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The effect of a bump in an elastic tube on wave propagation in a viscous fluid of variable viscosity

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Abstract

In the present work, treating the arteries as a thin walled prestressed elastic tube with a bump, and the blood as a Newtonian fluid of variable viscosity, we have studied the propagation of weakly nonlinear waves in such a medium by employing the reductive perturbation method, in the longwave approximation. Korteweg–deVries–Burgers equation with variable coefficients is obtained as the evolution equation. Seeking a progressive wave type of solution to this evolution equation, it is observed that the wave speed is variable. The numerical calculations show that the wave speed reaches to its maximum value at the center of the bump but it gets smaller and smaller as we go away from the center of the bump. Such a result seems to be reasonable from physical considerations.

Keywords: Solitary waves; Elastic tubes with bump

1. Introduction

Due to its applications in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers [1,2]. Most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focused on the dispersive character of waves (see, [3–5]). However, when the nonlinear terms arising from the constitutive equations and kinematical relations are introduced, one has to consider either finite amplitude, or small-but-finite amplitude waves, depending on the order of nonlinearity.

The propagation of finite amplitude waves in fluid-filled elastic or viscoelastic tubes has been examined, for instance, by Rudinger [6], Ling and Atabek [7], Anliker et al [8] and Tait and Moodie [9] by using the method of characteristics, in studying the shock formation. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson [10], Hashizume [11], Yomosa [12], and Demiray [13,14] by employing various asymptotic methods. In all these works [10–14], depending on the balance between the nonlinearity, dispersion and dissipation, the Korteweg–de Vries (KdV), Burgers'

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or KdV–Burgers' equations are obtained as the evolution equations. In obtaining such evolution equations, they treated the arteries as circularly cylindrical long thin tubes with a constant cross-section. In essence, the arteries have variable radius along the axis of the tube.

In the present work, treating the arteries as a thin walled prestressed elastic tube with a bump, and the blood as a Newtonian fluid of variable viscosity, we have studied the propagation of weakly nonlinear waves in such a medium by employing the reductive perturbation method, in the longwave approximation. Korteweg–deVries–Burgers equation with variable coefficients is obtained as the evolution equation. Seeking a progressive wave type of solution to this evolution equation, it is observed that the wave speed is variable. The numerical calculations show that the wave speed reaches to its maximum value at the center of the bump but it gets smaller and smaller as we go away from the center of the bump. Such a result seems to be reasonable from physical considerations.

2. Basic equations and theoretical preliminaries

2.1. Equations of tube

In this section, we shall derive the basic equations governing the motion of a prestressed thin elastic tube, with an axially symmetric bump (stenosis), and filled with a viscous fluid. For that purpose, we consider a circularly cylindrical tube of radius R_0, Fig. 1. It is assumed that such a tube is subjected to an axial stretch λ_z and the static pressure $P_0(Z^*)$. Under the effect of such a variable pressure the position vector of a generic point on the tube is assumed to be described by

$$\mathbf{r_0} = [r_0 - f^*(z^*)]\mathbf{e}_r + z^*\mathbf{e}_z, \quad z^* = \lambda_z Z^*, \tag{1}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are the unit base vectors in the cylindrical polar coordinates, r_0 is the deformed radius at the origin of the coordinate system, Z^* is the axial coordinate before the deformation, z^* is the axial coordinate after static deformation and $f^*(z^*)$ is a function that characterizes the axially symmetric bump on the surface of the arterial wall and will be specified later.

Upon this initial static deformation, we shall superimpose a dynamical radial displacement $u^*(z^*, t^*)$, where t^* is the time parameter, but, in view of the external tethering, the axial displacement is assumed to be negligible. Then, the position vector $\mathbf{r}$ of a generic point on the tube may be described by

$$\mathbf{r} = [r_0 - f^*(z^*) + u^*]\mathbf{e}_r + z^*\mathbf{e}_z, \tag{2}$$

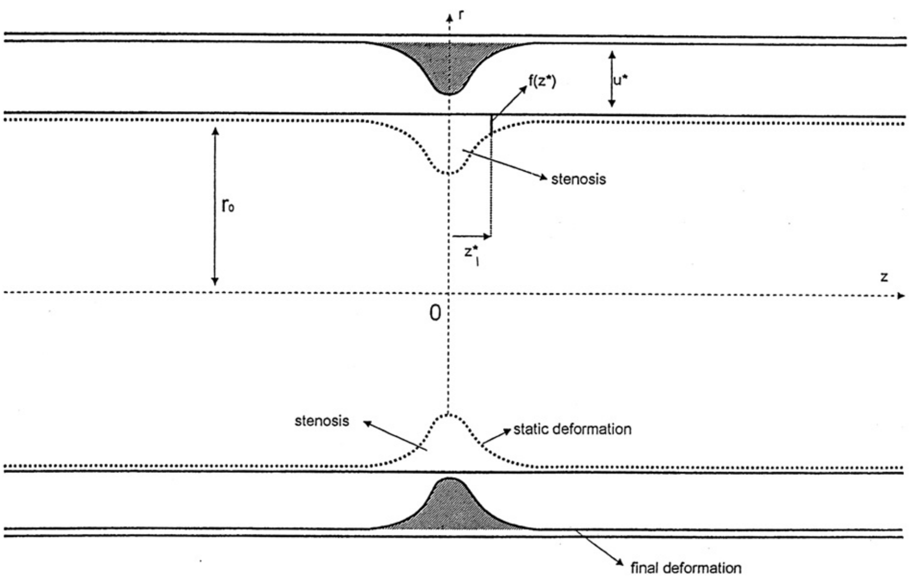

Fig. 1. The geometry of the tube in various configuration.

The arclengths along the meridional and circumferential curves are, respectively, given by

$$\mathrm{d}s_z = \left[1 + \left(-f^{*\prime} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} \mathrm{d}z^*, \quad \mathrm{d}s_\theta = (r_0 - f^* + u^*)\mathrm{d}\theta. \tag{3}$$

Then, the stretch ratios along the meridional and circumferential curves, respectively, may be given by

$$\lambda_1 = \lambda_z[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}, \quad \lambda_2 = \frac{1}{R_0}(r_0 - f^* + u^*), \tag{4}$$

where the prime denotes the differentiation of the corresponding quantity with respect to z^*. The unit tangent vector $\mathbf{t}$ along the deformed meridional curve and the unit exterior normal vector $\mathbf{n}$ to the deformed tube are given by

$$\mathbf{t} = \frac{(-f^{*\prime} + \partial u^*/\partial z^*)\mathbf{e}_r + \mathbf{e}_z}{[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}}, \quad \mathbf{n} = \frac{\mathbf{e}_r - (-f^{*\prime} + \partial u^*/\partial z^*)\mathbf{e}_z}{[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}}. \tag{5}$$

Let T_1 and T_2 be the membrane forces along the meridional and circumferential curves, respectively. hen, the equation of the radial motion of a small tube element placed between the planes $z^* = \text{const}$, $z^* + \mathrm{d}z^* = \text{const}$, $\theta = \text{const}$ and $\theta + \mathrm{d}\theta = \text{const}$ may be given by

$$\begin{aligned} &- T_2\left[1 + \left(-f^{*\prime} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} + \frac{\partial}{\partial z^*}\left\{\frac{(r_0 - f^* + u^*)(-f^{*\prime} + \partial u^*/\partial z^*)}{[1 + (f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}} T_1\right\} \\ &\quad \times P_r^*(r_0 - f^* + u^*)\left[1 + \left(-f^{*\prime} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} = \rho_0 \frac{HR_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}, \end{aligned} \tag{6}$$

where ρ_0 is the mass density of the tube, H is the thickness in the undeformed configuration and P_r^* is the fluid reaction force to be specified later.

Let $\mu\Sigma$ be the strain energy density function of the membrane, where μ is the shear modulus of the tube material. Then, the membrane forces may be expressed in terms of the stretch ratios as

$$T_1 = \frac{\mu H}{\lambda_2}\frac{\partial\Sigma}{\partial\lambda_1}, \quad T_2 = \frac{\mu H}{\lambda_1}\frac{\partial\Sigma}{\partial\lambda_2}. \tag{7}$$

Introducing (6)into Eq. (5), the equation of motion of the tube in the radial direction takes the following form:

$$-\frac{\mu}{\lambda_z}\frac{\partial\Sigma}{\partial\lambda_2} + \mu R_0\frac{\partial}{\partial z^*}\left\{\frac{(-f^{*\prime} + \partial u^*/\partial z^*)}{[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}}\frac{\partial\Sigma}{\partial\lambda_1}\right\} + \frac{P_r^*}{H}(r_0 - f^* + u^*)[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2} = \rho_0\frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}. \tag{8}$$

2.2. Equations of fluid

In general, blood is known to be an incompressible non-Newtonian fluid. However, in the course of flow in large arteries, the red blood cells in the vicinity of arterial wall move to the central region of the artery so that hematocrit ratio becomes quite low near the arterial wall, which results in lower viscosity in this region. Moreover, due to high shear rate near the arterial wall the viscosity of blood is further reduced. Therefore, for flow problems in large blood vessels, the blood may be treated as a Newtonian fluid with variable viscosity, which vanishes on the arterial wall and it takes the maximum value at the center of the artery. Because of the vanishing viscosity on the arterial wall, the non-slip condition of the viscous fluid will be violated, i.e. the tangential velocity of the fluid will not be set equal to the tangential velocity of the tube.

Let V_r^* and V_z^* denote the radial and the axial velocity components of the fluid body. In this work we shall be concerned with the symmetrical motion of the fluid. Then, the physical components of the stress tensor of the fluid read

$$\begin{aligned} &\sigma_{rr} = -\bar{p} + 2\mu_v(r)\frac{\partial V_r^*}{\partial r}, \quad \sigma_{r\theta} = 0, \quad \sigma_{rz} = \mu_v(r)\left(\frac{\partial V_r^*}{\partial z^*} + \frac{\partial V_z^*}{\partial r}\right), \\ &\sigma_{\theta\theta} = -\bar{p} + 2\mu_v(r)\frac{V_r^*}{r}, \quad \sigma_{z\theta} = 0, \quad \sigma_{zz} = -\bar{p} + 2\mu_v(r)\frac{\partial V_z^*}{\partial z^*}, \end{aligned} \tag{9}$$

where $\bar{p}$ is the pressure function and $\mu_v(r)$ is the variable viscosity function. The equations of motion in the cylindrical polar coordinates may be given by

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z^*}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=\rho_f a_r^*, \tag{10}$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z^*}+\frac{\sigma_{rz}}{r}=\rho_f a_z^*, \tag{11}$$

$$\frac{\partial V_r^*}{\partial r}+\frac{V_r^*}{r}+\frac{\partial V_z^*}{\partial z^*}=0 \quad \text{(incompressibility)}, \tag{12}$$

where ρ_f is the mass density of the fluid and a_r^*, a_z^* are the components of the acceleration vector in the cylindrical coordinates and given by

$$a_r^*=\frac{\partial V_r^*}{\partial t^*}+V_r^*\frac{\partial V_r^*}{\partial r}+V_z^*\frac{\partial V_r^*}{\partial z^*}, \quad a_z^*=\frac{\partial V_z^*}{\partial t^*}+V_r^*\frac{\partial V_z^*}{\partial r}+V_z^*\frac{\partial V_z^*}{\partial z^*}. \tag{13}$$

In Eqs. (10)–(12), the effect of the body force is neglected. Introducing (9) and (13) into Eqs. (10) and (11) we have

$$\frac{\partial V_r^*}{\partial t^*}+V_r^*\frac{\partial V_r^*}{\partial r}+V_z^*\frac{\partial V_r^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial r}-\hat{v}\gamma(r)\left(\frac{\partial^2 V_r^*}{\partial r^2}+\frac{1}{r}\frac{\partial V_r^*}{\partial r}-\frac{V_r^*}{r^2}+\frac{\partial^2 V_r^*}{\partial z^{*2}}\right)-2\hat{v}\gamma'(r)\frac{\partial V_r^*}{\partial r}=0, \tag{14}$$

$$\frac{\partial V_z^*}{\partial t^*}+V_r^*\frac{\partial V_z^*}{\partial r}+V_z^*\frac{\partial V_z^*}{\partial z^*}+\frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial z^*}-\hat{v}\gamma(r)\left(\frac{\partial^2 V_z^*}{\partial r^2}+\frac{1}{r}\frac{\partial V_z^*}{\partial r}+\frac{\partial^2 V_z^*}{\partial z^{*2}}\right)-\hat{v}\gamma'(r)\left(\frac{\partial V_r^*}{\partial z^*}+\frac{\partial V_z^*}{\partial r}\right)=0, \tag{15}$$

$$\frac{\partial V_r^*}{\partial r}+\frac{V_r^*}{r}+\frac{\partial V_z^*}{\partial z^*}=0, \tag{16}$$

with the boundary conditions

$$\begin{aligned} V_r^*|_{r=r_f}&=\frac{\partial u^*}{\partial t^*}+\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)V_z^*\bigg|_{r=r_f}, \\ S_r^*&=\frac{1}{\Lambda}\left[\bar{P}-2\rho_f\hat{v}\gamma(r)\frac{\partial V_r^*}{\partial r}+\rho_f\hat{v}\gamma(r)\left(-f^{*\prime}+\frac{\partial u^*}{\partial z^*}\right)\left(\frac{\partial V_r^*}{\partial z^*}+\frac{\partial V_z^*}{\partial r}\right)\right]\bigg|_{r=r_f}, \end{aligned} \tag{17}$$

where we have defined

$$\mu_v(r)=\rho_f\hat{v}\gamma(r), \quad r_f=r_0-f^*(z^*)+u^*. \tag{18}$$

Here, $\hat{v}$ is the kinematical viscosity of the fluid at the center of the tube. The fluid reaction force density S_r^* is obtained from the stress boundary condition in the radial direction

$$S_r^*=-(\sigma_{rr}n_r+\sigma_{rz}n_z)|_{r=r_f}. \tag{19}$$

At this stage it is convenient to introduce the following non-dimensional quantities:

$$\begin{aligned} &t^*=\left(\frac{R_0}{c_0}\right)t, \quad z^*=R_0 z, \quad u^*=R_0 u, \quad V_r^*=c_0 V_r, \quad V_z^*=c_0 V_z, \quad r=R_0 x, \quad f^*=R_0 f, \\ &\bar{P}=\rho_f c_0^2\bar{p}, \quad S_r^*=\rho_f c_0^2 p_r/\Lambda, \quad m=\frac{\rho_0 H}{\rho_f R_0}, \quad c_0^2=\frac{\mu H}{\rho_f R_0}, \quad \hat{v}=c_0 R_0\bar{v}, \end{aligned} \tag{20}$$

where c_0 is the Moens–Korteweg speed. Introducing (20) into Eqs. (8), (14)–(17), the following non-dimensional equations are obtained:

$$p_r=\frac{m}{\lambda_z(\lambda_\theta-f+u)}\frac{\partial^2 u}{\partial t^2}+\frac{1}{\lambda_z(\lambda_\theta-f+u)}\frac{\partial \Sigma}{\partial \lambda_2}-\frac{1}{(\lambda_\theta-f+u)}\frac{\partial}{\partial z}\left\{\frac{(-f'+\partial u/\partial z)}{[1+(-f'+\partial u/\partial z)^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\}, \tag{21}$$

$$\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial x}+V_z\frac{\partial V_r}{\partial z}+\frac{\partial \bar{p}}{\partial x}-\bar{v}\gamma(x)\left(\frac{\partial^2 V_r}{\partial x^2}+\frac{1}{x}\frac{\partial V_r}{\partial x}-\frac{V_r}{x^2}+\frac{\partial^2 V_r}{\partial z^2}\right)-2\bar{v}\gamma'(x)\frac{\partial V_r}{\partial x}=0, \tag{22}$$

$$\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial x}+V_z\frac{\partial V_z}{\partial z}+\frac{\partial \bar{p}}{\partial z}-\bar{v}\gamma(x)\left(\frac{\partial^2 V_z}{\partial x^2}+\frac{1}{x}\frac{\partial V_z}{\partial x}+\frac{\partial^2 V_z}{\partial z^2}\right)-\bar{v}\gamma'(x)\left(\frac{\partial V_r}{\partial z}+\frac{\partial V_z}{\partial x}\right)=0, \tag{23}$$

$$\frac{\partial V_r}{\partial x}+\frac{V_r}{x}+\frac{\partial V_z}{\partial z}=0 \tag{24}$$

with the boundary conditions

$$V_r\big|_{x=\lambda_\theta-f+u}=\frac{\partial u}{\partial t}+\left(-f'+\frac{\partial u}{\partial z}\right)V_z\bigg|_{x=\lambda_\theta-f+u}, \tag{25}$$

$$p_r=\left[\bar{p}-2\bar{v}\gamma(x)\frac{\partial V_r}{\partial x}+\bar{v}\gamma(x)\left(-f'+\frac{\partial u}{\partial z}\right)\left(\frac{\partial V_r}{\partial z}+\frac{\partial V_z}{\partial x}\right)\right]\bigg|_{x=\lambda_\theta-f+u}. \tag{26}$$

Eqs. (21)–(26) give sufficient relations to determine the field quantities u, V_r, V_z and $\bar{p}$ completely.

3. Longwave approximation

In this section we shall examine the propagation of small-but-finite amplitude waves in a fluid-filled thin elastic tube with a bump, whose dimensionless governing equations are given in Eqs. (21)–(26). For this, we adopt the long wave approximation and employ the reductive perturbation method [15,16]. For this type of problems, it is convenient to introduce the following type of stretched coordinates:

$$\xi=\epsilon^{1/2}(z-ct),\quad \tau=\epsilon^{3/2}z, \tag{27}$$

where ϵ is a small parameter measuring the weakness of nonlinearity and dispersion and c is the scale parameter to be determined from the solution. Solving z in terms of τ we get

$$z=\epsilon^{-3/2}\tau. \tag{28}$$

Introducing (28) into the expression of $f(z)$ we obtain

$$f(\epsilon^{-3/2}\tau)=h(\epsilon,\tau). \tag{29}$$

In order to take the effect of bump into account, the function $f(z)$ must be of order $\epsilon^{5/2}$. Thus we can write

$$h(\epsilon,\tau)=\epsilon h(\tau). \tag{30}$$

Introducing the following differential relations:

$$\frac{\partial}{\partial t}=-\epsilon^{1/2}c\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial z}=\epsilon^{1/2}\frac{\partial}{\partial \xi}+\epsilon^{3/2}\frac{\partial}{\partial \tau}, \tag{31}$$

into Eqs. (21)–(26) we obtain

$$\begin{aligned}p_r={}&\epsilon\frac{mc^2}{\lambda_z(\lambda_\theta-\epsilon h+u)}\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{\lambda_z(\lambda_\theta-\epsilon h+u)}\frac{\partial \Sigma}{\partial \lambda_2}\\&-\frac{\epsilon}{(\lambda_\theta-\epsilon h+u)}\left(\frac{\partial}{\partial \xi}+\epsilon\frac{\partial}{\partial \tau}\right)\left\{\frac{(-\epsilon^2h'+\partial u/\partial \xi)}{[1+(-\epsilon^2h'+\partial u/\partial \xi)^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\},\end{aligned} \tag{32}$$

$$\begin{aligned}&-\epsilon^{1/2}c\frac{\partial V_r}{\partial \xi}+V_r\frac{\partial V_r}{\partial x}+\epsilon^{1/2}V_z\frac{\partial V_r}{\partial \xi}+\frac{\partial \bar{p}}{\partial x}-\bar{v}\gamma(x)\left(\frac{\partial^2 V_r}{\partial x^2}+\frac{1}{x}\frac{\partial V_r}{\partial x}-\frac{V_r}{x^2}\right)\\&\quad+\epsilon\left(\frac{\partial^2 V_r}{\partial \xi^2}+2\epsilon\frac{\partial^2 V_r}{\partial \xi\partial \tau}+\frac{\partial^2 V_r}{\partial \tau^2}\right)-2\bar{v}\gamma'(x)\frac{\partial V_r}{\partial x}=0,\end{aligned} \tag{33}$$

$$\begin{aligned}&-\epsilon^{1/2}c\frac{\partial V_z}{\partial \xi}+V_r\frac{\partial V_z}{\partial x}+\epsilon^{1/2}V_z\left(\frac{\partial V_z}{\partial \xi}+\epsilon\frac{\partial V_z}{\partial \tau}\right)+\epsilon^{1/2}\left(\frac{\partial \bar{p}}{\partial \xi}+\epsilon\frac{\partial \bar{p}}{\partial \tau}\right)\\&\quad-\bar{v}\gamma(x)\left[\frac{\partial^2 V_z}{\partial x^2}+\frac{1}{x}\frac{\partial V_z}{\partial x}+\epsilon\left(\frac{\partial^2 V_z}{\partial \xi^2}+2\epsilon\frac{\partial^2 V_z}{\partial \xi\partial \tau}+\epsilon^2\frac{\partial^2 V_z}{\partial \tau^2}\right)\right]\\&\quad-\bar{v}\gamma'(x)\left[\left(\epsilon^{1/2}\frac{\partial V_r}{\partial \xi}+\epsilon\frac{\partial V_r}{\partial \tau}\right)+\frac{\partial V_z}{\partial x}\right]=0,\end{aligned} \tag{34}$$

$$\frac{\partial V_r}{\partial x}+\frac{V_r}{x}+\epsilon^{1/2}\left(\frac{\partial V_z}{\partial \xi}+\epsilon\frac{\partial V_z}{\partial \tau}\right)=0, \tag{35}$$

with the boundary conditions

$$\begin{aligned} &V_r|_{x=\lambda_\theta-\epsilon h+u}=-\epsilon^{1/2}c\frac{\partial u}{\partial \xi}+\epsilon^{1/2}\left(-\epsilon^2 h'+\frac{\partial u}{\partial \xi}\right)V_z\Big|_{x=\lambda_\theta-\epsilon h+u},\\ &p_r=\left[\bar{p}-2\bar{v}\gamma(x)\frac{\partial V_r}{\partial x}+\epsilon\bar{v}\gamma(x)\left(-\epsilon^2 h'+\frac{\partial u}{\partial \xi}\right)\right]\left[\left(\frac{\partial V_r}{\partial \xi}+\epsilon\frac{\partial V_r}{\partial \tau}\right)+\frac{\partial V_z}{\partial x}\right]\Big|_{x=\lambda_\theta-\epsilon h+u}. \end{aligned} \tag{36}$$

For the long wave limit, it is assumed that the field quantities may be expanded into asymptotic series as

$$\begin{aligned} &u=\epsilon u_1+\epsilon^2 u_2+\cdots,\\ &V_r=\epsilon^{1/2}(\epsilon V_r^{(1)}+\epsilon^2 V_r^{(2)}+\cdots),\\ &V_z=\epsilon V_z^{(1)}+\epsilon^2 V_z^{(2)}+\cdots,\\ &\bar{p}=\bar{p}_0+\epsilon\bar{p}_1(\xi,\tau)+\epsilon^2\bar{p}_2(\xi,\tau)+\cdots,\\ &p_r=p_r^{(0)}+\epsilon p_r^{(1)}+\epsilon^2 p_r^{(2)}+\cdots,\\ &\gamma(x)=\gamma_0(x)+\epsilon\gamma_1(x)+\epsilon^2\gamma_2(x)+\cdots, \end{aligned} \tag{37}$$

where $\gamma_0(x)$, $\gamma_1(x)$ and $\gamma_2(x)$ are defined by

$$\gamma_0(x)=1-\frac{x}{\lambda_\theta},\quad \gamma_1(x)=\frac{x}{\lambda_\theta^2}(u_1-h),\quad \gamma_2(x)=\frac{x}{\lambda_\theta^2}\left[u_2-\frac{(u_1-h)^2}{\lambda_\theta}\right]. \tag{38}$$

Here, we assumed that the function $\gamma(x)$, characterizing the variation of the viscosity, is of the form

$$\gamma(x)=1-\frac{x}{\lambda_\theta-\epsilon h+u}. \tag{39}$$

Introducing the expansions (37) and (38) into Eqs. (32)–(36), the following sets of differential equations are obtained:

O(ϵ) *equations*

$$\begin{aligned} &-c\frac{\partial V_z^{(1)}}{\partial \xi}+\frac{\partial \bar{p}_1}{\partial \xi}-v\gamma_0'(x)\frac{\partial V_z^{(1)}}{\partial x}-v\gamma_0(x)\left(\frac{\partial^2 V_z^{(1)}}{\partial x^2}+\frac{1}{x}\frac{\partial V_z^{(1)}}{\partial x}\right)=0,\\ &\frac{\partial \bar{p}_1}{\partial x}=0,\quad \frac{\partial V_r^{(1)}}{\partial x}+\frac{1}{x}V_r^{(1)}+\frac{\partial V_z^{(1)}}{\partial \xi}=0, \end{aligned} \tag{40}$$

and the boundary conditions

$$V_r^{(1)}|_{x=\lambda_\theta}+c\frac{\partial u_1}{\partial \xi}=0,\quad p_r^{(1)}=\bar{p}_1|_{x=\lambda_\theta}. \tag{41}$$

O(ϵ^2) *equations*

$$\begin{aligned} &-c\frac{\partial V_r^{(1)}}{\partial \xi}+\frac{\partial \bar{p}_2}{\partial x}-2v\gamma_0'(x)\frac{\partial V_r^{(1)}}{\partial x}-v\gamma_0(x)\left(\frac{\partial^2 V_r^{(1)}}{\partial x^2}+\frac{1}{x}\frac{\partial V_r^{(1)}}{\partial x}-\frac{V_r^{(1)}}{x^2}\right)=0,\\ &-c\frac{\partial V_z^{(2)}}{\partial \xi}+V_r^{(1)}\frac{\partial V_z^{(1)}}{\partial x}+V_z^{(1)}\frac{\partial V_z^{(1)}}{\partial \xi}+\frac{\partial \bar{p}_2}{\partial \xi}+\frac{\partial \bar{p}_1}{\partial \tau}-v\gamma_0(x)\left(\frac{\partial^2 V_z^{(2)}}{\partial x^2}+\frac{1}{x}\frac{\partial V_z^{(2)}}{\partial x}+\frac{\partial^2 V_z^{(1)}}{\partial \xi^2}\right)\\ &-v\gamma_1(x)\left(\frac{\partial^2 V_z^{(1)}}{\partial x^2}+\frac{1}{x}\frac{\partial V_z^{(1)}}{\partial x}\right)-v\gamma_1'(x)\frac{\partial V_z^{(1)}}{\partial x}-v\gamma_0'(x)\left(\frac{\partial V_r^{(1)}}{\partial \xi}+\frac{\partial V_z^{(2)}}{\partial x}\right)=0,\\ &\frac{\partial V_r^{(2)}}{\partial x}+\frac{V_r^{(2)}}{x}+\frac{\partial V_z^{(2)}}{\partial \xi}+\frac{\partial V_z^{(1)}}{\partial \tau}=0, \end{aligned} \tag{42}$$

and the boundary conditions

$$V_r^{(2)}|_{x=\lambda_\theta} + (u_1 - h)\frac{\partial V_r^{(1)}}{\partial x} + c\frac{\partial u_2}{\partial \xi} - \frac{\partial u_1}{\partial \xi}V_z^{(1)}\Big|_{x=\lambda_\theta} = 0,$$
$$p_r^{(2)} = \left[\bar{p}_2 + (u_1 - h)\frac{\partial \bar{p}_1}{\partial x} - 2\nu\gamma(x)\frac{\partial V_r^{(1)}}{\partial x}\right]\Big|_{x=\lambda_\theta}. \tag{43}$$

Here, it is assumed that the viscosity is of order of $\epsilon^{1/2}$, i.e. $\bar{\nu} = \epsilon^{1/2}\nu$.

In order to complete the equations, one must know the expressions of $p_r^{(1)}$ and $p_r^{(2)}$ in terms of the radial displacement u. For that purpose we need the series expansion of the stretch ratios λ_1 and λ_2, which read

$$\lambda_1 \simeq \lambda_z, \quad \lambda_2 = \lambda_\theta + \epsilon(u_1 - h) + \epsilon^2 u_2. \tag{44}$$

Using the expansion (44) in the expression of p_r, given in (32), we have

$$p_r^{(1)} = \beta_1(u_1 - h), \tag{45}$$
$$p_r^{(2)} = \beta_2(u_1 - h)^2 + \beta_1 u_2 + \left(\frac{mc^2}{\lambda_\theta\lambda_z} - \alpha_0\right)\frac{\partial^2 u_1}{\partial \xi^2}, \tag{46}$$

where the coefficients α_0, β_1 and β_2 are defined by

$$\alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_z}, \quad \beta_1 = \frac{1}{\lambda_\theta\lambda_z}\left(\frac{\partial^2\Sigma}{\partial\lambda_\theta^2} - \frac{\partial \Sigma}{\partial \lambda_\theta}\right), \quad \beta_2 = \frac{1}{2\lambda_\theta\lambda_z}\frac{\partial^3\Sigma}{\partial\lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta}. \tag{47}$$

3.1. Solution of the field equations

From the solution of Eqs. (40) under the boundary conditions (41) we have

$$u_1 = U(\xi,\tau), \quad V_z^{(1)} = \frac{\beta_1}{c}(U + w), \quad V_r^{(1)} = -\frac{\beta_1}{2c}\frac{\partial U}{\partial \xi}x, \quad p_1 = \beta_1(U - h), \tag{48}$$

provided that the following condition holds true:

$$\beta_1 = 2c^2/\lambda_\theta. \tag{49}$$

Here $U(\xi,\tau)$ is an unknown function whose governing equation will be obtained later and $(\beta_1/c)w(\tau)$ corresponds to the axial steady flow resulting from the pressure $-\beta_1 h(\tau)$.

To obtain the solution of $O(\epsilon^2)$ equations we introduce (48) into Eqs. (42) and the boundary conditions Eq. (43) we obtain

$$\frac{\beta_1}{2}\frac{\partial^2 U}{\partial \xi^2}x - \frac{\nu\beta_1}{\lambda_\theta c}\frac{\partial U}{\partial \xi} + \frac{\partial \bar{p}_2}{\partial x} = 0, \tag{50}$$

$$-c\frac{\partial V_z^{(2)}}{\partial \xi} + 2\frac{\beta_1}{\lambda_\theta}(U + w)\frac{\partial U}{\partial \xi} + \frac{\partial \bar{p}_2}{\partial \xi} + \frac{\partial \bar{p}_1}{\partial \tau} - \beta_1\frac{\mathrm{d}h}{\mathrm{d}\tau}$$
$$-\nu\left(1 - \frac{x}{\lambda_\theta}\right)\left(\frac{\partial^2 V_z^{(2)}}{\partial x^2} + \frac{1}{x}\frac{\partial V_z^{(2)}}{\partial x} + \frac{\beta_1}{c}\frac{\partial^2 U}{\partial \xi^2}\right) + \frac{\nu}{\lambda_\theta}\left(-\frac{\beta_1}{2c}\frac{\partial^2 U}{\partial \xi^2}x + \frac{\partial V_z^{(2)}}{\partial x}\right) = 0, \tag{51}$$

$$\frac{\partial V_r^{(2)}}{\partial x} + \frac{V_r^{(2)}}{x} + \frac{\beta_1}{c}\frac{\partial U}{\partial \tau} + \frac{\beta_1}{c}\frac{\mathrm{d}w}{\mathrm{d}\tau} = 0, \tag{52}$$

and the boundary conditions

$$V_r^{(2)}|_{x=\lambda_\theta} - \frac{3\beta_1}{2c}U\frac{\partial U}{\partial \xi} + \frac{\beta_1}{2c}(h - 2w)\frac{\partial U}{\partial \xi} + c\frac{\partial u_2}{\partial \xi} = 0, \quad p_r^{(2)} = \bar{p}_2|_{x=\lambda_\theta}. \tag{53}$$

Here we noted that $\gamma_0(\lambda_\theta) = 0$. From the integration of (50) and the use of the boundary condition $(53)_2$ we have

$$\bar{p}_2 = -\frac{\beta_1}{4}\frac{\partial^2 U}{\partial \xi^2}x^2 + \frac{v\beta_1}{\lambda_\theta c}\frac{\partial U}{\partial \xi}x + p_2(\xi,\tau), \tag{54}$$

$$p_2 = \left(\frac{mc^2}{\lambda_\theta \lambda_z} + \frac{\beta_1 \lambda_\theta^2}{4} - \alpha_0\right)\frac{\partial^2 U}{\partial \xi^2} - \frac{v\beta_1}{c}\frac{\partial U}{\partial \xi} + \beta_1 u_2 + \beta_2 (U - h)^2. \tag{55}$$

From the solution of Eq. (51) and (52) one can get

$$\begin{aligned} V_z^{(2)} &= -\frac{\beta_1}{4c}\frac{\partial^2 U}{\partial \xi^2}x^2 + w_2(\xi,\tau), \\ V_r^{(2)} &= \frac{\beta_1}{16c}\frac{\partial^3 U}{\partial \xi^3}x^3 - \frac{x}{2}\left(\frac{\partial w_2}{\partial \xi} + \frac{\beta_1}{c}\frac{\partial U}{\partial \tau} + \frac{\beta_1}{c}\frac{\mathrm{d}w}{\mathrm{d}\tau}\right), \end{aligned} \tag{56}$$

provided that the following relation holds true:

$$\begin{aligned} &-c\frac{\partial w_2}{\partial \xi} + 2\left(\beta_2 + \frac{\beta_1}{\lambda_\theta}\right)U\frac{\partial U}{\partial \xi} - \frac{v\beta_1}{c}\frac{\partial^2 U}{\partial \xi^2} + \left(\frac{mc^2}{\lambda_\theta\lambda_z} + \frac{\beta_1\lambda_\theta^2}{4} - \alpha_0\right)\frac{\partial^3 U}{\partial \xi^3} \\ &\quad + 2\left(\frac{\beta_1}{\lambda_\theta}w - \beta_2 h\right)\frac{\partial U}{\partial \xi} + \beta_1\frac{\partial U}{\partial \tau} + \beta_1\frac{\partial u_2}{\partial \xi} - \beta_1\frac{\mathrm{d}h}{\mathrm{d}\tau} = 0, \end{aligned} \tag{57}$$

where $w_2(\xi,\tau)$ is another unknown function to be determined from the solution.

The use of the boundary condition $(53)_1$ yields

$$\frac{\beta_1\lambda_\theta^3}{16c}\frac{\partial^3 U}{\partial \xi^3} - \frac{\lambda_\theta}{2}\left(\frac{\partial w_2}{\partial \xi} + \frac{\beta_1}{c}\frac{\partial U}{\partial \tau} + \frac{\beta_1}{c}\frac{\mathrm{d}w}{\mathrm{d}\tau}\right) - \frac{3\beta_1}{2c}U\frac{\partial U}{\partial \xi} + \frac{\beta_1}{2c}(h - 2w)\frac{\partial U}{\partial \xi} + c\frac{\partial u_2}{\partial \xi} = 0. \tag{58}$$

Eliminating u_2 between Eqs. (57) and (58) we obtain the following evolution equation:

$$\begin{aligned} &2\beta_1\frac{\partial U}{\partial \tau} + \left(5\frac{\beta_1}{\lambda_\theta} + 2\beta_2\right)U\frac{\partial U}{\partial \xi} - \frac{v\beta_1}{c}\frac{\partial^2 U}{\partial \xi^2} + \left(\frac{mc^2}{\lambda_\theta\lambda_z} + \frac{\beta_1\lambda_\theta^2}{8} - \alpha_0\right)\frac{\partial^3 U}{\partial \xi^3} \\ &\quad + \left[-\left(\frac{\beta_1}{\lambda_\theta} + 2\beta_2\right)h + 4\frac{\beta_1}{\lambda_\theta}w\right]\frac{\partial U}{\partial \xi} + \beta_1\frac{\mathrm{d}}{\mathrm{d}\tau}(h - w) = 0. \end{aligned} \tag{59}$$

The Eq. (59) must even be valid when $U = 0$, which results in

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(h - w) = 0. \tag{60}$$

The solution of (60) gives $w(\tau) = h(\tau)$. Introducing this expression of $w(\tau)$ into Eq. (59) we obtain the following Korteweg–de Vries–Burgers equation with variable coefficients:

$$\frac{\partial U}{\partial \tau} + \mu_1 U\frac{\partial U}{\partial \xi} - \mu_2\frac{\partial^2 U}{\partial \xi^2} + \mu_3\frac{\partial^3 U}{\partial \xi^3} - \mu_4 h(\tau)\frac{\partial U}{\partial \xi} = 0, \tag{61}$$

where the coefficients μ_1, μ_2, μ_3 and μ_4 are defined by

$$\mu_1 = \left(\frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}\right), \quad \mu_2 = \frac{v}{2c}, \quad \mu_3 = \left(\frac{m}{4\lambda_z} + \frac{\lambda_\theta^2}{16} - \frac{\alpha_0}{2\beta_1}\right), \quad \mu_4 = \left(\frac{\beta_2}{\beta_1} - \frac{3}{2\lambda_\theta}\right). \tag{62}$$

3.2. Progressive wave solution

In this sub-section we shall present the progressive wave solution to the KdV–Burgers equation with variable coefficients given in (61). For that purpose we introduce the following coordinate transformation:

$$\tau' = \tau, \quad \xi' = \xi + \mu_4 \int_0^\tau h(s)\,\mathrm{d}s. \tag{63}$$

The use of (63) in (61) leads to the following conventional KdV–Burgers equation:

$$\frac{\partial U}{\partial \tau'} + \mu_1 U \frac{\partial U}{\partial \xi'} - \mu_2 \frac{\partial^2 U}{\partial \xi'^2} + \mu_3 \frac{\partial^3 U}{\partial \xi'^3} = 0, \tag{64}$$

Following Demiray [17], the progressive wave solution to the evolution (64) may be given by

$$U = \frac{a}{\mu_1} + \frac{3}{25}\frac{\mu_2^2}{\mu_3}(\mathrm{sech}^2\zeta + 2\tanh\zeta), \tag{65}$$

where a is a constant and the phase function ζ is defined by

$$\zeta = \frac{\mu_2}{10\mu_3}(\xi' - a\tau'). \tag{66}$$

Using the coordinate transformation (63), the phase function ζ takes the following form:

$$\zeta = \frac{\mu_2}{10\mu_3}\left[\xi - a\tau + \mu_4 \int_0^\tau h(s)\,\mathrm{d}s\right]. \tag{67}$$

As is seen from the expression of the phase function ζ, the trajectory of the wave is not a straight line anymore, it is rather a curve in the (ξ, τ) plane. This is the result of the stenosis in the tube. As a matter of fact, the existence of stenosis causes the variable wave speed. Noting that τ is space variable and ξ is temporal variable, the wave speed may be defined by

$$v_p = \frac{\mathrm{d}\tau}{\mathrm{d}\xi} = \frac{1}{[a - \mu_4 h(\tau)]}. \tag{68}$$

4. Numerical results and discussion

In order to see the effects of a stenosis on the wave speed one has to know the sign of the coefficient μ_4. For that reason, one must know the constitutive relation of the tube material. In this work we shall utilize the constitutive relation proposed by Demiray [18] for soft biological tissues. Following Demiray [18], the strain energy density function may be expressed as

$$\Sigma = \frac{1}{2\alpha}\{\exp[\alpha(I_1 - 3)] - 1\}, \tag{69}$$

where α is a material constant and I_1 is the first invariant of Finger deformation tensor and defined by $I_1 = \lambda_z^2 + \lambda_\theta^2 + 1/\lambda_z^2\lambda_\theta^2$. Introducing (69) into Eq. (47), the coefficients α_0, β_0, β_1 and β_2 are obtained as

$$\begin{aligned} \alpha_0 &= \left(\lambda_z^2 - \frac{1}{\lambda_\theta^2\lambda_z^2}\right)F(\lambda_\theta, \lambda_z), \quad \beta_0 = \left(\frac{1}{\lambda_z} - \frac{1}{\lambda_\theta^4\lambda_z^3}\right)F(\lambda_\theta, \lambda_z), \\ \beta_1 &= \left[\left(\frac{1}{\lambda_\theta\lambda_z} + \frac{3}{\lambda_\theta^5\lambda_z^3}\right) + 2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]F(\lambda_\theta, \lambda_z). \end{aligned} \tag{70}$$

where the function $F(\lambda_\theta, \lambda_z)$ is defined by

$$F(\lambda_\theta, \lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right]. \tag{71}$$

As is seen from Eq. (68) the effect of the stenosis is closely related to the sign of the coefficient μ_4. Therefore, it might be instructive to study the variation of the coefficient μ_4 with the initial deformation. In order to study the variation of the coefficient μ_4 with the initial deformation, we need the value of material constant α. For the static case, the present model was compared by Demiray [19] with the experimental measurements by Simon et al [20] on canine abdominal artery with the characteristics $R_i = 0.31$ cm, $R_0 = 0.38$ cm and $\lambda_z = 1.53$, and the value of the material constant α was found to be $\alpha = 1.948$. Using this numerical value of the coefficient α, the value of the coefficient μ_4 is calculated for the initial deformation $\lambda_\theta = \lambda_z = 1.6$ and the value is found to be $\mu_4 = 1.21$. Then, the wave speed takes the following form:

$$v_p = \frac{1}{[a - 1.21h(\tau)]}. \tag{72}$$

As is seen from Eq. (72) the wave speed reaches to its maximum value at the center of the stenosis, and it gets smaller and smaller as we go away from the center of the stenosis. Such a result is to be expected from physical consideration.

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