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CONFORMAL AND GENERALIZED CONCIRCULAR MAPPINGS OF EINSTEIN-WEYL MANIFOLDS*

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Abstract In this article, after giving a necessary and sufficient condition for two Einstein-Weyl manifolds to be in conformal correspondence, we prove that any conformal mapping between such manifolds is generalized concircular if and only if the covector field of the conformal mapping is locally a gradient. Using this fact we deduce that any conformal mapping between two isotropic Weyl manifolds is a generalized concircular mapping. Moreover, it is shown that a generalized concircularly flat Weyl manifold is generalized concircular to an Einstein manifold and that its scalar curvature is prolonged covariant constant.

Key words Einstein-Weyl manifold; conformal mapping; generalized concircular mapping; isotropic manifold

2000 MR Subject Classification 53A30

1 Introduction

A geodesic circle in a Riemannian manifold was defined in [1] as a curve whose first curvature is constant and second curvature vanishes identically. Circles and spheres in Riemannian geometry are defined and studied from the point of view of development by K. Nomizu and K. Yano in [2].

In a series of papers (cf. [3]), K. Yano gave the foundations of Concircular Geometry, the geometry in which concircular transformations and spaces admitting such transformations are considered. Some problems concerning the concircular transformations of Riemannian manifolds were also studied in [4, 5].

In [6], as a generalization of a geodesic circle in a Riemannian manifold, by using the prolonged covariant differentiation we defined the so-called generalized circle in a Weyl manifold as a curve whose first curvature is prolonged covariant constant and its second curvature vanishes identically.

After introducing the notion of generalized circle, it seems natural to study the conformal mapping of a Weyl manifold upon another which preserves the generalized circles. Such a conformal mapping is named as a generalized concircular mapping or a generalized concircular transformation. Then, we can speak of generalized concircular Weyl geometry.

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It is well known that any conformal mapping of an Einstein manifold upon another is a concircular mapping and that an Einstein manifold can be transformed into an Einstein manifold by a concircular mapping [3, 4]. However, the first part of this statement is not in general true for Einstein-Weyl manifolds of dim > 2 while the second part holds true, which will be proved in Theorem 3.1.

In this work, we first give a necessary and sufficient condition in order that two Einstein-Weyl manifolds of dim > 2 may be in conformal correspondence (Lemma 3.1) and then prove that any conformal mapping between two Einstein-Weyl manifolds is generalized concircular if and only if the covector field of the conformal mapping is locally a gradient (Theorem 3.1(a)). However, an Einstein-Weyl manifold is transformed into an Einstein-Weyl manifold under a generalized concircular mapping (Theorem 3.1 (b)). In Riemannian geometry, it was proved that a manifold of constant curvature is transformed into a manifold of constant curvature by a concircular transformation [3]. In this work, as a corollary to Theorem 3.1,we prove that any conformal mapping between two isotropic Weyl manifolds is a generalized concircular one (Corollary 3.1). Finally, within the framework of generalized concircular geometry, we give a sufficient condition for a Weyl manifold to be generalized concircular to an Einstein manifold (Theorem 3.2).

2 Preliminaries

A differentiable manifold of dimension n having a conformal class C of metrics and a torsion- free connection ∇ preserving the conformal class C is called a Weyl manifold, denoted by $W_n(g,w)$, where $g \in C$ and w is a 1-form satisfying the so-called compatibility condition

$$\nabla g = 2(g \otimes w). \tag{1}$$

Under the conformal re-scaling (re-normalisation)

$$\bar{g} = \lambda^2 g \qquad (\lambda > 0) \tag{2}$$

of the representative metric tensor g, w is transformed by the law

$$\bar{w} = w + d\ln\lambda. \tag{3}$$

A quantity A defined on $W_n(g,w)$ is called a satellite of g of weight $\{p\}$ if it admits a transformation of the form

$$\bar{A} = \lambda^p A \tag{4}$$

under the conformal re-scaling (2) of g [7–9].

It can be easily seen that the pair $(\bar{g},\bar{w})$ generates the same Weyl manifold. The process of passing from (g,w) to $(\bar{g},\bar{w})$ is called a gauge transformation.

The curvature tensor, covariant curvature tensor, the Ricci tensor and the scalar curvature of $W_n(g,w)$ are respectively defined by

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)v^p = v^j W^p_{jkl}\ , \tag{5}$$

$$W_{hjkl} = g_{hp}\ W^p_{jkl}\ , \tag{6}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \tag{7}$$

$$W = g^{ij}\ W_{ij}. \tag{8}$$

From (5) it follows that

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}, \partial_k\ = \frac{\partial}{\partial x^k}, \tag{9}$$

where Γ^i_{kl} are the coefficients of the Weyl connection ∇ given by

$$\Gamma^i_{kl} = \left\{ {i \atop kl} \right\} - (\delta^i_k w_l + \delta^i_l w_k - g_{kl} g^{im} w_m), \tag{10}$$

in which $\left\{ {i \atop kl} \right\}$ are the coefficients of the Levi-Civita connection.

By straightforward calculations it is easy to see that the antisymmetric part of W_{ij} has the property

$$W_{[ij]} = n \nabla_{[i} w_{j]}, \tag{11}$$

where brackets indicate antisymmetrization.

The prolonged (extended) covariant derivative of the satellite A of weight $\{p\}$ in the direction of the vector X is defined by

$$\dot{\nabla}_X A = \nabla_X A - p w(X) A \tag{12}$$

from which it follows that $\dot{\nabla}_X\ g = 0$ for any X [7, 9].

A satellite of g is called prolonged covariant constant if its prolonged covariant derivative vanishes identically.

A Riemannian manifold is called an Einstein manifold if its Ricci tensor is proportional to its metric.

A Weyl manifold is said to be an Einstein-Weyl manifold if the symmetric part of the Ricci tensor is proportional to the metric $g \in C$ [10, 11], and hence

$$W_{(ij)} = \frac{W}{n}\ g_{ij}. \tag{13}$$

We call a manifold an isotropic manifold if at each point of the manifold the sectional curvature is independent of the plane chosen [12].

In [6], as a generalization of geodesic circles in a Riemannian manifold, we defined the so-called generalized circles by means of prolonged covarinat differentiation as follows.

Definition 2.1 [6] Let C be a smooth curve belonging to the Weyl manifold $W_n(g, w)$ and ξ_1 be the tangent vector to C at the point P, normalized by the condition $g(\xi_1, \xi_1) = 1$. C is called a generalized circle in $W_n(g, w)$ if there exist a vector field ξ_2, normalized by the condition $g(\xi_2, \xi_2) = 1$ and a positive prolonged covarinat constant scalar function κ_1 of weight $\{-1\}$ along C, such that

$$\dot{\nabla}_{\xi_1} \xi_1 = \kappa_1 \xi_2, \qquad \dot{\nabla}_{\xi_1} \xi_2 = -\kappa_1 \xi_1. \tag{14}$$

According to the Frenet formulas

$$\dot{\nabla}_{\xi_1} \xi_m = -\kappa_{m-1} \xi_{m-1} + \kappa_m \xi_{m+1},\ m = 1, 2, \cdots, n; \quad \kappa_0 = \kappa_n = 0$$

given in [7], the equations (14) imply that C is a generalized circle if and only if the first curvature κ_1 of C is prolonged covariant constant and the second curvature κ_2 is zero along C. Namely,

$$\dot{\nabla}_{\xi_1}\kappa_1 = \nabla_{\xi_1}\kappa_1 + \kappa_1 w(\xi_1) = 0, \qquad \kappa_2 = 0. \tag{15}$$

We note that equations (15) are invariant under a conformal re-scaling of the metric g.

A conformal mapping of a Weyl manifold upon another Weyl manifold is called generalized concircular if it preserves the generalized circles [6]. Concerning generalized concircular mappings we have the following two theorems.

Theorem 2.1 [6] The conformal mapping $\tau : W_n(g,w)\to\tilde{W}_n(\tilde{g},\tilde{w})$ will be generalized concircular if and only if

$$P_{kl} = \phi\ g_{kl}, P_{kl} = \nabla_l P_k - P_k P_l + \frac{1}{2} g_{kl} g^{rs} P_r P_s, \tag{16}$$

where

$$P = w - \tilde{w} \tag{17}$$

is the covector field of the conformal mapping of weight zero and ϕ is a smooth scalar function of weight $\{-2\}$ defined on $W_n(g,w)$.

Theorem 2.2 [6] The tensor Z of type $(1,3)$ whose components are given by

$$Z^p_{jkl} = W^p_{jkl} - \frac{W}{n(n-1)}(\delta^p_l g_{jk} - \delta^p_k g_{jl}) \tag{18}$$

is invariant under a generalized concircular mapping of $W_n(g,w)$. Such a tensor is called the generalized concircular curvature tensor of $W_n(g,w)$. Contraction on the indices p and l in (18) gives the generalized concircularly invariant tensor

$$Z^p_{jkp} = Z_{jk} = W_{jk} - \frac{W}{n} g_{jk}. \tag{19}$$

3 Conformal and Generalized Concircular Mappings of Einstein-Weyl Manifolds

In this section, we first study the conformal mappings of Einstein-Weyl manifolds and prove a lemma which will be needed in our subsequent work. Let τ be a conformal mapping of the Weyl manifold $W_n(g,w)$ upon another Weyl manifold $\tilde{W}(\tilde{g},\tilde{w})$. It is clear that the case $n=1$ is of no interest. By straightforward calculations it can be shown that every 2-dimensional Weyl manifold is an Einstein-Weyl manifold and that any two 2-dimensional Weyl manifolds can be locally mapped conformally upon each other. So, in what follows we assume that $n>2$.

At corresponding points of $W_n(g,w)$ and $\tilde{W}(\tilde{g},\tilde{w})$ we can make [8, 9]

$$g = \tilde{g}. \tag{20}$$

It is clear that the covector field $P = w - \tilde{w}$ of τ is of zero weight.

Let ∇ and $\tilde{\nabla}$ be the connections of $W_n(g,w)$ and $\tilde{W}(\tilde{g},\tilde{w})$ and let the connection coefficients be denoted by Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$, respectively. Then, by (10) and (20), we have

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j P_k + \delta^i_k P_j - g^{im} P_m g_{jk}. \tag{21}$$

Replacing Γ^i_{jk} in (9) by $\tilde{\Gamma}^i_{jk}$ in (21), we obtain the curvature tensor of $\tilde{W}(\tilde{g}, \tilde{w})$ as [6]

$$\tilde{W}^p_{jkl} = W^p_{jkl} + \delta^p_l P_{jk} - \delta^p_k P_{jl} + g_{jk} g^{pm} P_{ml} - g_{jl} g^{pm} P_{mk} + 2\delta^p_j \nabla_{[k} P_{l]}, \tag{22}$$

where $\nabla_{[k} P_{l]}$ is the antisymmetric part of $\nabla_k P_l$ and

$$P_{kl} = \nabla_l P_k - P_k P_l + \frac{1}{2} g_{kl} g^{rs} P_r P_s. \tag{23}$$

Contraction on the indices p and l in (22) gives

$$\tilde{W}_{jk} = W_{jk} + (n-2)P_{jk} + g_{jk} g^{lm} P_{ml} + 2\nabla_{[k} P_{j]} \tag{24}$$

in which we have used the relation $g_{jk} g^{km} = \delta^m_j$. Remembering that $g^{mk} P_{[mk]} = 0$, we may conclude that

$$g^{mk} P_{mk} = g^{mk} P_{(mk)}. \tag{25}$$

In view of (24) and (25) we obtain

$$\tilde{W}_{(jk)} = W_{(jk)} + (n-2)P_{(jk)} + g_{jk} g^{lm} P_{(ml)}. \tag{26}$$

Multiplying (26) by $\tilde{g}^{jk} = g^{jk}$ and summing up, and using the fact that $\tilde{g}^{jk}\tilde{W}_{(jk)} = \tilde{W}$, $g^{jk} W_{(jk)} = W$, we obtain $\tilde{W} = W + 2(n-1) g^{jk} P_{(jk)}$, from which it follows that

$$g^{jk} P_{(jk)} = \frac{\tilde{W} - W}{2(n-1)}. \tag{27}$$

By virtue of (27), (26) becomes

$$\tilde{W}_{(jk)} = W_{(jk)} + (n-2)P_{(jk)} + \frac{\tilde{W} - W}{2(n-1)} g_{jk}. \tag{28}$$

Suppose now that $W(g, w)$ and $\tilde{W}(\tilde{g}, \tilde{w})$ are Einstein-Weyl manifolds. Then, since

$$W_{(jk)} = \frac{W}{n} g_{jk}, \qquad \tilde{W}_{(jk)} = \frac{\tilde{W}}{n} \tilde{g}_{jk},$$

(28) transforms into

$$(n-2)\left[P_{(jk)} - \frac{\tilde{W} - W}{2n(n-1)} g_{jk}\right] = 0 \tag{29}$$

or, for $n > 2$, we get

$$P_{(jk)} = \frac{\tilde{W} - W}{2n(n-1)} g_{jk} \quad (n > 2). \tag{30}$$

Conversely, suppose that $W_n(g, w)$ is an Einstein-Weyl manifold and that the condition (30) is satisfied. Then, from (28) we obtain

$$\tilde{W}_{(jk)} - \frac{\tilde{W}}{n} \tilde{g}_{jk} = W_{(jk)} - \frac{W}{n} g_{jk} = 0$$

showing that $\tilde{W}_n(\tilde{g}, \tilde{w})$ is also an Einstein-Weyl manifold. We have thus proved

Lemma 3.1 Let $W_n(g, w)$ and $\tilde{W}_n(\tilde{g}, \tilde{w})$ be two Einstein-Weyl manifolds of dim > 2 which are in conformal correspondence. Then, the condition $P_{(jk)} = \frac{\tilde{W}-W}{2n(n-1)} g_{jk}$ holds true.

Suppose that $W_n(g,w)$ is an Einstein-Weyl manifold and that the condition $P_{(jk)} = \frac{\tilde{W}-W}{2n(n-1)} g_{jk}$ is fulfilled. Then, the conformal transformation of $W_n(g,w)$ is the Einstein-Weyl manifold $\tilde{W}_n(\tilde{g},\tilde{w})$, $\tilde{g}=g$.

K. Yano [3] and Y. Tashiro [4] proved, among other things, the following theorem for (Riemannian) Einstein manifolds.

Theorem (a) If an Einstein manifold is conformal to another Einstein manifold, then such an Einstein manifold must admit a concircular transformation.

(b) An Einstein manifold is transformed into an Einstein manifold by a concircular transformation.

We now generalize this theorem to Einstein-Weyl manifolds.

Thereom 3.1 (a) Let $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$ be two Einstein-Weyl manifolds and let τ be a conformal mapping of $W_n(g,w)$ into $\tilde{W}_n(\tilde{g},\tilde{w})$. Then, τ is a generalized concircular mapping if and only if the covector field P of τ is locally a gradient.

(b) An Einstein-Weyl manifold is transformed into an Einstein-Weyl manifold under any generalized concircular mapping.

Proof of (a) Necessity. Suppose that τ is generalized concircular. Then, by (16), the tensor P_{ij} is symmetric and consequently its antisymmetric part $P_{[ij]}$ becomes zero. On the other hand, from (23) we obtain

$$P_{[ij]} = \nabla_{[j} P_{i]} = \frac{1}{2}(\partial_j P_i - \partial_i P_j) = 0,$$

which implies that P is locally a gradient.

To prove the sufficiency of the condition, let us assume that the covector field P of τ is locally a gradient. Then we have $P_{[ij]} = 0$ and so the tensor P_{ij} becomes symmetric. Since $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$ are supposed to be conformal, by Lemma 3.1, we obtain

$$P_{(ij)} = P_{ij} = \frac{\tilde{W}-W}{2n(n-1)} g_{ij},$$

which, according to Theorem 2.1, states that $W_n(g,w)$ must admit a generalized concircular mapping with $\phi = \frac{\tilde{W}-W}{2n(n-1)}$.

Proof of (b) Let $W_n(g,w)$ be an Einstein-Weyl manifold and let it be transformed into the Weyl manifold $\hat{W}_n(\hat{g},\hat{w})$ by the generalized concircular mapping $\hat{\tau}$. Since the generalized concircular tensor Z^p_{jkl}, defined by (18), and its contracted tensor Z_{jk} are generalized concircularly invariant, we have from (19) that

$$Z_{jk} = \hat{Z}_{jk} \Rightarrow \hat{W}_{jk} - \frac{\hat{W}}{n}\hat{g}_{jk} = W_{jk} - \frac{W}{n} g_{jk},$$

from which it follows that

$$\hat{W}_{(jk)} - \frac{\hat{W}}{n}\hat{g}_{jk} = W_{(jk)} - \frac{W}{n} g_{jk}. \tag{31}$$

Since $W_n(g,w)$ is supposed to be an Einstein-Weyl manifold, the right-hand member of (31) vanishes. Consequently, $\hat{W}_{jk} - \frac{\hat{W}}{n}\hat{g}_{jk} = 0$, showing that $\hat{W}_n(\hat{g},\hat{w})$ is also an Einstein-Weyl manifold.

As a corollary to the above theorem we may state

Corollary 3.1 Any conformal mapping between two isotropic Weyl manifolds of dim > 2 is a generalized concircular mapping.

Proof According to Theorem 2.1, proved in [13], any isotropic Weyl manifold is Einstein-Weyl and its covector field (the 1-form ω) is locally a gradient. Then, by the first part of Theorem 3.1, the result follows.

In [13], a sufficient condition for a Weyl manifold to be locally conformal to an Einstein manifold was given in terms of sectional curvatures. Within the framework of generalized concircular Weyl geometry, we now give another sufficient condition for a Weyl manifold to be locally generalized concircular to an Einstein manifold.

Theorem 3.2 A generalized concircularly flat Weyl manifold is generalized concircular to an Einstein manifold and its scalar curvature is prolonged covariant constant.

Proof Let the Weyl manifold $W_n(g, w)$ be generalized concircularly flat. Then, according to (18), we have

$$Z^p_{jkl} = W^p_{jkl} - \frac{W}{n(n-1)}(\delta^p_l g_{jk} - \delta^p_k g_{jl}) = 0.$$

Contraction on the indices p and l gives

$$Z^p_{jkp} = Z_{jk} = W_{jk} - \frac{W}{n} g_{jk} = 0 \tag{32}$$

or, equivalently,

$$W_{(jk)} - \frac{W}{n} g_{jk} = -R_{[jk]} = -n\nabla_{[j}w_{k]} \tag{33}$$

in which we have made use of (11). Then, (33) reduces to $W_{(jk)} - \frac{W}{n} g_{jk} = 0$, $\nabla_{[j}w_{k]} = 0$, stating that $W_n(g, w)$ is an Einstein-Weyl manifold and that w is locally a gradient. By a conformal re-scaling w can be made zero. Therefore, $W_n(g, w)$ is generalized concircular to an Einstein manifold. By using the generalized Einstein tensor for $W_n(g, w)$ it was proved in [13] that the scalar curvature of $W_n(g, w)$ is prolonged covariant constant.

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