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Computing finite time non-ruin probability and some joint distributions in discrete time risk model with exchangeable claim occurrences

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ABSTRACT

In this paper, we study a discrete time risk model based on exchangeable dependent claim occurrences. In particular, we obtain expressions for the finite time non-ruin probability, and the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. An illustration of the results is given and some implications of the results are provided. Comparisons are made with the corresponding results for the classical compound binomial model of independent and identically distributed claim occurrences.

1. Introduction

Binomial models are used in finance and insurance on many topics like stock price analysis, pricing of financial derivatives, credit risk assessments and solvency oriented bankruptcy studies. For the last two, loss causing default events are the main concern in the modeling where the commonly envisaged important components are risk capital, income generating cash inflows, and default driven cash outflows. A general view of risk modeling on these topics can be found in the works of Kijima [1], Melnikov [2] and Franke et al. [3]. On the finance side, the loan and debt security markets are the vast grounds to observe default losses of debtors upon their failures in upfront scheduled payments. In the insurance market, claims of insureds are default events that may create loss burdens on insurers. In either case, each individual loss generating event can be modeled in probability with a Bernoulli distribution so that their aggregation in partial or complete sums follows a Binomial distribution when individual losses are independent. Time dynamic representations of such events are done with Binomial processes.

Concentrating on the insurance markets, the bankruptcy, as a solvency measure, for an insurer is modeled by a surplus or reserve process $\{U_n, n \in \mathbb{N}\}$ in order to express the evolution of surplus checked. When time is measured in discrete units, the process is a discrete one and the surplus at the end of time period n is defined by

$$U_n = u + n - \sum_{j=1}^{n} Y_j \tag{1}$$

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with $U_0 = u$, where u is the initial surplus, the periodic premium is one, and Y_j is the eventual claim amount in period j which is defined by

$$Y_j = \begin{cases} X_j, & I_j = 1 \\ 0, & I_j = 0, \end{cases}$$

where $I_j = 1$ if a claim occurs in period j and $I_j = 0$, otherwise, $j = 1, 2, \ldots$. The random variable I_j and the individual claim amount random variable X_j are independent in each time period. Eq. (1) can be equivalently defined as

$$U_n = u + n - \sum_{j=1}^{N_n} X_j, \tag{2}$$

where N_n is a random variable that stands for the number of claims up to time n. If the random indicators $I_1, I_2, \ldots$ are independent with $p = P\left\{I_j = 1\right\}$, then the above model is called the compound binomial model and

$$P\{N_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

There is a rich literature on the surplus processes in actuarial risk modeling. Consideration of the compound Binomial model for the surplus processes in analogy of the classical compound Poisson model is a relatively new approach. Gerber [4] investigates compound Binomial process modeling for the surplus processes. Then follow the studies on the subject by Michel [5], Shiu [6], Willmot [7], Dickson [8], Cheng et al. [9], Liu and Zhao [10], Lefèvre and Loisel [11], Wei and Hu [12]. Cheng et al. [9] studied the "discounted" probability $f(x, y; u)$ of ruin for an initial surplus u, such that the surplus just before ruin is x and the deficit at ruin is y under the compound binomial model. They have obtained both recursive and asymptotic formulas for $f(x, y; u)$. Liu and Zhao [10] derived the joint distribution of the time of ruin, the surplus immediately before ruin and the deficit at ruin in the compound binomial model. Recently, Li and Sendova [13] derived the probability distribution of the time to ruin under the compound binomial model following the idea in [14]. They have also obtained the distribution of the time the surplus process hits a specific level.

Define

$$T = \inf\{t > 0 : U_t \le 0\}$$

to be the random time to ruin. Let $\alpha(u; n) = P_u\{T > n\}$ denote the finite time non-ruin (survival) probability, $u > 0$. Then it can be computed recursively from

$$\alpha(u; n) = \sum_{t=1}^{n} pq^{t-1} \sum_{x=1}^{u+t-1} f(x)\alpha(u + t - x; n - t) + (1-p)^n, \tag{3}$$

for $n > 0$ with $\alpha(u; 0) = 1$ [4].

The assumption of independence between the claims occurrences may not be realistic for certain portfolios. Easing this assumption, Cossette et al. [15] studied the discrete time risk model (1) by assuming a Markovian type dependence between the claim occurrences. That is, $\{I_k, k \in \mathbb{N}\}$ is a Markovian process with a given transition probability matrix. Under this assumption, the model (1) is referred as the compound Markov binomial model. Cossette et al. [15] provided recursive formulas for computing the ruin probabilities over finite and infinite time horizons. Further computational extensions on this model is given by another work of Cossette et al. [16]. Yuen and Guo [17] extracted two discrete-time renewal risk processes from the compound Markov binomial model. They have investigated the Gerber–Shiu expected discounted penalty functions based on these renewal risk processes. Shizu et al. [18] obtained renewal equations for the conditional and unconditional Gerber–Shiu discounted penalty function under the Markov binomial model.

In this paper, we study the model (1) when the claim occurrences are exchangeable and dependent. The sequence of claim occurrence indicators $\{I_k, k \in \mathbb{N}\}$ is exchangeable if for each $n > 0$, the joint distribution of $I_1, I_2, \ldots, I_n$ is invariant under any permutation of its indices, i.e.

$$P\left\{I_{\pi_1} = x_1, I_{\pi_2} = x_2, \ldots, I_{\pi_n} = x_n\right\} = P\left\{I_1 = x_1, I_2 = x_2, \ldots, I_n = x_n\right\}$$

for any permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the indices in $\{1, 2, \ldots, n\}$. Exchangeability in probability models is an important concept when joint distributions are sought for permutations of the elements of a sample itself or equal sized subsets of it without any judgement of independence or without requiring existence of any limit of relative frequencies. We refer to Schervish [19, pp. 5–52] for details on exchangeability and for some of its applications on parametric models. In this context, according to the fundamental Theorem of de Finetti, restated in the same reference, there is a random variable Θ supported on $(0, 1)$ with cumulative distribution function (c.d.f.) $G(\theta)$ such that $E(\Theta) = \theta$ and

$$P\{I_1 = x_1, \ldots, I_n = x_n\} = \int_0^1 \theta^k (1-\theta)^{n-k} dG(\theta),$$

where $k = \sum_{i=1}^{n} x_i$, and $G(\theta) = P\{\Theta \le \theta\}$ may be regarded as a mixing distribution. Such a dependence in $\{I_k, k \in \mathbb{N}\}$ occurs when the portfolio of an insurance company is diversified in the sense that the claim occurrence probabilities associated with different groups of insureds in an insurance portfolio according to their risk proneness features are significantly different. This sort of differentiation may be necessary for credibility assessment and pricing of insurance contracts on the basis of demographic, economic and separated market policy reasons. This means that, in a portfolio the value p for an individual policy is one of the possible values of a random variable Θ. Risk models that include dependence which is incorporated through a mixing model in the individual claim amount distributions have been studied in [20–22]. In our model, the dependence is incorporated through a mixing model in the individual claim occurrence probability.

Eryilmaz [23] studied the distribution of various random variables in the compound binomial model when the claim occurrences are independent and identically distributed (i.i.d.). The exchangeability of claim occurrences is first considered in this paper, and the method for deriving the corresponding distributions is novel as presented in Section 2.

The rest of the paper is organized as follows. In Section 2, we derive an expression for the finite time non-ruin probability. Section 3 contains the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. In Section 4, we present a numerical illustration of the results and briefly indicate the potential risk management implications of the findings.

2. Finite time non-ruin probability

Let the claim occurrences $I_1^{(e)}, I_2^{(e)}, \ldots$ be exchangeable. Then the distribution of the total number of claims arrived up to time n can be expressed as

$$P\left\{N_n^{(e)} = k\right\} = P\left\{\sum_{i=1}^{n} I_i^{(e)} = k\right\} = \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \lambda_{k+i}, \tag{4}$$

where $\lambda_s = P\left\{I_1^{(e)} = I_2^{(e)} = \cdots = I_s^{(e)} = 1\right\}$ for $s > 0$ and $\lambda_0 = 1$. The proof of Eq. (4) is based on the well-known principle of inclusion–exclusion (see, e.g. [24]).

Throughout the paper, the random variable $I_j^{(e)}$ and the individual claim amount random variable X_j are assumed to be independent in each time period. The random variable X_j is strictly positive and $\left\{X_j, j \geq 1\right\}$ is a sequence of i.i.d. random variables with probability mass function (p.m.f.) $f(x) = P\{X = x\}$ and c.d.f. $F(x) = P\{X \leq x\}$.

Denote by $T^{(e)}$ and T time to ruin random variables which are respectively based on exchangeable $I_1^{(e)}, I_2^{(e)}, \ldots$ and i.i.d. $I_1, I_2, \ldots$ claim occurrence indicators. By conditioning on the total number of claims arrived up to time n,

$$P_u\left\{T^{(e)} > n\right\} = \sum_{k=0}^{n} P_u\left\{T^{(e)} > n \mid N_n^{(e)} = k\right\} P\left\{N_n^{(e)} = k\right\}. \tag{5}$$

Because of exchangeability, all sequences with the same length and the same number of ones are equally likely. Hence the conditional distributions of $T^{(e)}$ and T, given the number of claims, are identical, i.e.

$$P_u\left\{T^{(e)} > n \mid N_n^{(e)} = k\right\} = P_u\left\{T > n \mid N_n = k\right\}. \tag{6}$$

Therefore, we have

$$P_u\left\{T^{(e)} > n\right\} = \sum_{k=0}^{n} P_u\left\{T > n \mid N_n = k\right\} P\left\{N_n^{(e)} = k\right\}. \tag{7}$$

The conditional distribution in (6) is independent of claim occurrence probabilities, and only depends on the claim size distribution. According to Eq. (7), the derivation of the finite time non-ruin probability for an exchangeable sequence of claim occurrences needs the usage of the conditional distribution of T given the total number of claims N_n when the claim occurrences are i.i.d.

Eryilmaz [23] obtained the following recursive equation for the joint probability of $T > n$ and $N_n = k$. For $u = 1, 2, \ldots$ and $k = 0, 1, \ldots, n$,

$$\beta(u; n, k) = P_u\{T > n, N_n = k\}$$
$$= \begin{cases} 1, & \text{if } n = 0 \text{ and } k = 0 \\ (1-p)^n, & \text{if } n \geq k \text{ and } k = 0 \\ \displaystyle\sum_{t=1}^{n} pq^{t-1} \sum_{x=1}^{u+t-1} f(x)\beta(u+t-x; n-t, k-1), & \text{if } n \geq k \text{ and } k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain the following result for the finite time non-ruin probability for an exchangeable sequence of claim occurrences.

Theorem 1. *Let the claim occurrences $I_1^{(e)}, I_2^{(e)}, \ldots$ be exchangeable with*

$$\lambda_s = P\left\{I_1^{(e)} = I_2^{(e)} = \cdots = I_s^{(e)} = 1\right\}.$$

Then for $u = 1, 2, \ldots$ and $k = 0, 1, \ldots, n$,

$$P_u\left\{T^{(e)} > n\right\} = \sum_{k=0}^{n}\sum_{i=0}^{n-k}(-1)^i C(u; n, k)\binom{n-k}{i}\lambda_{k+i},$$

where

$$C(u; n, k) = \frac{\beta(u; n, k)}{p^k(1-p)^{n-k}}.$$

Proof. The proof immediately follows by substituting

$$P_u\{T > n \mid N_n = k\} = \frac{\beta(u; n, k)}{\binom{n}{k}p^k(1-p)^{n-k}}$$

in (7), and using (4). ■

Remark 1. Although the term $C(u; n, k)$ seems to be dependent on p, it is independent of p because of the cancellations of the terms $p^k(1-p)^{n-k}$ in the numerator and denominator of $C(u; n, k)$. Indeed, if for example $n = 2$ and $k = 1$, we have

$$\begin{aligned} C(u; 2, 1) &= \frac{(1-p)pP\{X \le u+1\} + p(1-p)P\{X \le u\}}{p(1-p)} \\ &= F(u+1) + F(u). \end{aligned}$$

3. Joint distribution of $(T^{(e)}, U_{T^{(e)}-1}, U_{T^{(e)}})$

In this section, we derive an equation for the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. Let $P(n, i, j)$ denote the joint p.m.f. of the random vector $(T^{(e)}, U_{T^{(e)}-1}, \left|U_{T^{(e)}}\right|)$ based on a sequence of exchangeable claim indicators $I_1^{(e)}, I_2^{(e)}, \ldots$, i.e.

$$P(n, i, j) = P_u\left\{T^{(e)} = n, U_{T^{(e)}-1} = i, \left|U_{T^{(e)}}\right| = j\right\}.$$

We first prove the following lemma which will be useful in the sequel. Let

$$\gamma(u; n, j, k) = P_u\{T > n, U_n = j, N_n = k\}$$

be the joint p.m.f. of (T, U_n, N_n) for a sequence of i.i.d. claim occurrences.

Lemma 1. *Let the claim occurrences $I_1, I_2, \ldots$ be i.i.d. with $p = P\{I_i = 1\}$. For $u = 1, 2, \ldots$ and $k = 0, 1, \ldots, n$,*

$$\gamma(u; n, j, k) = \begin{cases} (1-p)^n & \text{if } k = 0, j = u+n \\ \sum_{t=1}^{n} pq^{t-1}\sum_{x=1}^{u+t-1} f(x)\gamma(u+t-x; n-t, j, k-1), & \text{if } k > 0, j < u+n \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proof is based on conditioning on the waiting time for the occurrence of the first claim. Let W_1 denote the waiting time for the first claim. Then for $k > 0$ and $j < u + n$,

$$\begin{aligned} P_u\{T > n, U_n = j, N_n = k\} &= \sum_{t=1}^{n} P_u\{T > n, U_n = j, N_n = k \mid W_1 = t\}\, pq^{t-1} \\ &\quad + \sum_{t=n+1}^{\infty} P_u\{T > n, U_n = j, N_n = k \mid W_1 = t\}\, pq^{t-1}. \end{aligned}$$

If $t > n$ and $j < u+n$ then $P_u\{T > n, U_n = j, N_n = k \mid W_1 = t\} = 0$. Thus noting that $U_{W_1=t} > 0$ for $t \le n$ since ruin occurs after period n and then by conditioning on the value of the first individual claim one obtains

$$P_u\{T > n, U_n = j, N_n = k\} = \sum_{t=1}^{n} pq^{t-1}\sum_{x=1}^{u+t-1} f(x)P_{u+t-x}\{T > n-t, U_{n-t} = j, N_{n-t} = k-1\},$$

for $k > 0$ and $j < u + n$. For $k = 0$ and $j = u + n$, $P_u\{T > n, U_n = j, N_n = k\} = P\{I_1 = 0, \ldots, I_n = 0\}$. Therefore the proof is completed. ■

Theorem 2. *Let the claim occurrences* $I_1^{(e)}, I_2^{(e)}, \ldots$ *be exchangeable with*

$$\lambda_s = P\left\{I_1^{(e)} = I_2^{(e)} = \cdots = I_s^{(e)} = 1\right\}.$$

Then for $u, i = 1, 2, \ldots$ *and* , $j = 0, 1, \ldots,$

$$P(n, i, j) = P\{X = i + j + 1\} \sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} (-1)^m D(u; n, i, k) \binom{n-k-1}{m} \lambda_{k+m+1},$$

where

$$D(u; n, i, k) = \frac{\gamma(u; n-1, i, k)}{p^k (1-p)^{n-k-1}},$$

for $n = 1, 2, \ldots,$ *and* $P(1, i, j) = \lambda_1 P\{X = i + j + 1\} I\{u = i\}$, *where* $I\{A\} = 1$ *if* A *occurs and* $I\{A\} = 0$ *otherwise.*

Proof. The proof for $n = 1$ is immediate since

$$\begin{aligned} P(1, i, j) &= P\left\{I_1^{(e)} = 1, U_0 = i, |U_1| = j\right\} \\ &= \lambda_1 P\{X = i + j + 1\} I\{u = i\}. \end{aligned}$$

For $n > 1$,

$$\begin{aligned} P(n, i, j) &= P_u\left\{T^{(e)} = n, U_{T^{(e)}-1} = i, \left|U_{T^{(e)}}\right| = j\right\} \\ &= P_u\left\{T^{(e)} > n-1, U_{n-1} = i, I_n^{(e)} = 1, X = i + j + 1\right\} \\ &= P_u\left\{T^{(e)} > n-1, U_{n-1} = i, I_n^{(e)} = 1\right\} P\{X = i + j + 1\}. \end{aligned} \tag{8}$$

By conditioning on the total number of claims,

$$\begin{aligned} P_u\left\{T^{(e)} > n-1, U_{n-1} = i, I_n^{(e)} = 1\right\} &= \sum_{k=0}^{n-1} P_u\left\{T^{(e)} > n-1, U_{n-1} = i, I_n^{(e)} = 1, N_{n-1}^{(e)} = k\right\} \\ &= \sum_{k=0}^{n-1} P_u\left\{T^{(e)} > n-1, U_{n-1} = i \mid N_{n-1}^{(e)} = k, I_n^{(e)} = 1\right\} \\ &\quad \times P\left\{N_{n-1}^{(e)} = k, I_n^{(e)} = 1\right\}. \end{aligned} \tag{9}$$

Note that given $\left\{N_{n-1}^{(e)} = k, I_n^{(e)} = 1\right\}$, the conditional probability of $(T^{(e)} > n-1, U_{n-1} = i)$ is identical with the conditional probability of the same random variables for the i.i.d. case, and

$$\begin{aligned} P_u\left\{T^{(e)} > n-1, U_{n-1} = i \mid N_{n-1}^{(e)} = k, I_n^{(e)} = 1\right\} &= P_u\{T > n-1, U_{n-1} = i \mid N_{n-1} = k, I_n = 1\} \\ &= P_u\{T > n-1, U_{n-1} = i \mid N_{n-1} = k\} \\ &= \frac{\gamma(u; n-1, i, k)}{\binom{n-1}{k} p^k (1-p)^{n-k-1}}. \end{aligned} \tag{10}$$

On the other hand, under exchangeability

$$P\left\{N_{n-1}^{(e)} = k, I_n^{(e)} = 1\right\} = \binom{n-1}{k} \sum_{m=0}^{n-k-1} (-1)^m \binom{n-k-1}{m} \lambda_{k+m+1}. \tag{11}$$

Thus the proof is completed by substituting (10) and (11) first in (9), and then using (9) in (8). ∎

For $\lambda_s = p^s$ in Theorem 2, we obtain the joint distribution of $(T, U_{T-1}, |U_T|)$ for the classical compound Binomial model. Therefore the following result is an alternative to the expression for the probability $P(n, i, j)$ which is given in Theorem 3.2 of Liu and Zhao [10], and based on i.i.d. claim occurrences.

Corollary 1. *Let the claim occurrences* $I_1, I_2, \ldots$ *be i.i.d. with* $p = P\{I_i = 1\}$. *Then for* $u, i = 1, 2, \ldots$ *and* , $j = 0, 1, \ldots,$

$$P(n, i, j) = P\{X = i + j + 1\} \sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} (-1)^m \frac{\gamma(u; n-1, i, k)}{(1-p)^{n-k-1}} \binom{n-k-1}{m} p^{m+1},$$

for $n = 1, 2, \ldots,$ *and* $P(1, i, j) = pP\{X = i + j + 1\} I\{u = i\}$.

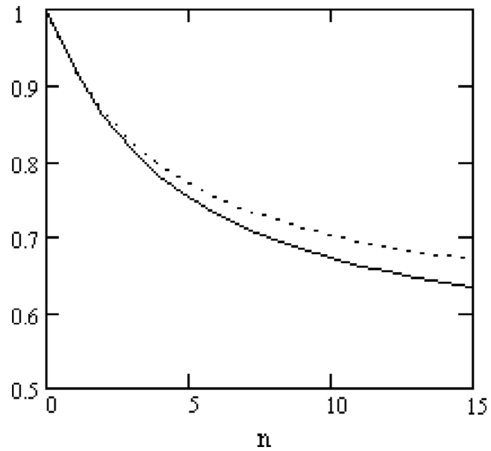

Fig. 1. Finite time non-ruin probabilities for i.i.d. (the solid line) and exchangeable (the dashed line) cases when $u = 1$, $\alpha = 4/5$, $p = 0.1$, $a = 1$, $b = 9$.

4. Applications of the results

With the findings here, default risk modelers are availed of using the exchangeability and exchangeable dependence concepts in their loss and ruin analysis. In this regard, the explicit expression of the joint mass function $P(n, i, j)$ of time to ruin, immediate surplus and surplus deficit quantities can lead risk managers to several policy decisions. Policy specifications about initial reserve as risk capital and amount of premiums as cash inflow have definite impacts on an insurance portfolio surplus. Surplus at a given time is known to depend positively on initial reserve, net premium income and reinsurance retention levels whilst it is negatively related with random claim amount [25, pp. 155–207]. Obviously, the latter for a written portfolio size is a function of default probability p and surges up as p increases.

A numerical illustration is given below for finite time non-ruin probabilities for i.i.d. and exchangeable claim occurrences. For the exchangeable case, we assume a beta-mixing distribution, i.e.

$$dG(\theta) = \frac{1}{B(a, b)}\theta^{a-1}(1-\theta)^{b-1}d\theta, \tag{12}$$

where $B(a, b)$ denotes the Beta function. In this case,

$$\lambda_s = P\left\{I_1^{(e)} = I_2^{(e)} = \cdots = I_s^{(e)} = 1\right\} = \int_0^1 \theta^s dG(\theta)$$
$$= \frac{1}{B(a, b)}\int_0^1 \theta^{a+s-1}(1-\theta)^{b-1}d\theta = \frac{B(a+s, b)}{B(a, b)},$$

for $s \geq 0$.

The distribution of the total number of claims arrived up to time n is

$$P\left\{N_n^{(e)} = k\right\} = \binom{n}{k}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\frac{B(a+k+i, b)}{B(a, b)},$$

for $k = 0, 1, \ldots, n$.

Assume that the individual claim size distribution is a zero-truncated geometric distribution with p.m.f. $f(x) = (1-\alpha)\alpha^{x-1}$ and c.d.f. function $F(x) = 1 - \alpha^x$, $x = 1, 2, \ldots$ In Figs. 1 and 2, we plot the finite time non-ruin probabilities $P_u\{T > n\}$ and $P_u\left\{T^{(e)} > n\right\}$ as a function of n when $p = 0.1$ for the i.i.d. case (the solid line), and when $a = 1$ and $b = 9$ for the exchangeable case (the dashed line). The values of a and b in (12) are chosen such that the probability of claim occurrence in each period are identical for both cases, i.e. $\lambda_1 = \frac{a}{a+b} = E(\Theta) = p$. Fig. 3 plots finite time non-ruin probabilities when $p = 0.2$ for the i.i.d. case (the solid line), and when $a = 1$ and $b = 4$ for the exchangeable case (the dashed line). In all cases $\alpha = \frac{4}{5}$.

From the Figures we observe that the finite time non-ruin probability is larger under the assumption of exchangeable and dependent claim occurrences, and the difference between $P_u\{T > n\}$ and $P_u\left\{T^{(e)} > n\right\}$ becomes larger as n tends to infinity. An increase in the initial reserve u makes the difference between the finite time non-ruin probabilities smaller. When compared Figs. 1 and 3, as expected, an increase in individual claim occurrence probability leads to a decrease in finite time non-ruin probabilities for both cases.

In the case of default losses of debtors in financial markets where loan and debt security interest payments are under concern, similar implications hold true. Thereof, credit risk of a financial asset is determined essentially by the probability

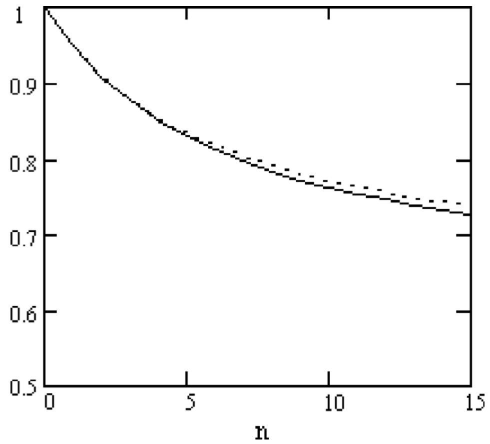

Fig. 2. Finite time non-ruin probabilities for i.i.d. (the solid line) and exchangeable (the dashed line) cases when $u = 3$, $\alpha = 4/5$, $p = 0.1$, $a = 1$, $b = 9$.

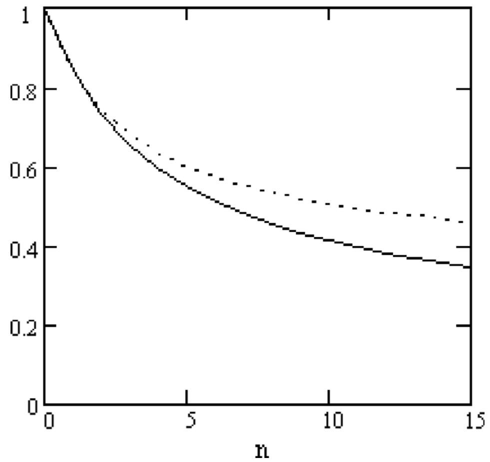

Fig. 3. Finite time non-ruin probabilities for i.i.d. (the solid line) and exchangeable (the dashed line) cases when $u = 1$, $\alpha = 4/5$, $p = 0.2$, $a = 1$, $b = 4$.

Table 1
The joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin when $\alpha = \frac{4}{5}$, $a = 1$ and $b = 9$.

n	j	i	$P(n, i, j)$	n	j	i	$P(n, i, j)$
5	0	1	0.00020	5	2	1	0.00013
		2	0.00035			2	0.00022
		3	0.00046			3	0.00029
10	0	1	0.00007	10	5	1	0.00002
		2	0.00012			2	0.00004
		3	0.00015			5	0.00006
		5	0.00017			8	0.00005
		10	0.00057			10	0.00019

p of default and the loss exposure at default. Among the factors that affect this risk, borrower characteristics have a major importance and these characteristics are reflected in the default probability p for exchangeable dependent default event occurrences. The loss amount at a default event then goes along with p as an item that depletes the surplus quantity.

Using Theorem 2, $P(n, i, j)$, the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin can be calculated for given n, i and j. In Table 1, we compute $P(n, i, j)$ when the individual claim size distribution is zero-truncated geometric distribution with $\alpha = \frac{4}{5}$, and when $a = 1$ and $b = 9$.

5. Summary and conclusions

This paper presents analytical derivations and useful results for realistic applications for a discrete time risk model where occurrences of exchangeable loss creating events are considered against the usual assumption of independent loss events.

Under the assumption of exchangeability, claim occurrences are dependent over the periods in many insurance and finance applications that focus on risk of loss at default events.

We have obtained explicit expressions for the finite time non-ruin probability, and the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. Our derivations make use of conditioning on the total number of claim occurrences. After this conditioning, we use the fact that the conditional distribution of a random variable defined on the risk model, given the number of claim occurrences are identical in both exchangeable and i.i.d. cases. Our results are consistent with the results previously obtained in the literature since exchangeability generalizes the i.i.d. case when $\lambda_s = p^s$ in Theorems 1 and 2.

Joint probability function for the variables of surplus immediate before ruin, deficit at ruin and time to ruin is obtained and potential applications of it are discussed. It is exhibited that non-ruin probability under the assumption of exchangeable and dependent claim occurrences is larger than that under independent and identically distributed claim occurrences. Increasing initial reserve or risk capital and increasing premium or payment income quantities are seen to lead to a decreasing difference between the finite time non-ruin probabilities of exchangeable dependent and i.i.d. cases.

The dependence models considered in [15,16] and in the present paper are significantly different. Under the assumption of exchangeable and dependent claim occurrences, the joint distribution of $I_1, I_2, \ldots, I_n$ is invariant under any permutation of its indices for all n while under Markov dependence such a property is not satisfied in general. On the other hand, the discrete time risk model with exchangeable claim occurrences can be linked to the binomial model in a Markovian environment by the help of partial exchangeability which is also known as Markov exchangeability. A sequence $\{I_k, k \in \mathbb{N}\}$ taking values on $\mathbf{I} = \{0, 1\}$ is said to be partially exchangeable, or Markov exchangeable if for two sequences $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbf{I}^n$ which have the same starting state and the same transition counts,

$$P\{I_1 = \sigma_1, \ldots, I_n = \sigma_n\} = P\{I_1 = \tau_1, \ldots, I_n = \tau_n\}$$

(see, e.g. [26]).

The results given in the present paper can be extended to this more general Markov exchangeable model. This will be among our future research problems.

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