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Joint Pricing and Ordering Problem with Charitable Donations

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Abstract: Finding the correct pricing strategy for a product with multiple versions is an issue for retailers from various industries. In this paper, joint pricing and ordering problem is considered for a product that has two versions at each selling period. Two models, namely with or without the donation option, are analyzed and optimality conditions and monotonicity properties of the decision variables are characterized. When demands of products depend on prices of both versions, donating part of old product inventory would be more profitable for the retailer. Moreover, the donation model would result in less wasted inventory, contributing to sustainability and goals of green economy. Analytical results are supported with numerical analysis of a realistic case.

Keywords: joint pricing and ordering; dynamic programming; donations; sustainability; circular economy

1. Introduction

Pricing problem of the same good at different quality levels/with different versions is an important issue for retailers from various industries. The demand values for both old and new (versions of the) products often depend on the prices of both versions and the implemented pricing strategies have a significant impact on the profit levels. Especially when perishable goods are considered, incorrect pricing strategies might result in significant rises of wasted inventory, leading to escalated environmental damages and a huge loss in terms of social welfare in addition to economic losses. FAO (Food and Agriculture Organization of the United Nations) indicates that approximately 1.3 billion tons of food (one-third of the total produced) is wasted globally every year ([1]). Meanwhile, there are millions of people who are on the verge of starvation and rely on food-banks to maintain a healthy diet ([2]). This issue has been worsened with the increasing unemployment rates and increasing food prices in a post-COVID world. Moreover, any excess food that is sent to landfill, rather than food banks, generates CO_2 and methane emissions during the disposition process and results in wasting drinkable water and energy during this process ([3]).

The problem is also pertinent in the sector of technological goods. When new versions of products enter the market, old versions of products are offered at discounted prices, as is prevalent in electronics and automobile industries ([4]). High-tech products can also be considered to be perishable as they have short product life cycles because of rapid technological innovation and shifts in customer preferences. In this case, product life cycles of new and old products are overlapped, and old and new products are sold at the same period. Thus, companies need to determine the order (or production) quantity for new products and often adjust the prices of old products by taking into account the inventory level of existing products ([4]). The unsold items are again sent to waste disposal, leading to several environmental concerns as they involve heavy metals. Awasthi et al. [5] draws attention to the fact that electronic waste is the fastest growing category of hazardous solid waste in the world, and the problem is exacerbated by illegal trading and rudimentary processing of e-waste. The latter occurs in many parts of the world, especially in emerging market economy countries such as China, Ghana, India and Nigeria, and the process generates toxic residues and emissions to air, soil and water. Although there can also be donation opportunities in the high-tech sector (e.g., donating technological products to public schools), these options are not entertained at desirable levels.

Despite a broad range of pricing strategies proposed to tackle with the issue of wasted products, the "donation strategy" is not considered widely by either the practitioners or researchers. The donation strategy would refer to donating some of the older (version) product to the people who cannot afford it, or to foundations who could transfer these products to low-income people. Healthcare and consumer staples are the leading industries in terms of the percentage they give in non-cash contributions ([6,7]). Furthermore, in many countries (e.g., Italy, UK, USA, and Austria, to name a few), there are laws encouraging donations by providing tax subsidies to donors ([8]). However, the employment of this strategy, in particular by large grocery chains or other perishable good vendors, is still not at the desired levels.

One concern for the firms towards adopting donation strategy is that this practice could cannibalize some of their demand and hurt their profits. Some of the existing research shows that this might indeed be the case ([9–11]). However, as Ozbilge et al. [12] argue, incorporating donation can improve inventory management practices and reduce waste, leading to higher profit levels for the firm. Similarly, in this paper, it is shown that donation could enable a firm to manage the inventories of old and new products better, leading to higher levels of profit than the case without donations. Plus, the increased social welfare and decreased environmental emissions are the side benefits.

That is, donations are only considered to be "good samaritan work" by retailers in practice, and it is mainly recognized as a way to reduce waste by researchers in the academic field. However, donations could also work as a useful tool towards increasing the profits of the seller while simultaneously reducing waste. Hence, donations would not only benefit the stakeholders in a social manner, but would also improve the viability of profit-maximizing firms. This paper aims to show that it is possible to achieve both profit increase and waste reduction, thus create more sustainable socio-economic systems, by adopting donations.

To this end, a mathematical model is formulated in this paper for a monopolist retailer who sells the old and new versions of the same good simultaneously. The retailer needs to determine the order quantity for the new product and the price for the old product. The assumption is that the price of new product is exogenous while the retailer can freely set the price for the old product (as is the case in various settings, i.e., online technology stores selling the newest version of the good at a predetermined price while the price of older versions can change from one store to another; sellers in a farmer's market selling their high-quality product at the price other sellers are offering while they can set the price for crushed/yesterday's leftover produce at any price they find fit, etc.). The demand for each version of the good depends both on its own price and the other version's price. First, the optimal order quantity and pricing decision for the retailer is explored in a setting without donation, and next the optimal order quantity, price, and donated amount are characterized when there is a donation option. Under the assumptions of the model, it is shown that donation option is always more profitable, and leads to a lower amount of product salvage. Note that Murray et al. [13] define "Circular Economy" as "an economic model wherein planning, resourcing, procurement, production and reprocessing are designed and managed, as both process and output, to maximize ecosystem functioning and human well-being". Therefore, the model developed here can be a good example of the business models that are compatible with Circular Economy concepts, as resource efficiency and competitive advantage are achieved simultaneously.

The remainder of paper is as follows: In Section 2, the relevant literature on joint pricing and quantity setting and on donations is presented. In Section 3, the mathematical model and the main findings of the paper are discussed. Next, the problem is numerically analyzed in Section 4 for a realistic setting to obtain further insights. Finally, Section 5 concludes the paper by stating limitations and discussing practical implications.

2. Literature Review

The joint pricing and ordering problem has been analyzed for more than seven decades, starting with Whitin [14], who defines the problem in a newsvendor setting. The majority of models focus on pricing and ordering of a single item ([15–17]). Petruzzi and Dada [18] account for several demand models to jointly determine quantity and price in newsvendor setting.

The joint pricing and inventory management problem for perishable products has also attracted attention of many researchers. Chen et al. [19] consider this problem in a setting where pricing is sequential, i.e., there is no replenishment, hence the products at different quality levels do not coexist, leading to the result that pricing decisions for the same product at its various lifetime stages are made in respective periods. Wang and Lee [20] propose a discrete pricing model for perishable goods in which the quality is dynamically traced using tracking technologies. Specifically, the authors compare single price and multiple markdown policies, and find that the timing of markdown impacts the firm's profits considerably. Moreover, multiple price markdowns are more appropriate for products that have longer shelf lives. Banerjee and Turner [21] develop a solvable set of differential equations whose solutions characterize the optimal pricing strategy for perishable assets. They incorporate group arrivals of customers into their model and investigate their impact on the prices and stock levels of the firm. The impact of using the optimal pricing policy becomes substantial especially in the group arrivals case. Chew et al. [4] consider a perishable product with multiple period-lifetime and decide the order quantity and the prices simultaneously. For the two-period problem, given the inventory level for the old product, the expected profit is a concave function of the order quantity, the price of the new product and the discounted price of the old product. Clearly, this paper differs from the model discussed in Chew et al.'s [4] work, as the donation factor is incorporated here, and the new product price is exogenous in the current setting. Li et al. [10] consider the same problem when a retailer does not sell new and old inventory at the same time. At the beginning of a period, the retailer makes replenishment and pricing decisions, and at the end of a period, the retailer decides whether to dispose of ending inventory or carry it forward to the next period. They find that dynamic pricing aids extending shelf life and when disposal incurs a lower cost (or positive salvage value), the retailer is induced to dispose earlier. Chen et al. [11], on the other hand, focus on the problem of a vendor who sells the same product at various stages of its life cycle and analyze joint replenishment and pricing decisions. Differently from he model of this work, in both papers, there is a single price in each period applicable to all versions of the same product. Kaya and Polat [22] investigate the problem for a deterministic perishable inventory system in which demand is time and price dependent. The seller has the opportunity to adjust prices for a discrete number of times at a certain cost during the sales season to influence demand and to improve revenues. The authors develop a mathematical model to find the optimal times to change the prices and analyze the efficiency of multiple pricing strategy by comparing the profits obtained by single pricing. However, inventory is replenished only at the beginning of a sales period in their model.

In durable goods literature, Dhebar [23], Fudenberg and Tirole [24], Kornish [25] investigate the pricing problem for sequential versions of the product, but neither of them analyzes pricing and ordering together. Jia and Zhang [26] consider the dynamic pricing and ordering decisions for a durable product with multiple generations in a supply chain with one manufacturer and one retailer. Demands for the product are quality- and price-sensitive. They show that the retailer's optimal pricing strategy exists and depends only on the consumer's quality- and price-sensitivities for a given product quality. Their work is different from this paper again by being modeled as a sequential game between a retailer and a wholesaler.

In short, in both perishable and durable goods literature, joint pricing and inventory management models almost never considered donation strategies as a viable option. Coming to the literature on donation models, Aiello et al. [8] focus on the food losses in the food supply chains and, drawing attention to the existence of undernourished people, propose a model to determine the optimal time to withdraw products from the shelves as well as the quantities to be donated to the non-profit organizations with an objective of maximizing the retailer profit. The authors do not consider pricing in their model and assume exogenous prices. Another research paper on the operational planning of charitable donations is that of Chu et al. [27]. The paper analyzes the impact of tax deductions arising from donations for a profit-maximizing firm. The mathematical model involves a two-period problem, where the firm sets the prices, the production quantity, and the donation quantities sequentially in each period. The authors find that two factors are prevalent in determining the firm's optimal donation behavior, namely fixed cost of production and uncertainty in demand distribution. The model differs from the model of this paper in several aspects; such as sequentiality in pricing decisions in two periods, no quality difference within the same product, a unique production/ordering decision (i.e., a newsvendor model with price markdowns), a fixed cost of production, to name a few. Along the same line of research, Ozbilge et al. [12] make use of condition tracking technologies in order to better manage inventory and reduce food waste across an agrifood supply chain. To this end, they define a quality-dependent newsvendor problem and study two mathematical models. In the former model, the firm performs joint pricing and ordering; whereas in the latter, donation and repricing is considered. Assuming both price- and quality-sensitive customers, they find that donations can improve the firm's profit and contribute to its social responsibility dimension. In their work, the product is at a homogeneous quality level at all times, which is the basic difference of their model from the model of this paper.

This work can be regarded as in the intersection of joint pricing and ordering literature and donation models. Although joint pricing and ordering has been analyzed extensively before, the impact of donation strategies on profitability levels has not been properly analyzed. Moreover, most of the previous work involve sequential pricing decisions, or do not consider co-existence of several versions/quality levels of the same product. This paper could also be helpful for understanding the dynamics of optimal pricing and inventory management practices of retailers when two versions of the same product co-exist. The donation argument elevates this discussion one more level, contributing to developing more socially responsible and sustainable strategies which would be beneficial to all stakeholders.

3. Mathematical Model

The problem involves a profit-maximizing vendor who wants to determine the ordering quantity (for the new version) and the price (for the old version) simultaneously for a product which has old and new versions. At each period t, the vendor sells both new and old products, and the selling season is comprised of T periods. Each newly ordered product can be sold in two consecutive periods, being new in the first period that it is acquired, and as an old product in the second period after replenishment. The products that cannot be sold in two consecutive periods are salvaged. The price for the new product is exogenous and determined by market dynamics. However, the vendor should decide the price for the old product himself/herself. The demands for both versions of the product are stochastic and dependent on the prices of both. Below, all the parameters and decision variables of the problem are defined.

Model parameters:

p_{1t} : price of the new (version of) product in period t; $t \in \{1, 2, \dots T\}$, $p_{1t} \in \{P_1\}$ it is determined by industry/competitors.
p_{2t} : price of the old (version of) product in period t; $t \in \{1, 2, \dots T\}$, $p_{2t} \in \{P_2\}$
$D_1(p_{1t}, p_{2t})$: stochastic demand of the new product in period t; it is decreasing in p_{1t} and increasing in p_{2t}.
$D_2(p_{1t}, p_{2t})$: stochastic demand of the old product in period t; it is decreasing in p_{2t} and increasing in p_{1t}.
x_{2t} : the inventory level of old product at the beginning of t
y_t : the amount of new order (of new product) in period t
c : ordering cost
h : holding cost

s : salvage cost

Although it is assumed that $s \geq 0$, the results continue to hold for a positive "salvage value", i.e., $s < 0$ in the revenue model provided that $p_{2t} + s \geq 0$ for the given price range $\forall t$. The ordering, holding and salvage costs might vary from one selling period to another in real life, but they are assumed to be constant during the entire selling horizon in order to keep the formulations as simple as possible.

A linear stochastic demand is assumed, i.e., $D_1(p_{1t}, p_{2t})$ and $D_2(p_{1t}, p_{2t})$ can be defined as: $D_{1t}(p_{1t}, p_{2t}) = \mu_{1t}(p_{1t}, p_{2t}) + \epsilon_{1t} = a_{1t} - b_{11t}p_{1t} + b_{12t}p_{2t} + \epsilon_{1t}$, $D_{2t}(p_{1t}, p_{2t}) = \mu_{2t}(p_{1t}, p_{2t}) + \epsilon_{1t} = a_{2t} + b_{21t}p_{1t} - b_{22t}p_{2t} + \epsilon_{2t}$, where $a_{it}, b_{iit}, b_{ijt} > 0$ $i, j \in \{1, 2\}$, $p_2 \in \mathbf{P_2} := \{p_{min}, p_{max}\}$ such that $\mathbf{P_2}$ allows only for nonnegative demand, and ϵ_{it} is a random variable with pdf $f_{it}(.)$ and cdf $F_{it}(.)$ on the support $[\epsilon_{min}, \epsilon_{max}]$, $i = 1, 2$. Although in real life demand of a product might not be linear in prices, or other factors might be prevalent in the demand function, this form of demand function is frequently used in retail settings with multiple products to sell ([4,12]).

The pricing and ordering problem of the vendor will be analyzed in two settings: In the first, no donation options are available, while in the latter, the vendor can choose to donate some of the old products at a predetermined price at the beginning of each selling period.

3.1. Case 1: No Donations

In this section, a dynamic programming model that computes seller's expected revenue for T periods is developed and analyzed.

Maximum expected profit for the remaining periods starting in period t with an inventory of x_{2t} can be formulated as:

$$\begin{aligned} V_t(x_{2t}) &= \max_{p_{2t} \in \mathbf{P_2}, y_t} \left[J_t(x_{2t}, y_t, p_{2t}) \right] , \\ &= \max_{p_{2t} \in \mathbf{P_2}, y_t} \left[\phi_t(x_{2t}, y_t, p_{2t}) + \mathbf{E}[V_{t+1}((y_t - D_{1t}(p_{1t}, p_{2t}))^+)] \right] \end{aligned} \tag{1}$$

for $t \in \{1, 2, \ldots T\}$, where $J_t(x_{2t}, y_t, p_{2t})$ is the expected profit for the remaining periods starting in period t and ϕ_t is the net revenue function in period t defined as:

$$\begin{aligned} \phi_t(x_{2t}, y_t, p_{2t}) &= p_{1t}\mathbf{E}[min\{D_{1t}(p_{1t}, p_{2t}), y_t\}] - cy_t - h\mathbf{E}[y_t - D_{1t}(p_{1t}, p_{2t})]^+ \\ &+ p_{2t}\mathbf{E}[min\{D_{2t}(p_{1t}, p_{2t}), x_{2t}\}] - s\mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p_{2t})]^+ \end{aligned} \tag{2}$$

To obtain a revenue function that is unimodal in the given range, it will be assumed that:

$$\frac{F_{2t}(x_{2t} - \mu_{2t})(1 - F_{2t}(x_{2t} - \mu_{2t}))}{f_{2t}(x_{2t} - \mu_{2t})} \leq \mathbf{E}[min\{D_{2t}, x_{2t}\}] + b_{12t}(p_{1t} - c)$$

holds for all $p_{2t} \in \{p_{min}, p_{max}\}$ $\forall t$, for the given parameters. This assumption is easily satisfied in many settings considering $1 - F(.) \leq 1$ and $F(.) \leq 1$, and simply requires p_{1t} to be relatively large with respect to c, which is often the case. The following Theorem states the concavity of multiple period problem and characterizes the monotonic properties of the optimal prices and ordered amounts in each period.

Theorem 1. *$J_t(x_{2t}, y_t, p_{2t})$ is jointly concave with respect to (y_t, p_{2t}) given an inventory level x_{2t}; and $V_t(x_{2t})$ is concave in x_{2t}, $\forall t \in \{1, 2 \ldots T\}$. Moreover, the following monotonic properties hold for the optimal price and ordering values:*

1. *p^*_{2t} is a non-increasing function of x_{2t} and a non-decreasing function of p_{1t}.*
2. *y^*_t is a non-increasing function of x_{2t} and p_{1t}.*

The proof of Theorem 1 can be found in Appendix A. Theorem 1 states that the solution for joint pricing and ordering problem is easy to solve due to joint concavity property. Moreover, the theorem results are mainly intuitive, stating that price of the old product decreases in the inventory level of the old product and increases in the new product price. Similarly, the order quantity decreases in the inventory level of the old product and the new product price. The expected profit function is concave in the current inventory level; namely it is desirable to have some inventory of old product at the start of the new selling season to obtain higher revenues. In the next subsection, the structure of the profit function and resulting monotonic properties of the decision variables will be stated for the donation model, and the two models will be compared in certain aspects.

3.2. Case 2: With Donations

In this setting, the firm can choose to donate a certain amount, q_t ($\leq x_{2t}$), of the inventory on hand in period t before ordering new product. The firm obtains a revenue d (e.g., tax deduction or direct monetary benefits) per unit donated. As in Case 1, a dynamic programming model that computes the expected revenue for T periods is developed.

Maximum expected profit for the remaining periods starting in period t with an inventory of x_{2t} becomes:

$$\begin{aligned} VD_t(x_{2t}) &= \max_{p_{2t}\in \mathbf{P_2}, q_t y_t} [JD_t(x_{2t}, y_t, q_t, p_{2t})]] \\ &= \max_{p_{2t}\in \mathbf{P_2}, q_t, y_t} \left[\psi_t(x_{2t}, y_t, q_t, p_{2t}) + \mathbf{E}[VD_{t+1}((y_t - D_{1t}(p_{1t}, p_{2t}))^+)]\right] \end{aligned} \tag{3}$$

where $JD_t(x_{2t}, y_t, q_t, p_{2t})$ is the expected profit for the remaining periods starting in period t, and ψ_t is the net revenue function in period t defined as:

$$\begin{aligned} \psi_t(x_{2t}, y_t, q_t, p_{2t}) &= p_{1t}\mathbf{E}[min\{D_{1t}(p_{1t}, p_{2t}), y_t\}] - cy_t - h\mathbf{E}[y_t - D_{1t}(p_{1t}, p_{2t})]^+ \\ +p_{2t}\mathbf{E}[min\{D_{2t}(p_{1t}, p_{2t}), x_{2t} - q_t\}] &+ dq_t - s\mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p_{2t}) - q_t]^+ \end{aligned} \tag{4}$$

The main contribution of this paper is stated as in the following Theorem, proving that multiple period problem with donations is still concave, and stating the monotonic properties of the optimal prices and ordered amounts in each period in terms of the problem parameters including donation coefficient d.

Theorem 2. *$JD_t(x_{2t}, y_t, q_t, p_{2t})$ is jointly concave with respect to (y_t, q_t, p_{2t}) given an inventory level x_{2t}; and $VD_t(x_{2t})$ is concave in x_{2t}, $\forall t \in \{1, 2 \ldots T\}$.*

Moreover, the following monotonic properties hold for the optimal price and ordering values:

1. *p^*_{2t} is a non-increasing function of x_{2t} and a non-decreasing function of p_{1t} and d.*
2. *y^*_t is a non-increasing function of x_{2t} and p_{1t}, and a non-decreasing function of d.*
3. *q^*_t is a non-decreasing function of x_{2t} and d, and a non-increasing function of p_{1t}.*

The proof of Theorem 2 can be found in Appendix A.

It is a nice observation to see that the expected net revenue function is still concave despite adding a new dimension. Moreover, the monotonic properties of decision variables still hold with an addition of the new dimension regarding parameter d. A nice corollary of the above Theorem follows from observing that $\frac{\partial VD_T(x_{2T})}{\partial x_{2T}} = \frac{\partial \psi_t(x_{2t})}{\partial x_{2t}} = d$. That is, no matter how much inventory is left from the previous period, this would only contribute positively to this period's revenue, since the seller could donate as much as he/she wants and keep the quantity and price for the lower-quality product at a desired level. Even if an upper limit were to be brought on the amount of donations, the donation model would adjust accordingly and still be more efficient than the non-donation model. This observation is the key point in the analysis, letting the seller obtain more revenue under the donation model in all cases, due to the concavity of $V(.)$ and $VD(.)$ functions. This result is stated without proof.

Corollary 1. $VD_t(x_{2t}) \geq V_t(x_{2t}), \forall t, \forall x_{2t}$.

Finally, the donation model also results in less product waste, i.e., $\mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p_{2t})]^+$ is higher in the first model than $\mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p_{2t}) - q_t]^+$ in the second model. This result is stated and proved below.

Corollary 2. *Letting $p^*_{2t,ND}$ be the optimal price value maximizing $V_t(x_{2t})$ and $p^*_{2t,D}$ be the optimal price value maximizing $VD_t(x_{2t})$. Then:*

$$\mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p^*_{2t,ND})]^+ \geq \mathbf{E}[x_{2t} - D_{2t}(p_{1t}, p^*_{2t,D}) - q^*_t]^+$$

Proof. The above claim will be proved for $t = T$. The rest of proof follows from induction. First, assume that optimal prices are found at the FOC (first order condition), i.e.,

$$\begin{aligned}
\frac{\partial \phi_T}{\partial p_{2T}} &= b_{12T}(p_{1T} + h)F_{1T}(y^*_{T,ND} - \mu_{1T,ND}) - b_{22T}(p^*_{2T,ND} + s)F_{2T}(x_{2T} - \mu_{2T,ND}) + \\
&\quad \mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T,ND}), x_{2T}\}] = 0 \\
\frac{\partial \psi_T}{\partial p_{2T}} &= b_{12T}(p_{1T} + h)F_{1T}(y^*_{T,D} - \mu_{1T,D}) - b_{22T}(p^*_{2T,D} + s)F_{2T}(x_{2T} - \mu_{2T,ND} - q^*_T) + \\
&\quad \mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T,D}), x_{2T} - q^*_T\}] = 0
\end{aligned}$$

(If this is not the case, i.e., if $p^*_{2t,ND} = p^*_{2t,D} = p_{min}$ or $p^*_{2t,ND} = p^*_{2t,D} = p_{max}$, the result becomes trivial.)

As p^*_{2t} decreases in x_{2t} and $q^*_t \geq 0$ for all t, it is clear that $p^*_{2T,D} \geq p^*_{2T,ND}$. Therefore, $D_{2T}(p_{1T}, p^*_{2T,D}) \leq D_{2T}(p_{1T}, p^*_{2T,ND})$, meaning that:

$$\mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T,D}), x_{2T} - q^*_T\}] \leq \mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T,ND}), x_{2T}\}].$$

The first terms in both models are the same (Given that $p^*_{2T,D} \geq p^*_{2T,ND}$, $\mu_{1T,ND} \leq \mu_{1T,D}$ and $y^*_{T,ND} \leq y^*_{T,D}$; and comparing the derivatives, the differences should be the same, i.e., $y^*_{T,ND} - \mu_{1T,ND} = y^*_{T,D} - \mu_{1T,D}$ making the two first terms equal.) Combining these results:

$$\begin{aligned}
b_{22T}(p^*_{2T,ND} + s)F_{2T}(x_{2T} - \mu_{2T,ND}) &\geq b_{22T}(p^*_{2T,D} + s)F_{2T}(x_{2T} - \mu_{2T,D} - q^*_T) \\
&\Rightarrow x_{2T} - \mu_{2T,ND} \geq x_{2T} - \mu_{2T,D} - q^*_T
\end{aligned}$$

which then leads to $\mathbf{E}[x_{2T} - D_{2T}(p_{1T}, p^*_{2T,ND})]^+ \geq \mathbf{E}[x_{2T} - D_{2T}(p_{1T}, p^*_{2T,D}) - q^*_T]^+$. The same result can be proved for all t by induction, as $\phi_t(.)$ and $\psi_t(.)$ have the same structure as $\phi_T(.)$ and $\psi_T(.)$ respectively, and $p^*_{2t,D} \geq p^*_{2t,ND}, \forall t$.

The above corollaries indicate that in addition to improving the welfare of seller and low-income citizens, the donation policy could also improve the system sustainability by reducing waste and associated CO_2 emissions. Hence, adopting a donation model could create a win-win situation for all stakeholders including the seller, donees and the environment. □

4. Numerical Analysis

In this section, the impact of varying parameters on seller revenues, optimal prices, order sizes and other variables of interest will be analyzed for both non-donation and donation models. To this end, similar values for parameters such as demand function coefficients, holding, salvage and ordering costs are used as in [4], slightly tailored to reflect the requirements of current setting while remaining consistent with existing literature. The parameters are stated in Table 1.

Table 1. Numerical Analysis Parameters.

Parameters	Values	Parameters	Values
a_1	200	a_2	100
b_{11}, b_{22}	5	b_{12}, b_{21}	3
h	5	c	10
p_1	50	s	15

The stochastic component of demand is assumed to be uniformly distributed, i.e., $\epsilon \sim U[-60, 60]$, for all periods and for both old and new product demand functions. The price range $\mathbf{P_2} := \{p_{min}, p_{max}\}$ is $\{0, p_1\}$. Finally, it is assumed that $d = 12$, which is lower than the optimal old product price in all cases, as expected in practice, in order not to make donations too appealing (The runs with d = 6 were also conducted, and although the results were less drastic (i.e., net revenue increases were usually within 1–4% range), the structures observed were similar). Tests are run for $T = 2$, since two periods are sufficient to capture the dynamic nature of the problem and also to allow for time efficient computation. The backward DP algorithm for both cases are coded and run in MATLAB R2019B on a computer with 16 GB RAM. Each individual run lasts less than three minutes.

First, one would like to understand how the initial inventory value might affect the optimal prices p_2 and the profit levels under no-donation and donation settings. Table 2 provides the net revenues, first period optimal ordering quantity (y_1^*), first period optimal price for the old product (p_{21}^*) and salvage and donation values for varying starting inventory levels.

Table 2. Optimal Expected Profits and Optimal Values of Decision Variables at Different Starting Inventory Levels (of the Old Product).

	No Donation				Donation					Profit Increase
I_0	V_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	VD_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	q_1^*	
0	5.429	47	132.37	8.44	5.914	47	127.97	8.44	0	8.9 %
20	5.858	44	122.12	10.42	6.154	44	118.97	10.42	0	5 %
40	6.177	42	115.27	15	6.394	42	112.97	15	0	3.5 %
60	6.388	40	108.42	20.42	6.634	42	112.97	16.62	16.84	3.8 %
80	6.496	38	101.56	26.67	6.874	42	112.97	16.62	36.84	5.8 %
100	6.506	36	94.69	33.75	7.114	42	112.97	16.62	56.84	9.3 %
120	6.424	34	87.81	41.67	7.354	42	112.97	16.62	76.84	14.5 %
140	6.253	32	80.92	50.42	7.594	42	112.97	16.62	96.84	21.5 %
160	5.999	30	74.03	60	7.834	42	112.97	16.62	116.84	30.6 %
180	5.667	28	67.12	70.42	8.074	42	112.97	16.62	136.84	42.5 %
200	5.263	26	60.21	81.67	8.314	42	112.97	16.62	156.84	57.9 %

As evident from the results, the donation model brings higher profits particularly as the starting inventory levels increase. Since no upper limit was imposed on the maximum allowable donations, it is no wonder that expected profits increase with increasing starting inventory, since the unnecessary amount can immediately be donated. While the non-donation model tries to deal with extra inventory by decreasing the old product's price and the ordered amount of new product, the donation model stabilizes after a particular initial inventory level by always picking the same optimal old product price and order amount and donating the extra inventory. Even if there was a constraint limiting maximum amount of donations, it is easy to predict that donation model would perform better by setting the donation amount to the largest possible allowable value and adjusting p_2 accordingly, for large values of starting inventory. The expected salvaged amount is therefore low under donation model in all cases while a significant portion of initial inventory is wasted and a resulting profit loss occurs under the non-donation case.

Next, it is possible to analyze the effect of exogenous new product price, p_1, on the optimal solution and optimal values of decision variables. Table 3 displays the results for various values of p_1. The starting inventory is assumed to be 100 at the beginning of first selling period.

Table 3. Optimal Expected Profits and Optimal Values of Decision Variables at Different Price p_1 values.

	No Donation				Donation					Profit Increase
p_1	V_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	VD_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	q_1^*	
30	5.516	23	131.69	30.10	6.106	29	153.19	8.96	68.63	10.7 %
35	6.219	26	123.49	30.10	6.768	32	144.33	10.86	63.94	8.8 %
40	6.617	29	113.72	30.10	7.130	35	133.89	12.7	59.8	7.7 %
45	6.715	33	106.33	33.75	7.193	38	122.33	14.44	56.13	7.1 %
50	6.506	36	94.69	33.75	7.114	42	112.97	16.62	56.84	9.3 %
55	5.986	39	82.45	33.75	6.406	45	100	18.15	54	7 %
60	5.154	42	69.75	33.75	5.550	48	86.56	19.59	51.43	7.7 %
65	4.008	45	56.69	33.75	4.383	51	72.76	20.95	49.09	9.4 %
70	2.546	48	43.33	33.75	2.903	54	58.67	22.23	46.96	14 %
75	1.005	52	33.11	37.60	1.452	57	44.33	23.44	45	44.4 %

As expected, the prices for old product p_2 increase and the order quantities y decrease with increasing p_1 in both models. However, donation model always allows for higher old-product prices, brings higher profits and lower salvage values compared to the non-donation model. The improvement over profits is larger at extremely low or extremely high values of p_1, while the relative gap is becoming smaller at moderate new product prices. That is, donation option could be even more beneficial under extreme market conditions when new product prices are expected to increase substantially or decrease almost to the level of average old-product prices.

Next, it will be explored how the volatility in demand affects the two models. Table 4 displays results for various ϵ ranges. The results are interesting in several dimensions: In the non-donation model, the optimal price for the old product tends to increase with volatility, whereas in the donation model, it is the opposite case. Order quantities increase in both settings, although at a faster pace in the non-donation case. A possible explanation could be the retailer under non-donation model trying to increase expected revenues by capturing more of the new product demand by increasing product availability and attractiveness of new product option, while the retailer who has donation option trying rather to balance the two demand streams. Still, the expected profits decrease in both cases as the uncertainty in demand increases.

Table 4. Optimal Expected Profits and Optimal Values of Decision Variables at Different Demand Ranges.

	No Donation				Donation					Profit Increase
$[\epsilon_{min}, \epsilon_{max}]$	V_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	VD_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	q_1^*	
−20, 20	8.081	33	64.89	15.31	8.705	43	100.56	5.71	63.62	7.7 %
−30, 30	7.725	34	73.46	20.83	8.272	42	100.55	8.31	58.42	7.1 %
−40, 40	7.341	35	81.78	26.40	7.833	42	104.45	11.08	57.89	6.7 %
−50, 50	6.931	35	86.55	28.12	7.392	42	108.65	13.85	57.37	6.7 %
−60, 60	6.506	36	94.69	33.75	7.114	42	112.97	16.62	56.84	9.3 %
−70, 70	6.067	36	99.35	35.71	6.515	41	114.36	18.77	52.50	7.4 %
−80, 80	5.615	36	103.98	37.81	6.080	41	118.79	21.45	52.14	8.3 %
−90, 90	5.158	37	112.05	43.40	5.646	41	123.26	24.14	51.78	9.5 %
−100, 100	4.693	37	116.65	45.56	5.213	41	127.73	26.82	51.43	11.1 %
−110, 110	4.224	37	121.24	47.78	4.782	40	129.73	28.51	48	13.2 %

Finally, it is possible to analyze how the price elasticity and demand correlation parameters, i.e., b_{ij} and b_{ii}, $i \neq j$, $i, j \in \{1, 2\}$, affect the results. First, the values of b_{ii} are varied keeping b_{ij} fixed, and then the opposite is performed. Results (displayed in Table 5) indicate that if the effect of competing product's price on the other product's demand increases, prices for the old product increase in both models; the rationale probably being that it is possible to capture higher revenues by setting both prices high in that case. On the other hand, prices for old product are set lower and the order quantity for new products are also lower when each product's demand is not seriously affected by the other's price. In this case, the models find it more profitable to shift attention to selling more of the old product at lower prices, which also means a decrease in the sales of the new product. Comparing the two models, donation model again sets higher p_2, orders higher y, and salvages less in all cases. The benefit of donation option is more evident in cases where demand elasticity is high, namely when demand decreases faster with its own price.

Table 5. Optimal Expected Profits and Optimal Values of Decision Variables at Different Demand Parameters.

	No Donation				Donation					Profit Increase
$b_{ii}, b_{ij}, j = 1,2$	V_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	VD_1 ($\times 10^3$)	p_{21}^*	y_1^*	salvage	q_1^*	
4, 3	13.466	46	178.96	36.82	13.661	50	186.97	20.51	39.85	1.4 %
4.5, 3	9.897	40	133.42	33.75	10.253	46	149.97	18.64	50.12	3.6 %
5, 3	6.506	36	94.69	33.75	7.114	42	112.97	16.62	56.84	9.3 %
5.5, 3	3.360	32	55.92	30.82	3.889	38	75.97	14.44	60.13	15.7 %
6, 3	0.606	29	20.58	29.40	1.256	35	41.97	12.70	64.80	107 %
5, 2	0.945	26	34.21	33.75	1.671	32	50.97	10.86	68.94	76.8 %
5, 2.5	3.495	31	61.98	33.75	4.066	37	79.47	13.86	62.31	16.3 %
5, 3	6.506	36	94.68	33.75	7.114	42	112.97	16.62	56.84	9.3 %
5, 3.5	9.902	41	132.35	33.75	10.236	46	147.97	18.64	48.11	3.4 %
5, 4	13.501	46	174.96	33.75	13.661	50	186.97	20.65	39.85	1.2 %

5. Discussion and Conclusions

This paper shows that retailers or sellers of perishable inventory could improve their profits by adopting a donation model while also helping low-income citizens and reducing environmental side effects. The numerical analysis provided here shows a glimpse of how much profit increase is possible in different settings. The results are particularly impressive if demand is highly volatile or highly elastic, or if the starting inventory of old products is large.

Donations are considered to be a viable option in achieving Circular Economy (CE) objectives. As defined by The Ellen MacArthur Foundation, CE is based on designing out waste and pollution, keeping products and materials in use, and regenerating natural systems [28]. Hence, donation is one of the ways to reduce waste, use resources more efficiently, and keep products in use. Several top-tier textile firms such as H&M, Levi Strauss Co. etc. have clothing "donation bins" in stores ([29]), and many grocery stores engage in donation activities in collaboration with local food banks ([30,31]). Similarly, there are several papers in the CE literature that propose donation as one of the possible ways to achieve resource efficiency. For instance, Awasthi et al. [5] state that one of the policies to more effectively manage electronic waste is to support humanitarian organizations donating used electronics to poorer countries. Chang [29], Mariani [32] and Hvass [33] develop distinct, innovative business models incorporating donation to reduce waste in textile industry and reach circular economy objectives. However, what the practical donation applications or previous research on the topic fail to recognize is that beside being a good-samaritan activity and creating good public recognition for the firms, it is also possible to raise additional profit by donations using correct pricing strategies. In this paper, this aspect of donations is emphasized, which clearly aligns with the aims of CE, as CE represents a development strategy that enables economic growth while aiming to optimize the chain of consumption of biological and technical materials ([34]).

One should however note that for the results to be applicable, the two demand streams (one for new and one for the old product) should already be in effect. For instance, Ferguson and Koenigsberg [9] show that for a monopolist perishable product vendor, carrying inventory to the next period could cannibalize the demand for the new product and reduce profitability. In the setting of this paper, it is assumed that regardless of the inventory holding or disposal decision, some price-sensitive customers will prefer old products and could find them in other vendors if not offered by the seller. Therefore, keeping the old product for one more period and pricing it appropriately to maximize profits makes sense. In this setting, donation would increase financial benefits substantially if the starting old product inventory is higher than a certain level, and disposal is not an option or costly for old products in their second life phase.

Another limitation of this work could be the assumption that products go through only two phases in their life cycle (new vs. old) as opposed to a continuous scale and that all products in each phase are regarded as having the same quality. These products in fact could be heterogeneous within themselves, and/or the quality degradation might not be piece-wise as it is assumed here. Still, despite analytical complexity, the model and results that are found in this work could be extended to situations with more than two quality levels (versions) of the products.

In any case, the insights in this paper could be beneficial mainly for retailers in fresh food or high-tech industries, who would like to perform good samaritan work while not hurting their profits, and also for public policy-makers who would like to support low-income citizens. Customers in today's world display a clear interest in supporting firms who are known to engage in sustainability-ensuring activities ([35]), and they feel more connected to this firm if they are personally engaged in these activities (e.g., such as by using the donation bins in H&M stores and getting a reimbursement in return) ([33]). If large grocery or high-tech product chains engage in donation activities, this would strengthen the brand image for these companies, leading to positive effects on the customer loyalty, market shares and share prices. On the top of improved brand image, these firms will have the chance to reduce costs (by reducing waste) and improve profitability via correct pricing strategies and tax benefits; which would clearly lead to more sustainable and effective corporations in the long run.

Similarly, municipalities of big cities could develop policies towards buying unsold products from small vendors in farmer markets at discounted prices and send them to soup kitchens. For instance, consider a small vendor selling vegetables in a farmer's market. These vendors often sell the vegetable only during a few (for instance, two) consecutive days, fresh product and old (or crushed, battered but still edible) products have different sales prices, and the unsold products at the end of these days go to waste bin. However, if the municipality of the region sets a particular purchase quota for the unsold old products at the end of every sales day at a preannounced, lower-than-market price, the small vendor could enjoy the increased profit levels even without changing the initial inventory and the sales price of old product. He or she can earn even more by appropriately revising the sales price of old product and the order quantity of fresh product, as the numerical results of this paper suggest. On the other side, the municipality could provide material for food banks at reduced prices, leading to a win-win situation for all the parties that are involved.

In this aspect, when donations are used properly, small vendors of the farmer markets, large retailers and low-income citizens will be the direct winners, but the citizens in the region will also enjoy indirect benefits such as more efficient municipality services and lower environmental side effects due to lessened waste. Hence, both public and private business models incorporating the donation element bear the potential to improve stakeholder benefits and elevate sustainability of economic systems. It is hoped that this work attracts more attention towards donation models and contributes to building innovative business systems that are compatible with the notions of green deal strategies and CE.

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Appendix A

Proof of Theorem 1. The proof follows following steps: 1. The net revenue function of the last period, ϕ_T is shown to be jointly concave with respect to (y_T, p_{2T}) given an inventory level x_{2T}; 2. p_{2T}, y_T are non-increasing in x_{2T}. 3. V_T is concave in x_{2T} 4. $J_t(x_{2t}, y_t, p_{2t})$ is jointly concave with respect to (y_t, p_{2t}) given an inventory level x_{2t} 5. p_{2t}, y_t are non-increasing in x_{2t}, $\forall t$. 6. V_t is concave in x_{2t}, $\forall t$.

Step 1. Consider the Hessian matrix for ϕ_T, $H(\phi_T)$. Its determinant is:

$$det(H(\phi_T)) = \begin{vmatrix} \frac{\partial^2\phi_T}{\partial p_{2T}^2} & \frac{\partial^2\phi_T}{\partial p_{2T}\partial y_T} \\ \frac{\partial^2\phi_T}{\partial y_T \partial p_{2T}} & \frac{\partial^2\phi_T}{\partial y_T^2} \end{vmatrix}$$

Note that $\frac{\partial\phi_T}{\partial y_T} = p_{1T}(1 - F_{1T}(y_T - \mu_{1T})) - hF_{1T}(y_T - \mu_{1T}) - c$ and $\frac{\partial\phi_T}{\partial p_{2T}} = \mathbf{E}[min\{D_{2T}, x_{2T}\}] + (p_{1T} + h)b_{12T}F_{1T}(y_T - \mu_{1T}) - (p_{2T} + s)b_{22T}F_{2T}(x_{2T} - \mu_{2T})$. Hence:

$$\begin{aligned} \frac{\partial^2\phi_T}{\partial p_{2T}^2} &= -2b_{22T}F_{2T}(x_{2T} - \mu_{2T}) - (p_{1T} + h)f_{1T}(y_T - \mu_{1T})b_{12T}^2 - (p_{2T} + s)f_{2T}(x_{2T} - \mu_{2T})b_{22T}^2 \leq 0 \\ \frac{\partial^2\phi_T}{\partial y_T^2} &= -(p_{1T} + h)f_{1T}(y_T - \mu_{1T}) \leq 0 \\ \frac{\partial^2\phi_T}{\partial y_T p_{2T}} &= b_{12T}(p_{1T} + h)f_{1T}(y_T - \mu_{1T}) \end{aligned}$$

The first leading principal minor of the Hessian matrix is negative, while the second leading principal minor, $\frac{\partial^2\phi_T}{\partial p_{2T}^2} \times \frac{\partial^2\phi_T}{\partial y_T^2} - (\frac{\partial^2\phi_T}{\partial y_T p_{2T}})^2 = 2b_{22T}F_{2T}(x_{2T} - \mu_{2T})(p_{1T} + h)f_{1T}(y_T - \mu_{1T}) + (p_{2T} + s)f_{2T}(x_{2T} - \mu_{2T})b_{22T}^2(p_{1T} + h)f_{1T}(y_T - \mu_{1T}) \geq 0$, proving that $H(\phi_T)$ is a negative semi-definite matrix. Thus, ϕ_T is jointly concave in p_{2T}, y_T.

Step 2. The optimal values of p_{2T}, y_T can be obtained by solving the Lagrangian problem and stating the KKT (Karush-Kuhn-Tucker) conditions.

$$\begin{aligned} \frac{\partial\phi_T}{\partial p_{2T}} &= \mathbf{E}[min\{D_{2T}, x_{2T}\}] + (p_{1T} + h)b_{12T}F_{1T}(y_T^* - \mu_{1T}) - (p_{2T}^* + s)b_{22T}F_{2T}(x_{2T} - \mu_{2T}) && \text{(A1)} \\ & +\lambda_{1T}^* - \lambda_{2T}^* = 0 && \text{(A2)} \\ \frac{\partial\phi_T}{\partial y_T} &= p_{1T}(1 - F_{1T}(y_T^* - \mu_{1T})) - hF_{1T}(y_T^* - \mu_{1T}) - c = 0 && \text{(A3)} \\ & \lambda_{1T}^*(p_{2T}^* - p_{min}) = 0 && \text{(A4)} \\ & \lambda_{2T}^*(-p_{2T}^* + p_{max}) = 0 && \text{(A5)} \\ \lambda_{1T}^*, \lambda_{2T}^* &\geq 0, p_{min} \leq p_{2T}^* \leq p_{max} && \text{(A6)} \end{aligned}$$

By (A3), optimal y_T^* can be found as $y_T^* = F_{1T}^{-1}(\frac{p_{1T}-c}{p_{1T}+h}) + \mu_{1T}$. It is easy to see y_T^* decreases in p_{1T} by simply checking the sign of the first derivative. To see that y_T^* decreases in x_{2T}, note that by implicit theorem, $\frac{\partial y_T^*}{\partial x_{2T}} - \frac{\partial \mu_{1T}}{\partial x_{2T}} = 0$, i.e., $\frac{\partial y_T^*}{\partial x_{2T}} - b_{12T}\frac{\partial p_{2T}^*}{\partial x_{2T}} = 0$, proving that $\frac{\partial y_T^*}{\partial x_{2T}}$ and $\frac{\partial p_{2T}^*}{\partial x_{2T}}$ have the same sign. Next consider all cases where $\lambda_{1T}^* = \lambda_{2T}^* = 0$, $\lambda_{1T}^* > 0, \lambda_{2T}^* = 0$ and $\lambda_{1T}^* = 0, \lambda_{2T}^* > 0$.

Case (i): $\lambda_{1T}^* = \lambda_{2T}^* = 0$: In this case:

$$\frac{\partial\phi_T}{\partial p_{2T}} = b_{12T}(p_{1T} - c) - b_{22T}(p_{2T} + s)F_{2T}(x_{2T} - \mu_{2T}) + \mathbf{E}[min\{D_{2T}, x_{2T}\}] = 0 \quad \text{(A7)}$$

Calling $A := y_T - \mu_{1T}$ and $B := x_{2T} - \mu_{2T}$:

$$\frac{\partial}{\partial x_{2T}}[\frac{\partial\phi_T}{\partial p_{2T}^*}] = (1 - F_{2T}(B)) - b_{22T}(p_{2T}^* + s)f_{2T}(B) - b_{22T}\frac{\partial p_{2T}^*}{\partial x_{2T}}(F_{2T}(B) + b_{22T}(p_{2T}^* + s)f_{2T}(B)) = 0 \quad \text{(A8)}$$

So, provided $(1 - F_{2T}(B)) - b_{22T}(p_{2T}^* + s)f_{2T}(B) \leq 0$, $\frac{\partial p_{2T}}{\partial x_{2T}} \leq 0$. Noting that $b_{12T}(p_{1T} - c) - b_{22T}(p_{2T}^* + s)F_{2T}(B) + \mathbf{E}[min\{D_{2T}, x_{2T}\}] = 0$, this requirement pours into: $\frac{F_{2T}(B)(1-F_{2T}(B))}{f_{2T}(B)} \leq \mathbf{E}[min\{D_{2T}, x_{2T}\}] + b_{12T}(p_{1T} - c)$, which was previously assumed to hold.

Case (ii): $\lambda_{1T}^* > 0, \lambda_{2T}^* = 0$:

In this case, optimal value is $p^*_{2T} = p_{min}$, indicating the situation where μ_{2T} is too large. Taking the derivative of Equation (A4), $\frac{\partial \lambda^*_{1T}}{\partial x_{2T}}(p^*_{2T} - p_{min}) + \lambda^*_{1T}\frac{\partial (p^*_{2T} - p_{min})}{\partial x_{2T}} = \lambda^*_{1T}\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$. Since $\lambda^*_{1T} > 0$, this means $\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$. The same result for optimal y^*_T follows as $\frac{\partial y^*_T}{\partial x_{2T}} - b_{12T}\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$.

Case (iii): $\lambda^*_{1T} = 0, \lambda^*_{2T} > 0$:

In this case, optimal value is $p^*_{2T} = p_{max}$, indicating the situation where μ_{2T} is too low. Taking the derivative of Equation (A5), $\frac{\partial \lambda^*_{2T}}{\partial x_{2T}}(-p^*_{2T} + p_{max}) + \lambda^*_{2T}\frac{\partial (-p^*_{2T} + p_{max})}{\partial x_{2T}} = -\lambda^*_{2T}\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$. Since $\lambda^*_{2T} > 0$, this means $\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$. The same result for optimal y^*_T follows as $\frac{\partial y^*_T}{\partial x_{2T}} - b_{12T}\frac{\partial p^*_{2T}}{\partial x_{2T}} = 0$.

Step 3. The optimal net revenue function V_T is concave in x_{2T}. Take the derivative of V_T with respect to x_{2T}:

$$\begin{aligned}\frac{\partial V_T(x_{2T})}{\partial x_{2T}} &= \frac{\partial \phi_T(x_{2T}, y^*_T, p^*_{2T})}{\partial x_{2T}} = p_{1T}\Big[\int_{\epsilon_{min}}^{y^*_T-\mu_{1T}} b_{12T}\frac{\partial p^*_{2T}}{\partial x_{2T}} f_{1T}(\epsilon_{1T})d\epsilon_{1T} + \int_{y^*_T-\mu_{1T}}^{\epsilon_{max}} b_{12T}\frac{\partial p^*_{2T}}{\partial x_{2T}} f_{1T}(\epsilon_{1T})d\epsilon_{1T}\Big] \\ &+ p^*_{2T}\Big[\int_{\epsilon_{min}}^{x_{2T}-\mu_{2T}} -b_{22T}\frac{\partial p^*_{2T}}{\partial x_{2T}} f_{2T}(\epsilon_{2T})d\epsilon_{2T} + \int_{x_{2T}-\mu_{2T}}^{\epsilon_{max}} 1\frac{\partial p^*_{2T}}{\partial x_{2T}} f_{2T}(\epsilon_{2T})d\epsilon_{2T}\Big] \\ &- cb_{12T}\frac{\partial p^*_{2T}}{\partial x_{2T}} + \frac{\partial p^*_{2T}}{\partial x_{2T}}\mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T}), x_{2T}\}] - s\int_{\epsilon_{min}}^{x_{2T}-\mu_{2T}} (1 + b_{22T})\frac{\partial p^*_{2T}}{\partial x_{2T}} f_{2T}(\epsilon_{2T})d\epsilon_{2T}\end{aligned}$$

Rearranging the terms:

$$\begin{aligned}\frac{\partial V_T(x_{2T})}{\partial x_{2T}} &= \frac{\partial p^*_{2T}}{\partial x_{2T}}\Big[p_{1T}b_{12T} - cb_{12T} + \mathbf{E}[min\{D_{2T}(p_{1T}, p^*_{2T}), x_{2T}\}] \\ &- s(1 + b_{22T})F_{2T}(x_{2T} - \mu_{2T}) + p^*_{2T}\big[-b_{22T}F_{2T}(x_{2T} - \mu_{2T}) + 1 - F_{2T}(x_{2T} - \mu_{2T})\big]\Big]\end{aligned}$$

However, then, inserting (A2) and (A3) in the above equation, one obtains:

$$\frac{\partial V_T(x_{2T})}{\partial x_{2T}} = \Big[p^*_{2T} - F_{2T}(x_{2T} - \mu_{2T})(p^*_{2T} + s)\Big]$$

Taking the second derivative of $V_T(x_{2T})$ with respect to x_{2T}:

$$\frac{\partial^2 V_T(x_{2T})}{\partial x^2_{2T}} = \Big[\frac{\partial p^*_{2T}}{\partial x_{2T}}(1 - F_{2T}(x_{2T} - \mu_{2T})) - f_{2T}(x_{2T} - \mu_{2T})(p^*_{2T} + s)(1 + b_{22T}\frac{\partial p^*_{2T}}{\partial x_{2T}})\Big]$$

Observing that $\frac{\partial^2 V_T(x_{2T})}{\partial x^2_{2T}} \leq 0$ is not immediate at first glance. However, we know from (A8) that $(1 - F_{2T}(B)) - b_{22T}(p^*_{2T} + s)f_{2T}(B) = b_{22T}\frac{\partial p^*_{2T}}{\partial x_{2T}}(F_{2T}(B) + b_{22T}(p^*_{2T} + s)f_{2T}(B))$. Hence, incorporating this value in the above equation, one finds that $\frac{\partial^2 V_T(x_{2T})}{\partial x^2_{2T}} = \frac{\partial p^*_{2T}}{\partial x_{2T}} - (1 - F_{2T}(B))/b_{22T} \leq 0$ as p^*_{2T} is a decreasing function of x_{2T}, proving the concavity of this function.

Step 4. Given that V_{t+1} is concave in $x_{2(t+1)}$, it can be shown that $J_t(x_{2t}, y_t, p_{2t})$ is jointly concave with respect to (y_t, p_{2t}) given an inventory level x_{2t}. Note that

$$J_t(x_{2t}, y_t, p_{2t}) = \phi_t(x_{2t}, y_t, p_{2t}) + \int_{\epsilon_{min}}^{y_t-\mu_{1t}} V_{t+1}(y_t - \mu_{1t} - \epsilon)f_{1t}(\epsilon)d\epsilon + \int_{y_t-\mu_{1t}}^{\infty} V_{t+1}(0)f_{1t}(\epsilon)d\epsilon$$

The determinant of Hessian matrix $H(J_t)$ of this function can be stated as:

$$det(H(J_t)) = \begin{vmatrix} \frac{\partial^2 J_t}{\partial p^2_{2t}} & \frac{\partial^2 J_t}{\partial p_{2t}\partial y_t} \\ \frac{\partial^2 J_t}{\partial y_t \partial p_{2t}} & \frac{\partial^2 J_t}{\partial y^2_t} \end{vmatrix}$$

which takes the form:

$$det(H(J_t)) = \begin{vmatrix} \frac{\partial^2\phi_t}{\partial p_{2t}^2} + b_{21t}^2 A_t & \frac{\partial^2\phi_t}{\partial p_{2t}\partial y_t} - b_{21t}A_t \\ \frac{\partial^2\phi_t}{\partial p_{2t}\partial y_t} - b_{21t}A_t & \frac{\partial^2\phi_t}{\partial y_t^2} + A_t \end{vmatrix}$$

where $A_t = \int_{-\infty}^{y_t-\mu_{1t}} V''_{t+1}(y_t - \mu_{1t} - \epsilon)f_{1t}(\epsilon)d\epsilon \leq 0$ by concavity of V_{t+1}. Hence, Hessian matrix $H(J_t)$ is negative semi-definite, proving concavity of J_t in (y_t, p_{2t}).

Step 5. The claims that p_{2t}^* is a non-increasing function of x_{2t} and a non-decreasing function of p_{1t}, and y_t^* is a non-increasing function of x_{2t} and p_{1t}, are again obtained from the KKT conditions. The details are skipped as they follow along the same lines as in Step 2.

$$\frac{\partial\phi_t}{\partial p_{2t}} = \mathbf{E}[minD_{2t}, x_{2t}] + (p_{1t}+h)b_{12t}F_{1t}(y_t^* - \mu_{1t}) - (p_{2t}^* + s)b_{22t}F_{2t}(x_{2t} - \mu_{2t}) \tag{A9}$$

$$+\lambda_{1t}^* - \lambda_{2t}^* = 0 \tag{A10}$$

$$\frac{\partial\phi_t}{\partial y_t} = p_{1t}(1 - F_{1t}(y_t^* - \mu_{1t})) - hF_{1t}(y_t^* - \mu_{1t}) = 0 \tag{A11}$$

$$\lambda_{1t}^*(p_{2t}^* - p_{min}) = 0 \tag{A12}$$

$$\lambda_{2t}^*(-p_{2t}^* + p_{max}) = 0 \tag{A13}$$

$$\lambda_{1t}^*, \lambda_{2t}^* \geq 0, p_{min} \leq p_{2t}^* \leq p_{max} \tag{A14}$$

Step 6. Finally, the optimal net revenue function V_t needs to be shown to be concave in x_{2t}. Consider the first derivative of V_t with respect to x_{2t}:

$$\begin{aligned} \frac{\partial V_t(x_{2t})}{\partial x_{2t}} &= \frac{\partial\phi_t(x_{2t}, y_t^*, p_{2t}^*)}{\partial x_{2t}} + \int_{\epsilon_{min}}^{y_t^*-\mu_{1t}} \frac{\partial V_{t+1}(y_t^* - \mu_{1t} - \epsilon_{1t})}{\partial x_{2(t+1)}}(\frac{\partial y_t^*}{\partial x_{2t}} - b_{12t}\frac{\partial p_{2t}^*}{\partial x_{2t}})f_{2t}(\epsilon_{1t})d\epsilon_{1t} \\ &= \frac{\partial\phi_t(x_{2t}, y_t^*, p_{2t}^*)}{\partial x_{2t}} \end{aligned}$$

as $\frac{\partial y_t^*}{\partial x_{2t}} = b_{12t}\frac{\partial p_{2t}^*}{\partial x_{2t}}$. Hence, the second derivative of V_t with respect to x_{2t} is equivalent to the second derivative of the net revenue function in t, which can be shown to be concave in x_{2t} by a similar argument as in Step 3. The results follows. □

Proof of Theorem 2. We will follow the same sequence as in the Proof of Theorem 1.

First, to show the concavity of ψ_T, consider the Hessian matrix H_T. Its determinants are shown as:

$$det(H(T)) = \begin{vmatrix} \frac{\partial^2\psi_T}{\partial p_{2T}^2} & \frac{\partial^2\psi_T}{\partial p_{2T}\partial y_T} & \frac{\partial^2\psi_T}{\partial p_{2T}\partial q_T} \\ \frac{\partial^2\psi_T}{\partial y_T\partial p_{2T}} & \frac{\partial^2\psi_T}{\partial y_T^2} & \frac{\partial^2\psi_T}{\partial y_T\partial q_T} \\ \frac{\partial^2\psi_T}{\partial q_T\partial p_{2T}} & \frac{\partial^2\psi_T}{\partial q_T\partial y_T} & \frac{\partial^2\psi_T}{\partial q_T^2} \end{vmatrix}$$

Clearly, the first leading principal minor of this Hessian matrix is negative, i.e., $\frac{\partial^2\psi_T}{\partial p_{2T}^2} = -b_{12T}^2(p_{1T} + h)f_{1T}(y_T - \mu_{1T}) - b_{22T}^2(p_{2T}+s)f_{2T}(x_{2T} - q - \mu_{1T}) - b_{22T}F_{2T}(x_{2T} - q - \mu_{1T}) \leq 0$. Calling $A := y_T - \mu_{1T}$ and $B := x_{2T} - q - \mu_{1T}$, note that:

$$\begin{aligned} \begin{vmatrix} \frac{\partial^2\psi_T}{\partial p_{2T}^2} & \frac{\partial^2\psi_T}{\partial p_{2T}\partial y_T} \\ \frac{\partial^2\psi_T}{\partial y_T\partial p_{2T}} & \frac{\partial^2\psi_T}{\partial y_T^2} \end{vmatrix} &= \begin{vmatrix} -b_{12T}^2(p_{1T}+h)f_{1T}(A) - b_{22T}^2(p_{2T}+s)f_{2T}(B) - b_{22T}F_{2T}(B) & b_{12T}(p_{1T}+h)f_{1T}(A) \\ b_{12T}(p_{1T}+h)f_{1T}(A) & -(p_{1T}+h)f_{1T}(A) \end{vmatrix} \\ &= b_{22T}^2(p_{2T}+s)f_{2T}(B) - b_{22T}F_{2T}(B)(p_{1T}+h)f_{1T}(A) \geq 0 \end{aligned}$$

Thus, the second leading principal minor is positive. Finally, to show that the third order leading principal minor is negative, consider its determinant, i.e.,

$$det(H(T)) = \frac{\partial^2 \psi_T}{\partial q_T \partial p_{2T}} \begin{vmatrix} \frac{\partial^2 \psi_T}{\partial y_T \partial p_{2T}} & \frac{\partial^2 \psi_T}{\partial y_T^2} \\ \frac{\partial^2 \psi_T}{\partial q_T \partial p_{2T}} & \frac{\partial^2 \psi_T}{\partial q_T \partial y_T} \end{vmatrix} - \frac{\partial^2 \psi_T}{\partial q_T \partial y_T} \begin{vmatrix} \frac{\partial^2 \psi_T}{\partial p_{2T}^2} & \frac{\partial^2 \psi_T}{\partial p_{2T} \partial y_T} \\ \frac{\partial^2 \psi_T}{\partial q_T \partial p_{2T}} & \frac{\partial^2 \psi_T}{\partial q_T \partial y_T} \end{vmatrix} + \frac{\partial^2 \psi_T}{\partial q_T^2} \begin{vmatrix} \frac{\partial^2 \psi_T}{\partial y_T \partial p_{2T}} & \frac{\partial^2 \psi_T}{\partial y_T^2} \\ \frac{\partial^2 \psi_T}{\partial p_{2T}^2} & \frac{\partial^2 \psi_T}{\partial p_{2T} \partial y_T} \end{vmatrix}$$

First, note that $\frac{\partial^2 \psi_T}{\partial q_T \partial p_{2T}} = -(1 - F_{2T}(B)) \leq 0$, $\frac{\partial^2 \psi_T}{\partial q_T^2} = -(p_{2T} + s) f_{2T}(B) \leq 0$ and $\frac{\partial^2 \psi_T}{\partial y_T \partial q_T} = 0$. Thus, the above determinant pours into the value:

$$det(H(T)) = f_{1T}(A)(p_{1T} + h)\left[(1 - F_{2T}(B))^2 + (p_{2T} + s) f_{2T}(B)(b_{22T}^2 (p_2 + s) + b_{22T} F_{2T}(B)\right] \geq 0$$

ensuring the negative semi-definiteness of the Hessian matrix.

Next, note that $\frac{\partial^2 \psi_T}{\partial p_{2T}^2} \leq 0$, $\frac{\partial^2 \psi_T}{\partial y_T^2} \leq 0$, and $\frac{\partial^2 \psi_T}{\partial q_T^2} = 0$ showing the concavity of the function in all three terms. Hence, optimal p_{2T}^*, y_T^* and q_T^* satisfy:

$$\frac{\partial \psi_T}{\partial p_{2T}} = b_{12T}(p_{1T} + h) F_{1T}(A) - b_{22T}(p_{2T}^* + s) F_{2T}(B) + \mathbf{E}[min\{D_{2T}(p_{1T}, p_{2T}), x_{2T} - q_T\}] = 0 \quad (A15)$$

$$\frac{\partial \psi_T}{\partial y_T} = (p_{1T} - c) + (p_{1T} + h) F_{1T}(A) = 0 \quad (A16)$$

$$\frac{\partial \psi_T}{\partial q_T} = -(p_{2T}^* - d) + F_{2T}(B)(s + p_{2T}^*) = 0 \quad (A17)$$

where $A := y_T^* - \mu_{1T}$ and $B := x_{2T} - \mu_{2T} - q_T^*$. That is:

$$y_T^* = F_{1T}^{-1}\left(\frac{p_{1T} - c}{p_{1T} + h}\right) + \mu_{1T} \quad (A18)$$

$$q_T^* = x_{2T} - F_{2T}^{-1}\left(\frac{p_{2T}^* - d}{p_{2T}^* + s}\right) - \mu_{2T} \quad (A19)$$

The monotonicity properties can be shown as in Theorem 1. To see the relationship of the variables with d, observe that q_T^* increases with d by (A19), and that the partial derivative of $\frac{\partial \psi_T}{\partial p_{2T}}$ can be rewritten as:

$$\frac{\partial \psi_T}{\partial p_{2T}} = b_{12T}(p_{1T} - c) - b_{22T}(p_{2T} - d) + \mathbf{E}[min\{D_{2T}(p_{1T}, p_{2T}), x_{2T} - q_T\}] = 0$$

Thus, increasing d would increase the third term in the above equation, leading to an increase in the term $b_{22T}(p_{2T} - d)$, which means optimal p_{2T}^* should also increase to satisfy the equality. Similarly, y_T^* increases in d as μ_{1T} increases in d, as evident by Equation (A18).

Thus:

$$\begin{aligned} \frac{\partial VD_T(x_{2T})}{\partial x_{2T}} &= \frac{\partial \psi_T(x_{2T}, y_T^*, p_{2T}^*)}{\partial x_{2T}} = p_{1T}(1 - F_{1T}(A)) \frac{\partial y_T^*}{\partial x_{2T}} - c \frac{\partial y_T^*}{\partial x_{2T}} + p_{1T} b_{12T} F_{1T}(A) \frac{\partial p_{2T}^*}{\partial x_{2T}} \\ &- h\left[F_{1T}(A) \frac{\partial y_{2T}^*}{\partial x_{2T}} - b_{12T} \frac{\partial p_{2T}^*}{\partial x_{2T}}\right] + p_{2T}^* \left[F_{2T}(B)(-b_{22T} \frac{\partial p_{2T}^*}{\partial x_{2T}}) + (1 - F_{2T}(B))(1 - \frac{\partial q_T^*}{\partial x_{2T}})\right] \\ &+ \frac{\partial p_{2T}^*}{\partial x_{2T}} \mathbf{E}[min\{D_{2T}(p_{1T}, p_{2T}), x_{2T} - q_T\}] + d \frac{\partial q_T^*}{\partial x_{2T}} - s F_{2T}(B)\left[1 - \frac{\partial q_T^*}{\partial x_{2T}} + b_{22T} \frac{\partial p_{2T}^*}{\partial x_{2T}}\right] \end{aligned}$$

By the fact that $\frac{\partial y_T^*}{\partial x_{2T}} = b_{12T}\frac{\partial p_{2T}^*}{\partial x_{2T}}$ and $\frac{\partial q_T^*}{\partial x_{2T}} = 1 + b_{22T}\frac{\partial p_{2T}^*}{\partial x_{2T}} - \frac{\partial F_{2T}^{-1}(B)}{\partial x_{2T}}$, the above equation pours into:

$$\begin{aligned}\frac{\partial VD_T(x_{2T})}{\partial x_{2T}} &= (p_{1T}-c)b_{12T}\frac{\partial p_{2T}^*}{\partial x_{2T}} + \frac{\partial p_{2T}^*}{\partial x_{2T}}\mathbf{E}[min\{D_{2T}(p_{1T},p_{2T}),x_{2T}-q_T\}] \\ &+ p_{2T}^*\frac{\partial F_{2T}^{-1}(B)}{\partial x_{2T}}(1-F_{2T}(B)) - b_{22T}p_{2T}^*\frac{\partial p_{2T}^*}{\partial x_{2T}} - sF_{2T}(B)\frac{\partial F_{2T}^{-1}(B)}{\partial x_{2T}} + d[1-\frac{\partial F_{2T}^{-1}(B)}{\partial x_{2T}} + b_{22T}\frac{\partial p_{2T}^*}{\partial x_{2T}}]\end{aligned}$$

However, then, inserting (A18) and (A19) in the above equation, and noting that $\mathbf{E}[min\{D_{2T}^*, x_{2T} - q_T^*\}] = -b_{12T}(p_{1T}-c) + b_{22T}(p_{2T}^* - d)$ one obtains:

$$\frac{\partial VD_T(x_{2T})}{\partial x_{2T}} = d$$

That is, clearly VD_T is a concave function of x_{2T}.

To show that $JD_t(x_{2t}, y_t, q_t, p_{2t})$ is a concave function of (y_t, q_t, p_{2t}) for a given x_{2t}, we observe its Hessian matrix $H(JD_t)$. The determinant of this matrix can be stated as:

$$det(H(JD_t)) = \begin{vmatrix} \frac{\partial^2 JD_t}{\partial p_{2t}^2} & \frac{\partial^2 JD_t}{\partial p_{2t}\partial y_t} & \frac{\partial^2 JD_t}{\partial p_{2t}\partial q_t} \\ \frac{\partial^2 JD_t}{\partial y_t \partial p_{2t}} & \frac{\partial^2 JD_t}{\partial y_t^2} & \frac{\partial^2 JD_t}{\partial y_t\partial q_t} \\ \frac{\partial^2 JD_t}{\partial q_t\partial p_{2t}} & \frac{\partial^2 JD_t}{\partial q_t \partial y_t} & \frac{\partial^2 JD_t}{\partial q_t^2} \end{vmatrix}$$

which takes the form:

$$det(H(JD_t)) = \begin{vmatrix} \frac{\partial^2 \psi_t}{\partial p_{2t}^2} + b_{12t}^2 C & \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C & \frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t} \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C & -f_{1t}(A)(p_{1t}+h) & 0 \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t} & 0 & -f_{2t}(B)(p_{2t}+s) \end{vmatrix}$$

where $A := y_t^* - \mu_{1t}$ and $B := x_{2t} - \mu_{2t} - q_t^*$, and $C := \int \epsilon_{min}{}^{y_t - \mu_{1t}} VD_{t+1}''(y_t - \mu_{1t} - \epsilon_{1t}) f_{1(t+1)}(\epsilon_{1t}) d\epsilon_{1t}$. Clearly, $C \le 0$ by the concavity of V_{t+1} function. The above Hessian clearly has a negative first order leading principal minor. The second order leading principal minor is given as:

$$\begin{vmatrix} \frac{\partial^2 \psi_t}{\partial p_{2t}^2} + b_{12t}^2 C & \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C & -f_{1t}(A)(p_{1t}+h) \end{vmatrix} = f_{1t}(A)(p_{1t}+h)(b_{22t}^2(p_{2t}^*+s)f_{2t}(B) + 2b_{22t}F_{2t}(B)) + b_{12t}^2 C f_{1t}(A)(p_{1t}+h) - b_{12t}^2 C^2$$

The above equation is not necessarily positive for all C values. However, as will be evident within the course of the proof, $\frac{\partial^2 VD_t(x_{2t})}{\partial x_{2t}^2} = \frac{\partial^2 \psi_t(x_{2t}, y_t^*, q_t^*, p_{2t}^*)}{\partial x_{2t}^2}$ and that $\frac{\partial^2 \psi_t(x_{2t}, y_t^*, q_t^*, p_{2t}^*)}{\partial x_{2t}^2} = \frac{\partial d}{\partial x_{2t}} = 0\ \forall t$, i.e., $C = 0$. It is immediately observable that the above equation is positive for VD_{T-1}, since $\frac{\partial^2 VD_T(x_{2T})}{\partial x_{2T}^2} = 0$.

Finally, the third order leading principal minor is given by:

$$\begin{aligned}&\frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t}\begin{vmatrix} \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C & -f_{1t}(A)(p_{1t}+h) \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t} & 0 \end{vmatrix} - 0 \begin{vmatrix} \frac{\partial^2 \psi_t}{\partial p_{2t}^2} + b_{12t}^2 C & \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t} & 0 \end{vmatrix} \\ &+(-f_{2t}(B)(p_{2t}+s))\begin{vmatrix} \frac{\partial^2 \psi_t}{\partial p_{2t}^2} + b_{12t}^2 C & \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C \\ \frac{\partial^2 \psi_t}{\partial p_{2t}\partial y_t} - b_{12t}C & -f_{1t}(A)(p_{1t}+h) \end{vmatrix}\end{aligned}$$

which is negative since $(\frac{\partial^2 \psi_t}{\partial p_{2t}\partial q_t})^2 - (f_{1t}(A)(p_{1t}+h)) \le 0$, the second term is zero, and the third term is a negative value multiplied with the second order leading principal minor (which is positive). Hence, the matrix $H(JD_t)$ is negative semi-definite, proving the concavity of JD_t.

Finally, note that:

$$\begin{aligned}\frac{\partial VD_t(x_{2t})}{\partial x_{2t}} &= \frac{\partial \psi_t(x_{2t}, y_t^*, q_t^*, p_{2t}^*)}{\partial x_{2t}} + (\frac{\partial y_t^*}{\partial x_{2t}} - b_{12t}\frac{\partial p_{2t}^*}{\partial x_{2t}}) \int_{\epsilon_{min}}^{y_t-\mu_{1t}} VD'_{t+1}((y_t - D_{1t})f_{1t}(\epsilon_{1t})d\epsilon_{1t} \\ &= \frac{\partial \psi_t(x_{2t}, y_t^*, q_t^*, p_{2t}^*)}{\partial x_{2t}}\end{aligned}$$

Hence, $\frac{\partial^2 VD_t(x_{2t})}{\partial x_{2t}^2} = \frac{\partial^2 \psi_t(x_{2t}, y_t^*, q_t^*, p_{2t}^*)}{\partial x_{2t}^2} \leq 0$, proving the concavity of VD_t. □

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