{0.681\gamma^2}{\rho}\right)^2, \tag{23}$$

which is valid in the impact parameter region of $\lambda_C \leqslant \rho \leqslant 0.681\gamma\lambda_C$, as

$$W_0(\rho) = \frac{28}{9\pi^2}\frac{Z_A^2 Z_B^2\alpha^4}{\rho^2}[2\ln\gamma^2 - 3\ln(\rho)]\ln\rho \tag{24}$$

valid for the region $\lambda_C \leqslant \rho \leqslant \gamma\lambda_C$ and

$$W_0(\rho) = \frac{28}{9\pi^2}\frac{Z_A^2 Z_B^2\alpha^4}{\rho^2}\left(\ln\frac{\gamma^2}{\rho}\right)^2 \tag{25}$$

valid for $\gamma\lambda_C \leqslant \rho \leqslant \gamma^2$. This improvement alone does not solve the inadequacies of the equivalent photon approximation because it is still invalid for impact parameters less than λ_C, the Compton wavelength of the electron. On the other hand, Monte Carlo calculation gives an equation for the impact parameter dependence cross section valid for all impact parameters. In these impact parameter regions, the electromagnetic field is very high and a detailed study of this region is important for nonperturbative effects.

By using Eq. (7), we can write the lowest-order probability of producing the electron-positron pairs as

$$W_0^{MC}(\rho) = \frac{1}{2\pi\rho}\frac{d\sigma}{d\rho} = C_0\frac{1}{2\pi}\lambda_C^2 Z_A^2 Z_B^2\alpha^4\ln^3(\gamma)\frac{\rho_0}{(\rho_0^2+\rho^2)^{3/2}}. \tag{26}$$

The pair production probability is a continuous, well-behaved function and valid for all impact parameters. When we compare the Monte Carlo calculations, Born approximation with Coulomb corrections, and Racah equation results in Fig. 1, we see that for low energies $\leqslant$100 GeV the Monte Carlo calculation is higher than the Born approximation with Coulomb corrections and higher than the Racah equation. For the higher energies $\geqslant$100 GeV Monte Carlo results are lower than the Born and Racah calculations. However, in general the three results agree with each other for the impact parameter region of $\lambda_C \leqslant \rho \leqslant \gamma\lambda_C$. From this agreement we can assume that the Born approximation should also be valid for the small-impact-parameter region mainly less than one Compton wavelength of the electron. As an ansatz we can write the probability of electron-positron pair production as

$$W_0^{Born} = \frac{1}{2\pi\rho}\frac{d\sigma}{d\rho} = \frac{1}{2\pi}(\sigma^b + \sigma_A^c + \sigma_B^c + \sigma_{AB}^c)\frac{\rho_0}{(\rho_0^2+\rho^2)^{3/2}}. \tag{27}$$

When we integrate Eqs. (27) and (26) over the impact parameter ρ, we obtain the total cross-section equations (13) and (1), respectively, and the parameter ρ_0 becomes a

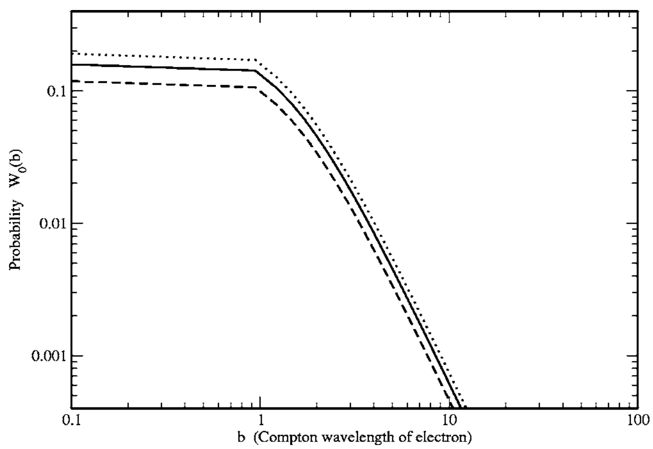

FIG. 5. Probability of pair production as a function of impact parameter for energies $\gamma=10$ and Au+Au collisions. The solid line is the Monte Carlo calculation, the dotted line is the Born approximation, and the dashed line is the Born approximation with Coulomb corrections.

dummy parameter which comes from the Monte Carlo calculation. In Eq. (27), instead of the Born results, when we insert the Racah formula for the total cross section, we can obtain very similar results for the probabilities of electron-positron pair production, since both results agree with each other very well. In Figs. 5–7 we compare the Monte Carlo calculation of the probability and our suggestion of an approximate impact parameter dependence probability of the Born calculation with Coulomb corrections. The agreement is very good for the valid impact parameters, and it strongly suggests that for impact parameters less than one Compton wavelength of the electron, the Born calculation should have finite values.

Since these probabilities are greater than 1 for high energies, the multipair production cross section can be written as Poisson distribution

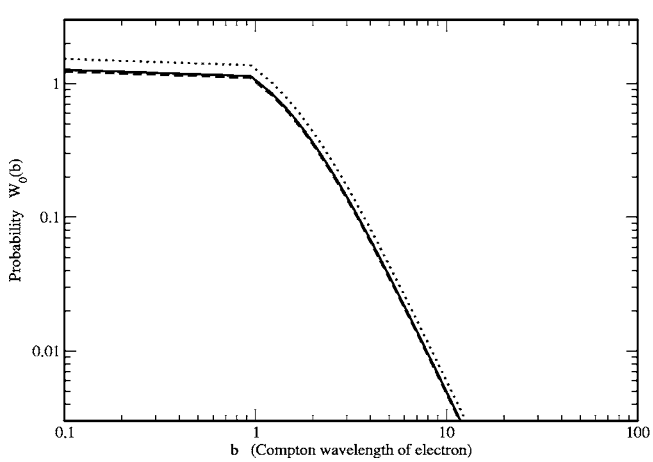

FIG. 6. Probability of pair production as a function of impact parameter for energies $\gamma=100$ and Au+Au collisions. The solid line is the Monte Carlo calculation, the dotted line is the Born approximation, and the dashed line is the Born approximation with Coulomb corrections.

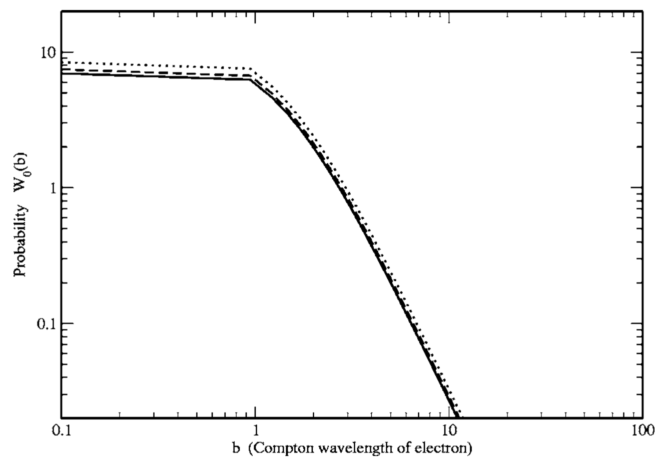

FIG. 7. Probability of pair production as a function of impact parameter for energies $\gamma=3400$ and Au+Au collisions. The solid line is the Monte Carlo calculation, the dotted line is the Born approximation, and the dashed line is the Born approximation with Coulomb corrections.

$$P_N(\rho) = \frac{W_0^N \exp(-W_0)}{N!} \tag{28}$$

We have compared the three calculations in Figs. 5–7 for energies of $\gamma=10$, 100, 3400 and for Au+Au collisions. In these figures, we clearly see that as the energy increases, the Born calculation alone and that with Coulomb correction results are higher than the Monte Carlo calculation. In Tables II–IV we have also calculated *N-pair* production cross sections as

$$\sigma_{N_{pair}} = \int_0^\infty 2\pi b db P_N(b). \tag{29}$$

In these tables, for $\gamma=10$ and 100, *N-pair* cross sections of Born results exceed the Monte Carlo calculation and Born results with Coulomb corrections are substantially lower than the Monte Carlo results. On the other hand, at LHC energies, both calculations are higher than the Monte Carlo calculation. The results are also very similar for the Racah equation.

IV. CONCLUSION

We have calculated the electron-positron pair production cross section exactly by using the Monte Carlo method. In this calculation, we have not made any approximation and calculated the cross sections exactly. On the other hand, the

TABLE II. Monte Carlo calculation of *N*-pair production cross sections for Au+Au collisions.

N	$\gamma=10$	$\gamma=100$	$\gamma=3400$
1	4140	24129	77035
2	123	3277	12839
3	5.8	944	5705
4	0.26	286	3324

TABLE III. N-pair production cross section calculations for Au+Au collisions with the Born approximation method.

N	$\gamma=10$	$\gamma=100$	$\gamma=3400$
1	4932	27602	87274
2	174	4013	14546
3	9.9	1297	6465
4	0.53	453	3770

TABLE IV. N-pair production cross section calculations for Au+Au collisions with the Born approximation including the Coulomb corrections method.

N	$\gamma=10$	$\gamma=100$	$\gamma=3400$
1	3142	23568	80553
2	71	3158	13425
3	2.5	891	5966
4	0.1	263	3478

authors of Ref. [10] made a small-momentum approximation (small k and small transverse momentum $p_{\perp}$) and obtained an analytic expression. Although most of the integration comes from the small-momentum range, the lowest order in transverse momentum is not adequate to obtain accurate Coulomb corrections and higher orders should be also included. This was first noticed by Baltz [12], and in this work we were also convinced that the small-momentum approximation alone is not adequate to obtain correct Coulomb corrections.

In addition to this, Monte Carlo calculations and the Born approximation give a similar total cross section and impact parameter dependence cross section. However, Born approximation results are valid for impact parameters only above the one-electron Compton wavelength. We made an assumption that, since both results agree for the valid impact parameter region, they should also behave similarly for the small-impact-parameter region. We have applied our results that are obtained by computational calculations to the Born approximation and obtain a well-behaved impact parameter dependence cross section and probabilities.

Recent publications about peripheral relativistic heavy-ion collisions [20–24] show that the impact parameter dependence cross sections of lepton-pair production are very important and detailed knowledge of impact parameter dependence cross sections particularly for small impact parameters can help to understand many physical events in STAR experiments.

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[1] C. F. von Weizsacker, Z. Phys. **88**, 612 (1934).
[2] S. J. Brodsky, T. Kinotshita, and H. Terazawa, Phys. Rev. D **4**, 1532 (1971).
[3] C. A. Bertulani and G. Baur, Phys. Rep. **163**, 299 (1988).
[4] A. J. Baltz, Phys. Rev. Lett. **78**, 1231 (1997).
[5] B. Segev and J. C. Wells, Phys. Rev. A **57**, 1849 (1998).
[6] Anthony J. Baltz and Larry McLerran, Phys. Rev. C **58**, 1679 (1998).
[7] B. Segev and J. C. Wells, Phys. Rev. C **59**, 2753 (1999).
[8] D. Y. Ivanov, A. Schiller, and V. G. Serbo, Phys. Lett. B **454**, 155 (1999).
[9] R. N. Lee and A. I. Milstein, Phys. Rev. A **61**, 032103 (2000).
[10] R. N. Lee and A. I. Milstein, Phys. Rev. A **64**, 032106 (2001).
[11] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).
[12] A. J. Baltz, Phys. Rev. C **68**, 034906 (2003).
[13] A. J. Baltz, Phys. Rev. C **71**, 024901 (2005).
[14] G. Racah, Nuovo Cimento **14**, 93 (1937).
[15] C. Bottcher and M. R. Strayer, Phys. Rev. D **39**, 1330 (1989).
[16] M. C. Guclu, J. C. Wells, A. S. Umar, M. R. Strayer, and D. J. Ernst, Phys. Rev. A **51**, 1836 (1995).
[17] M. C. Guclu, Nucl. Phys. A **668**, 149 (2000).
[18] M. C. Guclu, J. Li, A. S. Umar, D. J. Ernst, and M. R. Strayer, Ann. Phys. (N.Y.) **272**, 7 (1999).
[19] R. N. Lee, A. I. Milstein, and V. G. Serbo, Phys. Rev. A **65**, 022102 (2002).
[20] S. R. Klein, e-print hep-ex/0402028.
[21] S. R. Klein, e-print nucl-ex/0310020.
[22] V. B. Morozov, Ph.D. dissertation, UC Berkeley, 2003.
[23] G. Baur, K. Hencken, A. Aste, D. Trautmann, and S. R. Klein, Nucl. Phys. A **729**, 787 (2003).
[24] K. Hencken, G. Baur, and D. Trautmann, Phys. Rev. C **69**, 054902 (2004).