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On the classification of scalar evolution equations with non-constant separant

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Abstract

The 'separant' of the evolution equation $u_t = F$, where F is some differentiable function of the derivatives of u up to order m, is the partial derivative $\partial F/\partial u_m$, where $u_m = \partial^m u/\partial x^m$. As an integrability test, we use the formal symmetry method of Mikhailov–Shabat–Sokolov, which is based on the existence of a recursion operator as a formal series. The solvability of its coefficients in the class of local functions gives a sequence of conservation laws, called the 'conserved densities' $\rho^{(i)}$, $i = -1, 1, 2, 3, \ldots$. We apply this method to the classification of scalar evolution equations of orders $3 \leqslant m \leqslant 15$, for which $\rho^{(-1)} = [\partial F/\partial u_m]^{-1/m}$ and $\rho^{(1)}$ are non-trivial, i.e. they are not total derivatives and $\rho^{(-1)}$ is not linear in its highest order derivative. We obtain the 'top level' parts of these equations and their 'top dependencies' with respect to the 'level grading', that we defined in a previous paper, as a grading on the algebra of polynomials generated by the derivatives u_{b+i}, over the ring of C^∞ functions of $u, u_1, \ldots, u_b$. In this setting b and i are called 'base' and 'level', respectively. We solve the conserved density conditions to show that if $\rho^{(-1)}$ depends on $u, u_1, \ldots, u_b$, then, these equations are level homogeneous polynomials in $u_{b+i}, \ldots, u_m$, $i \geqslant 1$. Furthermore, we prove that if $\rho^{(3)}$ is non-trivial, then $\rho^{(-1)} = (\alpha u_b^2 + \beta u_b + \gamma)^{1/2}$, with $b \leqslant 3$ while if $\rho^{(3)}$ is trivial, then $\rho^{(-1)} = (\lambda u_b + \mu)^{1/3}$, where $b \leqslant 5$ and α, β, γ, λ and μ are functions of $u, \ldots, u_{b-1}$. We show that the equations that we obtain form commuting flows and we construct their recursion operators that are respectively of orders 2 and 6 for non-trivial and trivial $\rho^{(3)}$ respectively. Omitting lower order dependencies, we show that equations with non-trivial $\rho^{(3)}$ and $b = 3$

are symmetries of the ‘essentially non-linear third order equation’; for trivial $\rho^{(3)}$, the equations with $b = 5$ are symmetries of a non-quasilinear fifth order equation obtained in previous work, while for $b = 3, 4$ they are symmetries of quasilinear fifth order equations.

Keywords: classificaiton, differential polynomials, evolution equations, hierarchies

1. Introduction

The term ‘integrable equations’ refers to those equations that are either linearizable or solvable by inverse spectral transformation. The prototype of integrable equations is the Korteweg–de Vries (KdV) equation, that was proposed initially to describe shallow water waves. Its solution by the inverse spectral transformation [4] initiated a new line of research in the mathematics and physics community. The existence of solitons that are localized waves that exhibit particle-like properties was the initial focus of the theory. Conservation laws and generalized symmetries were extensively studied. The recursion operator [10] that sends symmetries to symmetries was a key concept in the understanding of hierarchies of integrable equations. The existence of a bi-Hamiltonian formulation and the expression of integrable equations as the compatibility conditions of auxiliary linear operators, i.e. Lax pairs were also primary characterizations of integrability. In this paper, we restrict our analysis to the expression of integrability in terms of generalized symmetries, recursion operators and conservation laws.

Research results on integrable systems are scattered in the mathematics and physics literature, over a spectrum ranging from functional analysis to applied optics. Solitary solutions of nonlinear PDEs facilitates the testing of numerical solvers and helps with stability analysis [15]. There is a relationship between quantum mechanical scattering from a potential and the KdV equation which is applied to reduce the reflection of a wave from an interface between two media of differing refractive indices [16]. Besides shallow water waves in a rectangular channel, the KdV equation also models ion-acoustic waves and magneto-hydrodynamic waves in plasmas, waves in elastic nods, mid-lattitude and equatorial planetary waves. Oceanographers and geophysicists are interested in solitary waves solutions while coastal engineers use cnoidal wave solutions in studies of sediment movement and erosion of sandy beaches [17–20]. Earlier work focused on mathematical properties of the KdV equation. Later on, the search for new integrable equations was an active research area that led to the discovery of two new hierarchies called the Sawada–Kotera [14] and Kaup–Kupershmidt [6] equations. Nevertheless, almost all of the ‘new’ equations were transformable to the KdV, Sawada–Kotera or Kaup–Kupershmidt equations by so-called Miura type transformations, the only exception being the (third order) Krichever–Novikov equation. We recall that the KdV hierarchy starts at order $m = 3$ and has symmetries at all odd orders, while the Sawada–Kotera and Kaup equations belong to hierarchies starting at order $m = 5$ and have symmetries at odd orders that are not multiples of 3. This line of research was marked by the negative results of Wang and Sanders [12]. They proved that polynomial equations of order $m \geqslant 7$ are symmetries of lower order equations and they extended this result to certain types of non-polynomial equations [13]. Recent investigations on integrable systems concentrate on physical and engineering applications of soliton propagation and on the mathematical abstractions of the notion of integrability.

We aimed to obtain a uniqueness result, similar to Wang–Sanders’, for general, non-polynomial integrable equations. We applied the method of formal symmetries, based on the existence of 'conserved densities' [7], to the classification of integrable evolution equations in $1+1$ dimensions [1, 8, 9, 11].

In [1] (2.11−13), we obtained the conserved densities $\rho^{(i)}$, $i=1,\ 2,\ 3$ for evolution equations of order $m \geqslant 7$ and we have shown that evolution equations admitting a conserved density of order $n>m$, are quasi linear. Then, in [8] we showed that evolution equations that admit at least 3 conserved densities of consecutive orders are polynomial in u_{m-1} and u_{m-2} and possess a certain scaling property that we called 'level grading' [9]. For or $m=5$, we obtained conserved densities $\rho^{(i)}$, $i=1,\ldots,5$ [11] $(A.3-7)$; we indicated the existence of a non-quasilinear 5th order equation [1] and gave a preliminary classification of quasilinear 5th order equation with that admit conserved densities of consecutive orders [11].

In subsequent work on the classification problem, we noticed that the existence of the conserved density $\rho^{(3)}$ was the key element that determines the form of integrable equations. In analogy with the fact that the KdV equation has conserved densities of all orders while the hierarchies of Sawada–Kotera and Kaup equations have missing conserved densities, we called those equations that admit an unbroken sequence of conserved densities as 'KdV-type' and those for which $\rho^{(3)}$ is trivial as 'Sawada–Kotera–Kaup-type'.

In the present work, we consider scalar evolution equations in 1 space dimension, $u_t=F(u,u_1,\ldots,u_m)$, $3\leqslant m\leqslant 15$ and we obtain their classifications up to 'Top Levels' and 'Top Dependencies' (to be defined in section 2), using the 'Formal Symmetry' method [7], assuming that the conserved densities $\rho^{(-1)}$ and $\rho^{(1)}$ are non-trivial, i.e. they are not total derivatives and $\rho^{(-1)}$ is not linear in its highest order derivative. We recall that

$$\rho^{(-1)}=\left[\frac{\partial F}{\partial u_m}\right]^{-1/m}$$

and use the notation

$$A=\frac{\partial F}{\partial u_m}=a^m=[\rho^{(-1)}]^{-m}.$$

In all cases, for reasons discussed in remark 2, we assume that

$$\partial\rho^{(-1)}/\partial u_3\neq 0.$$

The results can be summarized as below. If $\rho^{(3)}$ is non-trivial, then integrable equations are polynomial in u_k, $k\geqslant 4$, then $\rho^{(-1)}$ has the form

$$\rho^{(-1)}=(\alpha u_3^2+\beta u_3+\gamma)^{1/2}. \tag{1.1}$$

On the other hand, if $\rho^{(3)}$ is trivial, then integrable equations are non-polynomial only at odd orders that are not multiples of 3; they are polynomial in u_k, $k\geqslant b$, where $b=3$, 4, 5 and $\rho^{(-1)}$ has the form

$$\rho^{(-1)}=(\lambda u_b+\mu)^{1/3},\quad b=3,4,5. \tag{1.2}$$

In all cases, we prove that the equations that we obtain form commuting flows and we construct their recursion operators that are of orders 2 and 6 respectively for the KdV and Sawada–Kotera–Kaup types. Finally we use series of potentiations to convert the non-quasilinear equations to quasilinear ones, omitting lower order dependencies.

In section 2, we recall basic definitions and 'level grading'. In section 3, we describe our solution procedure and present the results in section 4. The construction of the recursion operator and explicit forms of the flows are given in section 5. Discussion of the results and a brief outline of the transformations to known equations are given in section 6.

2. Preliminaries

2.1. Notation and basic definitions

We work with evolution equations in 1 space dimension x and denote the dependent variable by u. The partial derivative of u with respect to time, t, is denoted as u_t, while the kth partial derivative of u with respect to x is denoted by u_k. For functions that depend on the derivatives of u, we reserve subscripts to the partial derivatives with respect to u_k's while we use superscripts in parenthesis as labels. For example, $\phi^{(j)}$ denotes some function of $u, u_1, \ldots u_k, \ldots u_n$, labelled by the index j; $\phi_k^{(j)}$ denotes its partial derivative with respect to u_k. Higher order partial derivatives are subscripts separated by commas, i.e. $\phi_{k,k}^{(j)} = \partial^2\phi^{(j)}/\partial u_k^2$. The total derivative with respect to x is denoted by D, while D^{-1} denotes its formal inverse. We consider evolution equations of arbitrary order m that are of the form $u_t = F$, where F is a C^∞ function of u, u_1, …, u_m.

A 'symmetry' σ, of the evolution equation is a solution of the linearized equation $\sigma_t = F_*\sigma$, where $F_* = \sum_i \partial F/\partial u_i D^i$ is the Frechet derivative of F, i.e.

$$\sigma_t = F_*\sigma = \sum_i \frac{\partial F}{\partial u_i} D^i(\sigma). \tag{2.1}$$

A 'conserved covariant' or a 'co-symmetry' γ, is a solution of $\gamma_t = -F_*^\dagger\gamma$, where $F_*^\dagger$ is the adjoint of F_*, i.e.

$$\gamma_t = -F_*^\dagger\gamma = \sum_i (-1)^{i+1} D^i\left(\frac{\partial F}{\partial u_i}\gamma\right). \tag{2.2}$$

A 'conserved density' ρ satisfies $\rho_t = D\eta$, for some η, that is

$$\rho_t = \sum_i \frac{\partial \rho}{\partial u_i} D^i F = D\eta. \tag{2.3}$$

A conserved density that is a total derivative, i.e. in the image of D is called 'trivial'. Using identities involving variational derivatives, it can be shown that [7], the variational derivative $\frac{\delta}{\delta u}$ of a conserved density is a conserved covariant

$$\gamma = \frac{\delta\rho}{\delta u} = \sum_i (-1)^i \frac{\partial\rho}{\partial u_i}. \tag{2.4}$$

Remark 1. If a conserved co-variant γ is the variational derivative of a conserved density, then γF is a total derivative, because if ρ is a conserved quantity, then

$$\rho_t = \sum \frac{\partial\rho}{\partial u_i} D^i F = \sum (-1)^i D^i\left(\frac{\partial\rho}{\partial u_i}\right) F = \gamma F = D\eta. \tag{2.5}$$

2.2. The recursion operator

A 'recursion operator' is defined as an integro-differential operator that sends symmetries to symmetries, i.e. $R\sigma$ should be a symmetry whenever σ is [10]. In particular, $R\sigma$ has to be a local function. It can be easily seen that if σ and $R\sigma$ are both symmetries, then,

$$(R_t + [R, F_*])\sigma = 0. \tag{2.6}$$

A 'formal symmetry' is a Laurent series that satisfies $R_t + [R, F_*] = 0$ up to a certain order. The solvability of $R_t + [R, F_*] = 0$ in the class of local functions is equivalent to the locality of the time evolution of certain quantities called the 'conserved densities'. The conserved densities are computable in terms of the partial derivatives of F and their conservation is proposed as an integrability test in [7].

In this work we assume that the conserved density $\rho^{(-1)} = (\partial F/\partial u_m)^{-1/m}$ and $\rho^{(1)}$ are non-trivial. Furthermore we assume that $\rho^{(-1)}$ is not linear in its highest order derivative u_b. The triviality or non-triviality of $\rho^{(3)}$ distinguishes between two classes of integrable equations that we called 'KdV-type' or 'Sawada–Kotera–Kaup-type'.

The derivation of the classification results depends heavily on the properties of 'level grading'.

2.3. The 'level grading'

Let K be the ring of differentiable functions of $u, u_1, \ldots u_k$. The module generated by $u_{k+1}, u_{k+2}, \ldots$ has a graded algebra structure. We called this grading as the 'level' of polynomials in u_{k+j} above the base level k.

As can be easily checked, differentiation increases the level by 1. For example, if $\varphi = \varphi(x, t, u, \ldots, u_k)$, then

$$D\varphi = \underbrace{\frac{\partial\varphi}{\partial u_k}u_{k+1}}_{\text{level 1}} + \alpha, \quad D^2\varphi = \underbrace{\frac{\partial\varphi}{\partial u_k}u_{k+2} + \frac{\partial^2\varphi}{\partial u_k^2}u_{k+1}^2}_{\text{level 2}} + \underbrace{\beta u_{k+1}}_{\text{level 1}} + \gamma,$$

where α, β, γ are certain expressions that depend on $u, u_1, \ldots u_k$. The crucial property that makes the level grading a useful tool is its invariance under integration by parts. Let $k < p_1 < p_2 < \cdots < p_l < s-1$. Integration by parts applied to monomials linear in the highest derivative gives either of the forms below.

$$\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s \cong -D\Big(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\Big) u_{s-1},$$

$$\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_{s-1}^{p} u_s \cong -\frac{1}{p+1} D\Big(\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l}\Big) u_{s-1}^{p+1}.$$

It can be seen that the level above k is preserved in both cases. Integrations by parts are repeated until one encounters a non-integrable monomial such as

$$u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s^{p}, \quad p > 1,$$

which has the same level as the original expression.

If P is a polynomial in u_k, $k > b$, with coefficients depending on u_i, $i \leqslant b$, then the monomials in P can be arranged according to their levels above the base level b. The part that has the highest level is called the 'top level' part of P and the dependency of the coefficients of the top level on u_b is called their 'top dependency' [9].

3. Solution procedure

We recall that, the classification problem for polynomial integrable equations was solved by Wang and Sanders [12], by proving that integrable polynomial evolution equations of order $m \geqslant 7$ are symmetries of a lower order equation. With the aim of obtaining a similar result, we undertook a program for classifying non-polynomial evolution equations by the formal symmetry method. The first result in this direction was the quasi-linearity, obtained in [1]. One of the main results of this paper is that integrable equations of order $m \geqslant 7$ admitting higher order conserved densities are quasilinear. We quote this proposition below.

Proposition 1. *Assume that the evolution equation* $u_t = F(u, u_1, \ldots, u_m)$*, with* $m \geqslant 7$*, admits a conserved density of order* $n = m + 1$*. Then F is linear in* u_m*.*

This result is not valid for $m = 5$ and there is in fact a non-quasilinear integrable evolution equation of order 5, as will be discussed in the next section.

Although we have obtained polynomiality in top 3 derivatives in [8], we start here with the quasilinear form and indicate the steps towards the classification of lower order evolution equations.

Since we deal with quasilinear equations, we start with,

$$u_t = Au_m + B, \tag{3.1}$$

where A and B depend on the derivatives of u up to u_{m-1}. As we work with top level terms, we let $F = Au_m$ and we assume only top dependency i.e. we let $A = A(u_{m-1})$ and we parametrize it as $A = a^m$. Then we use the conservation laws for the conserved densities $\rho^{(-1)}$ and $\rho^{(1)}$ to get $\partial A/\partial u_{m-1} = 0$. We finally show that $\partial^3 B/\partial u_{m-1}^3 = 0$, hence evolution equation has the form

$$u_t = Au_m + Bu_{m-1}^2 + Cu_{m-1} + E, \tag{3.2}$$

where the coefficients are functions of the derivatives of u up to order $m - 2$. As before, the conservation laws for $\rho^{(-1)} = a^{-1}$ and $\rho^{(1)}$ [1] (2.11) are used to solve the coefficient functions B and C in terms of the derivatives of a. For $m \geqslant 9$ we obtain $\partial a/\partial u_{m-2} = 0$. For $m = 7$, if $\rho^{(3)}$ is trivial, then a has a non-trivial dependency on u_5, but if $\rho^{(3)}$ is non-trivial, we get $a_5 = 0$. In the case where $a_{m-2} = 0$, the top level part is linear and we show that F is a sum of level homogeneous terms of level 3 above the base level $m - 3$, as given below.

$$u_t = Au_m + Bu_{m-1}u_{m-2} + Cu_{m-1}^3 + Eu_{m-2} + Gu_{m-1}^2 + Hu_{m-1} + K. \tag{3.3}$$

In this case also, for $m \geqslant 9$, we can show that the separant is independent of u_{m-3}, the top level part is linear and equation is of level 4 above base $m - 4$. For $m = 7$, the base level is $m - 3 = 4$. If $\rho^{(3)}$ is non-trivial, then the evolution equation is of level 4 above the base level 3. On the other hand, if $\rho^{(3)}$ is trivial, then we have integrable equations of order 7 with separants depending on u_4 and u_5, as presented in section 5.

F is now a sum of level homogeneous terms of level less than or equal to 4 above the base level $m - 4$. At this stage, it is more convenient to switch to the notation,

$$u_m = u_{b+4}, \quad u_{m-1} = u_{b+3}, \quad u_{m-2} = u_{b+2}, \quad u_{m-3} = u_{b+1}, \quad u_{m-4} = u_b, \tag{3.4}$$

and write u_t as

$$\begin{aligned} u_t = Au_{b+4} &+ Bu_{b+3}u_{b+1} + Cu_{b+2}^2 + Eu_{b+2}u_{b+1}^2 + Gu_{b+1}^4 \\ &+ Hu_{b+3} + Iu_{b+2}u_{b+1} + Ju_{b+1}^3 + Ku_{b+2} + Lu_{b+1}^2 \\ &+ Mu_{b+1} + N. \end{aligned} \tag{3.5}$$

For $m = 7$, the base level is 3 and $\partial A/\partial u_3$ is non-zero regardless of the triviality of $\rho^{(3)}$. The functional form of $A = (\rho^{-1})^{-m}$ depends on the non-triviality or triviality of $\rho^{(3)}$, as given respectively by equations (1.1) and (1.2). For $m \geqslant 9$, the base level is 5; the conserved density conditions imply that $a_5 = 0$ regardless of the triviality of $\rho^{(3)}$. For $m \geqslant 11$, $\rho^{(-1)}$ and $\rho^{(1)}$ only imply that $a_b = 0$, $b = m - 4$. It follows that u_t is a level homogeneous polynomial of level b above the base level $b = m - 5$.

For each order $m = 2k + 1$, $9 \leqslant m \leqslant 15$, and base level $b \leqslant m - 5$, we solve the conserved density conditions in a similar manner, the form of the evolution equation depending on the difference $m - b$ only.

The form of polynomials of level $j = m - b$ are obtained using partitions of the integer j. For low values of $j = m - b$, these forms can be found by inspection, but for higher values of j, the number of partitions grow quickly. We used an algorithm implemented by Errico [3]. The passage from the matrix of the partition of the integer j to the polynomial of level j above the base level b is achieved by a REDUCE program as described in appendix A, where we also present the results for $j \leqslant 4$.

In the case where $\rho^{(3)}$ is non-trivial, one can prove that its explicit form,which is given in [9], is not needed. The generic form of the conserved densities depends on the base level b only. They are of the form

$$\rho^{(-1)} = a^{-1}, \quad \rho^{(1)} = P^{(1)}u_{b+1}^2, \quad \rho^{(3)} = P^{(3)}u_{b+2}^2 + Q^{(3)}u_{b+1}^4, \tag{3.6}$$

where $P^{(i)}$, $Q^{(i)}$ depend on u_b. The computation of the integrability conditions depends on the order $m = 2k + 1$, as follows. As we use only the top dependency, i.e. the dependency on u_b, the time derivatives are

$$\begin{aligned}
\rho_t^{(-1)} &= (a^{-1})_b D^b F, \\
\rho_t^{(1)} &= 2P^{(1)}u_{b+1}D^{b+1}F + P_b^{(1)}u_{b+1}^2 D^b F, \\
\rho_t^{(3)} &= 2P^{(3)}u_{b+2}D^{b+2}F + 4Q^{(3)}u_{b+1}^3 D^{b+1}F + (P_b^{(3)}u_{b+2}^2 + Q_b^{(3)}u_{b+1}^4)D^b F.
\end{aligned} \tag{3.7}$$

These time derivatives are evaluated for each order and each base level using REDUCE programs. For higher orders, we had to use conserved density conditions for a generic expression of $\rho^{(5)}$, as level homogeneous polynomial of level 8 above the base level b.

For each order $7 \leqslant m \leqslant 15$, and each base level $b > 5$, we solved the conserved density conditions with REDUCE and we have seen that in all cases, the conserved density conditions imply that $\partial A/\partial u_b = 0$ and furthermore the flow is polynomial in u_b. For $b \leqslant 5$, the conserved density conditions for $\rho^{(-1)}$ and $\rho^{(1)}$ should be supplemented by the information on whether $\rho^{(3)}$ is trivial or not.

For the case with non-trivial $\rho^{(3)}$ we obtain the dependencies of the flows of order $m \leqslant 15$ on u_k, $k \geqslant 3$, as presented in the next section together with the recursion operator. These equations are polynomial in u_k, $k \geqslant 4$, their dependency on u_3 is given but dependencies on u_k, $k = 2,1,0$ are omitted. This hierarchy is characterized by the form of a given by equation (1.1),

$$a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}. \tag{3.8}$$

For the case where $\rho^{(3)}$ is trivial, we use its explicit expression and set the quantities $P^{(3)}$ and $Q^{(3)}$ to zero and add these to the constraints imposed by the conserved density conditions for $\rho^{(-1)}$ and $\rho^{(1)}$. For $b = 5$ and $b = 4$, the equations with non-constant separant exist only for $\rho^{(3)}$ trivial while for $b = 3$, we have a non-constant separant regardless of the triviality of $\rho^{(3)}$. The flows are polynomial on u_{b+j} and the dependency on u_b is via

$$a = (\lambda u_b + \mu)^{-1/3}, \quad b = 3, 4, 5 \tag{3.9}$$

and its derivatives. The first few flows are presented in the next section together with their recursion operators.

Remark 2. If the base level is 1, that is if the separant or equivalently $\rho^{(-1)}$ depends on u and u_1 only, then one can use transformations of type (1.4.13,14) given in [7] to transform the separant to 1. When $b=2$ and $\rho^{(3)}$ is non-trivial we get a in the form of (3.8) with $b=2$, but when $\rho^{(3)}$ is trivial, we obtain a third order ordinary differential equation for a, which admits (3.9) as a special solution. Since the candidates of integrable equations for $b=2$ form possibly a larger class, we omit this case in the present work and we assume that $b\geqslant 3$.

4. Results

4.1. Order 3

Evolution equations of order 3 are classified in [7]. These fall into 3 classes (3.3.7–9).

$$u_t = A_1 u_3 + A_2, \tag{4.1a}$$

$$u_t = (A_1 u_3 + A_2)^{-2} + A_3, \tag{4.1b}$$

$$u_t = (2A_1 u_3 + A_2)(A_1 u_3^2 + A_2 u_3 + A_3)^{-1/2} + A_4. \tag{4.1c}$$

We note that here, unlike our convention, subscripts refer to indices, not to differentiations. The first equation is quasilinear. The second one is characterized by the linearity of $\rho^{(-1)}$ in u_3, therefore it is excluded from our discussion. The third equation is known as the essentially non-linear third order equation studied further in [5].

When we start with the essentially non-linear equation in the form above, we see that $\rho^{(-1)}$ is

$$(2A_1A_3 - \frac{1}{2}A_2^2)^{-1/3}(A_1u_3^2 + A_2u_3 + A_3)^{1/2}. \tag{4.2}$$

This is not exactly the form that we want. In order to obtain $\rho^{(-1)}$ in the same form as the higher order equations, i.e. in the form (1.1), we should start with the evolution equation

$$u_t = (-\frac{1}{2}\beta^2 + 2\alpha\gamma)^{-1}(\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2}(2\alpha u_3 + \beta) + \delta. \tag{4.3}$$

We now prove that this choice is possible. In [1], proposition 4.4, we have shown that if $\rho^{(-1)} = F_m^{-1/m}$ is conserved and $\rho^{(-1)}_{m,m} \neq 0$, then

$$F_m = [c^{(1)}F^2 + c^{(2)}F + c^{(3)}]^{m/(m-1)}. \tag{4.4}$$

In order to show that the forms of F as given by (4.1c) and (4.3) are both consistent with (4.4), we start with these expressions, substitute them in

$$F_3^2 = [c^{(1)}F^2 + c^{(2)}F + c^{(3)}]^3$$

and solve for the $c^{(i)}$'s by equating to zero, the coefficients of linearly independent functions of u_3. For F given by equation (4.1c) we obtain, $c^{(1)} = -[4(4A_1A_3 - A_2^2)]^{-1/3}$, $c^{(2)} = -2A_4c^{(1)}$, $c^{(3)} = (A_4^2 - 4A_1)c^{(1)}$ while for F given by equation (4.3) we have, $c^{(1)} = \frac{1}{4}(\beta^2 - 4\alpha\gamma)$, $c^{(2)} = -2\delta c^{(1)}$, $c^{(3)} = -\alpha/c^{(1)} - \delta^2 c^{(1)}$. Thus we can start with the form (4.3) that we write as

$$u_t = \frac{4}{P}a^{-2}a_3, \qquad P = \beta^2 - 4\alpha\gamma, \tag{4.5}$$

at the top level and

$$a = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2} \tag{4.6}$$

Thus, the essentially nonlinear third order equation is characterized by

$$\rho^{(-1)} = (\alpha u_b^2 + \beta u_b + \gamma)^{1/2}, \tag{4.7}$$

with $b = 3$.

4.2. Order 5

If $\rho^{(3)}$ is non-trivial, than the top level part of the integrable equations of order 5 is of the form

$$u_t = a^5 u_5 + 5/2 a^4 a_3 u_4^2. \tag{4.8}$$

This equation is characterized by $\rho^{(-1)}$ of the form (4.7) above. We have in fact checked that it is a symmetry of the essentially nonlinear third order equation (4.5), by using the REDUCE program presented in appendix B. In [11], we obtained lower order terms and presented a special solution.

For the case where $\rho^{(3)}$ is trivial we have 3 classes of solutions with base levels $b = 5, 4, 3$. In [1], we have shown that for $m = 5$ there is a candidate of integrable equation of the form

$$u_t = -\frac{3}{2\lambda}(\lambda u_5 + \mu)^{-2/3} + \nu \tag{4.9}$$

where λ, μ and ν are independent of u_5. This form can also be obtained from the triviality of $\rho^{(3)}$, whose explicit expression is given in [11]. This expression involves $\int \rho_t^{(-1)}$, hence we first compute it up to some unknown function depending on at most u_5. Then we substitute this in the expression of $\rho^{(3)}$. The coefficient of u_7^2 gives $\partial^2 a/\partial u_5^2 = 4a_5^2 a^{-1}$, where $a_{(5,5)}$ means the second partial derivative of a with respect to u_5. This equation can be integrated twice to give

$$a = (\lambda u_5 + \mu)^{-1/3}. \tag{4.10}$$

Expressing λ in terms of a_5 we obtain the alternative form

$$u_t = \frac{1}{2}\frac{a^6}{a_5}. \tag{4.11}$$

Continuing with trivial $\rho^{(3)}$ and assuming $\frac{\partial a}{\partial u_5} = 0$, we obtain F simply as

$$u_t = a^5 u_5, \tag{4.12}$$

with

$$a = (\lambda u_4 + \mu)^{-1/3}. \tag{4.13}$$

Finally, again with trivial $\rho^{(3)}$ and $\frac{\partial a}{\partial u_4} = 0$, we obtain

$$u_t = a^5 u_5 + 5a^4 a_3 u_4^2, \tag{4.14}$$

with

$$a = (\lambda u_3 + \mu)^{-1/3}. \tag{4.15}$$

By remark 2, we omit the cases for $b \leqslant 2$.

4.3. The hierarchy structure

We obtain the explicit forms of integrable equations for $m \leqslant 15$ and $b = 3, 4, 5$ using REDUCE interactively as outlined in the appendix B. Furthermore, we also compute flows of order $m = 17$ as a symmetry of 5th order equations. We have explicitly checked that all evolution equations with non-trivial $\rho^{(3)}$ are symmetries of the essentially nonlinear third order equation (4.5) and they form a commuting flow. Similarly we have explicitly checked that equations with trivial $\rho^{(3)}$ over base levels $b = 3, 4, 5$ are symmetries of 5th order equations (4.14), (4.12) and (4.11) respectively and they form commuting flows.

5. Construction of the recursion operator

In [2], we have shown that if the recursion operator has the form

$$R = R^{(n)}D^n + R^{(n-1)}D^{n-1} + \cdots + R^{(1)}D + R^{(0)} + \sum_{i=1}^{N} \sigma_i D^{-1}\gamma_i, \tag{5.1}$$

then σ_i has to be a symmetry and γ_i has to be a conserved covariant. But in general, there is no guarantee that the recursion operator will have a *finite* expansion of this type.

By remark 1, if the conserved covariants γ_i's are chosen as variational derivatives of conserved densities, then it will follow that $R(F)$ will be a local function. The form of the (least) order of the recursion operator can be guessed by considering the orders of the symmetries in the hierarchy and by level grading arguments. Based on the form of the recursion operators for the KdV hierarchy and Sawada–Kotera and Kaup hierarchies [2], we start with recursion operators of the orders 2 and 6 respectively. For the KdV type equations, the recursion operator is proposed as

$$R^{(2)}D^2 + R^{(1)}D + R^{(0)} + \sigma D^{-1}\gamma, \tag{5.2}$$

where σ is proportional to the third order essentially non-linear equation and γ is the variational derivative of $1/a$. For the Sawada–Kotera–Kaup type equations, we start with

$$\begin{aligned} R = {} & a^6 D^6 + R^{(5)}D^5 + R^{(4)}D^4 + R^{(3)}D^3 + R^{(2)}D^2 + R^{(1)}D + R^{(0)} \\ & + \sigma^{(1)}D^{-1}\gamma^{(1)} + \sigma^{(2)}D^{-1}\gamma^{(2)}, \end{aligned} \tag{5.3}$$

where $\sigma^{(i)}$, $i = 1, 2$ are proportional 7th and 5th order flows, $\gamma^{(i)}$, $i = 1, 2$ are the variational derivatives of $\rho^{(-1)}$ and $\rho^{(1)}$ respectively.

We started with the form of the recursion operators as above, where the $R^{(i)}$'s were chosen as level homogeneous polynomials so that the operators R have levels 2 and 6 respectively. We determined the coefficient functions from the requirement that R acting on a symmetry produces the next order flow. In this procedure, for the Sawada–Kotera–Kaup type equations we needed the expression of the flow of order $m = 17$, which was obtained as a symmetry of lower order flows. We present the results below.

5.1. Non-trivial $\rho^{(3)}$

$$R = a^2D^2 + [(-a_3 a)u_4]\, D + [(a_3 a)u_5 + (3a_3^2)u_4^2] + \sigma D^{-1}\eta, \tag{5.4}$$

where

$$\sigma = \frac{4}{P} a_3 a^{-2}, \quad \eta = -D^3\left[\frac{\partial a^{-1}}{\partial u_3}\right], \quad P = \beta^2 - 4\alpha\gamma. \tag{5.5}$$

The first 4 flows for non-trivial $\rho^{(3)}$ and $b = 3$ are given by

$$\begin{aligned}
u_{t,3} &= \frac{4}{P}a^{-2}a_3, \\
u_{t,5} &= a^5u_5 + 5/2a^4a_3u_4^2, \\
u_{t,7} &= a^7u_7 + 14a^6a_3u_6u_4 + 21/2a^6a_3u_5^2 + a^5(98a_3^2 + 35/8Pa^6)u_5u_4^2 \\
&\quad + a^4a_3(189/2a_3^2 + 399/32Pa^6)u_4^4 \\
u_{t,9} &= a^9u_9 + 27a^8a_3u_8u_4 + 57a^8a_3u_7u_5 + 69/2a^8a_3u_6^2 \\
&\quad + a^7(360a_3^2 + 105/8Pa^6)u_7u_4^2 + a^7(1230a_3^2 + 189/4Pa^6)u_6u_5u_4 \\
&\quad + a^7(290a_3^2 + 91/8Pa^6)u_5^3 + 330a^6a_3(9a_3^2 + Pa^6)u_6u_4^3 \\
&\quad + a^6a_3(6105a_3^2 + 11\,187/16Pa^6)u_5^2u_4^2 \\
&\quad + a^5(16\,335a_3^4 + 29\,469/8Pa^6a_3^2 + 6699/128P^2a^{12})u_5u_4^4 \\
&\quad + a^4a_3(19\,305/2a_3^4 + 57\,915/16Pa^6a_3^2 + 39\,325/256P^2a^{12})u_4^6.
\end{aligned}$$

5.2. Trivial $\rho^{(3)}$

The recursion operator is chosen in the form (5.3), and the coefficients are solved in each case. If we write R by factoring our a^6, the recursion operators for $b = 3, 4, 5$ have the same functional form, up to constants.

$$\begin{aligned}
R = {} & a^6(D^6 + \tilde{R}^{(5)}D^5 + \tilde{R}^{(4)}D^4 + \tilde{R}^{(3)}D^3 + \tilde{R}^{(2)}D^2 + \tilde{R}^{(1)}D + \tilde{R}^{(0)}) \\
& + \sigma^{(1)}D^{-1}\gamma^{(1)} + \sigma^{(2)}D^{-1}\gamma^{(2)}.
\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
\tilde{R}^{(5)} &= k^{(5,1)}qu_{b+1}, \\
\tilde{R}^{(4)} &= k^{(4,1)}qu_{b+2} + k^{(4,2)}q^2u_{b+1}, \\
\tilde{R}^{(3)} &= k^{(3,1)}qu_{b+3} + k^{(3,2)}q^2u_{b+2}u_{b+1} + k^{(3,3)}q^3u_{b+1}^3, \\
\tilde{R}^{(2)} &= k^{(2,1)}qu_{b+4} + k^{(2,2)}q^2u_{b+3}u_{b+1} + k^{(2,3)}q^2u_{b+2}^2 \\
&\quad + k^{(2,4)}q^3u_{b+2}u_{b+1}^2 + k^{(2,5)}q^4u_{b+1}^4, \\
\tilde{R}^{(1)} &= k^{(1,1)}q\ u_{b+5} + k^{(1,2)}q^2u_{b+4}u_{b+1} + k^{(1,3)}q^2\ u_{b+3}u_{b+2} \\
&\quad + k^{(1,4)}q^3u_{b+3}u_{b+1}^2 + k^{(1,5)}q^3\ u_{b+2}^2u_{b+1} \\
&\quad + k^{(1,6)}q^4\ u_{b+2}u_{b+1}^3 + k^{(1,7)}q^5\ u_{b+1}^5, \\
\tilde{R}^{(0)} &= k^{(0,1)}qu_{b+6} + k^{(0,2)}q^2u_{b+5}u_{b+1} + k^{(0,3)}q^2u_{b+4}u_{b+2} \\
&\quad + k^{(0,4)}q^2u_{b+3}^2 + k^{(0,5)}q^3u_{b+4}u_{b+1}^2 + k^{(0,6)}q^3u_{b+3}u_{b+2}u_{b+1} \\
&\quad + k^{(0,7)}q^3u_{b+2}^3 + k^{(0,8)}q^4u_{b+3}u_{b+1}^3 + k^{(0,9)}q^4u_{b+2}^2u_{b+1}^2 \\
&\quad + k^{(0,10)}q^5u_{b+2}u_{b+1}^4 + k^{(0,11)}q^6u_{b+1}^6,
\end{aligned}$$

where the $k^{(i,j)}$'s are constants and $q = a_b/a$.

We present the explicit forms for each base level together with the first 4 flows.

5.3. Trivial $\rho^{(3)}$, $b = 5$

$$
\begin{aligned}
R^{(5)} &= 3a^5a_5u_6,\\
R^{(4)} &= 2u_7a_5a^5 + 7u_6^2a_5^2a^4,\\
R^{(3)} &= -u_8a_5a^5 - 16u_7u_6a_5^2a^4 - 42u_6^3a_5^3a^3,\\
R^{(2)} &= u_9a_5a^5 + 21u_8u_6a_5^2a^4 + 16u_7^2a_5^2a^4 + 262u_7u_6^2a_5^3a^3 + 490u_6^4a_5^4a^2,\\
R^{(1)} &= -u_{10}a_5a^5 - 28u_9u_6a_5^2a^4 - 51u_8u_7a_5^2a^4 - 462u_8u_6^2a_5^3a^3 - 660u_7^2u_6a_5^3a^3\\
&\quad - 5190u_7u_6^3a_5^4a^2 - 7560u_6^5a_5^5a,\\
R^{(0)} &= u_{11}a_5a^5 + 35u_{10}u_6a_5^2a^4 + 79u_9u_7a_5^2a^4 + 742u_9u_6^2a_5^3a^3 + 49u_8^2a_5^2a^4\\
&\quad + 2700u_8u_7u_6a_5^3a^3 + 11\,060u_8u_6^3a_5^4a^2 + 660u_7^3a_5^3a^3 + 23\,790u_7^2u_6^2a_5^4a^2\\
&\quad + 119\,040u_7u_6^4a_5^5a + 141\,400u_6^6a_5^6,
\end{aligned}
$$

$$
\begin{aligned}
\sigma^{(1)} &= -u_{t,7} = -a^7u_7 - \frac{7}{2}a^6a_5u_6^2,\\
\gamma^{(1)} &= \frac{\delta\rho^{(-1)}}{\delta u} = -D^5\frac{\partial a^{-1}}{\partial u_5},\\
\sigma^{(2)} &= -u_{t,5} = -\frac{1}{2}\frac{a^2}{a_5},\\
\gamma^{(2)} &= \frac{\delta\rho^{(1)}}{\delta u} = D^6\frac{\partial\rho^{(1)}}{\partial u_6} - D^5\frac{\partial\rho^{(1)}}{\partial u_5}, \quad \rho^{(1)} = \left(\frac{1}{a}a_5^2u_6^2\right).
\end{aligned}
$$

$$
\begin{aligned}
u_{t,5} &= \frac{1}{2}\frac{a^6}{a_5},\\
u_{t,7} &= a^7u_7 + \frac{7}{2}a^6a_5u_6^2,\\
u_{t,11} &= a^{11}u_{11} + 33a^{10}a_5u_{10}u_6 + 77a^{10}a_5u_9u_7 + 99/2a^{10}a_5u_8^2 + 682a^9a_5^2u_9u_6^2\\
&\quad + 2574a^9a_5^2u_8u_7u_6 + 1892/3a^9a_5^2u_7^3 + 10098a^8a_5^3u_8u_6^3 + 22066a^8a_5^3u_7^2u_6^2\\
&\quad + 107525a^7a_5^4u_7u_6^4 + 752675/6a^6a_5^5u_6^6
\end{aligned}
$$

5.4. Trivial $\rho^{(3)}$, base $b = 4$

$$
\begin{aligned}
R^{(5)} &= 9a^5a_4u_5,\\
R^{(4)} &= 5a^5a_4u_6 + 34a^4a_4^2u_5^2,\\
R^{(3)} &= a^5a_4u_7 + 16a^4a_4^2u_6u_5 + 42a^3a_4^3u_5^3,\\
R^{(2)} &= -4a^4a_4^2u_7u_5 - 56a^3a_4^3u_6u_5^2 - 140a^2a_4^4u_5^4,\\
R^{(1)} &= 2a^4a_4^2u_8 + 2a^4a_4^2u_7u_6 + 52a^3a_4^3u_7u_5^2\\
&\quad + 56a^3a_4^3u_6^2u_5 + 700a^2a_4^4u_6u_5^+1260aa_4^5u_5^5,\\
R^{(0)} &= -2a^4a_4^2u_9u_5 - 2a^4a_4^2u_7^2 - 56a^3a_4^3u_8u_5^2 - 156a^3a_4^3u_7u_6u_5 - 1060a^2a_4^4u_7u_5^3\\
&\quad - 1680a^2a_4^4u_6^2u_5^2 - 12180aa_4^5u_6u_5^4 - 17360a_4^6u_5^6
\end{aligned}
$$

$$\sigma^{(1)} = -u_{t,7} = -(a^7u_7 + 14a^6a_4u_6u_5 + 35a^5a_4^2u_5^3),$$
$$\gamma^{(1)} = \frac{\delta\rho^{(-1)}}{\delta u} = D^4\frac{\partial a^{-1}}{\partial u_4},$$
$$\sigma^{(2)} = -u_{t,5} = -a^5u_5,$$
$$\gamma^{(2)} = \frac{\delta\rho^{(1)}}{\delta u} = -D^5\frac{\partial\rho^{(1)}}{\partial u_5} + D^4\frac{\partial\rho^{(1)}}{\partial u_4}, \quad \rho^{(1)} = \left(\frac{1}{a}a_4^2u_5^2\right).$$

$$\begin{aligned}
u_{t,5} &= a^5u_5,\\
u_{t,7} &= a^7u_7 + 14a^6a_4u_6u_5 + 35a^5a_4^2u_5^3,\\
u_{t,11} &= a^{11}u_{11} + 44a^{10}a_4u_{10}u_5 + 110a^{10}a_4u_9u_6 + 176a^{10}a_4u_8u_7 + 1144a^9a_4^2u_9u_5^2\\
&\quad + 5016a^9a_4^2u_8u_6u_5 + 3267a^9a_4^2u_7^2u_5 + 4466a^9a_4^2u_7u_6^2 + 21\,692a^8a_4^3u_8u_5^3\\
&\quad + 118\,184a^8a_4^3u_7u_6u_5^2 + 164\,560/3a^8a_4^3u_6^3u_5 + 309\,485a^7a_4^4u_7u_5^4\\
&\quad + 871\,420a^7a_4^4u_6^2u_5^3 + 3225\,750a^6a_4^5u_6u_5^5 + 9784\,775/3a^5a_4^6u_5^7
\end{aligned}$$

5.5. *Trivial $\rho^{(3)}$, base $b = 3$*

$$\begin{aligned}
R^{(5)} &= 15a^5a_3u_4,\\
R^{(4)} &= 14a^5a_3u_5 + 115a^4a_3^2u_4^2,\\
R^{(3)} &= 6 * a^5a_3u_6 + 129a^4a_3^2u_5u_4 + 450a^3a_3^3u_4^3,\\
R^{(2)} &= a^5a_3u_7 + 21a^4a_3^2u_6u_4 + 16a^4a_3^2u_5^2 + 262a^3a_3^3u_5u_4^2 + 490a^2a_3^4u_4^4,\\
R^{(1)} &= -2a^4a_3^2u_7u_4 - 2a^4a_3^2u_6u_5 - 52a^3a_3^3u_6u_4^2 - 56a^3a_3^3u_5^2u_4 - 700a^2a_3^4u_5u_4^3 - 1260aa_3^5u_4^5\\
R^{(0)} &= 4a^4a_3^2u_7u_5 + 20a^3a_3^3u_7u_4^2 + 84a^3a_3^3u_6u_5u_4 + 420a^2a_3^4u_6u_4^3 + 56a^3a_3^3u_5^3\\
&\quad +1260a^2a_3^4u_5^2u_4^2 + 6720aa_3^5u_5u_4^4 + 9100a_3^6u_4^6,
\end{aligned}$$

$$\sigma^{(1)} = -u_{t,7} = -(a^7u_7 + 21a^6a_3u_6u_4 + 14a^6a_3u_5^2 + 245a^5a_3^2u_4^2u_5 + 455a^4a_3^3u_4^4),$$
$$\gamma^{(1)} = \frac{\delta\rho^{(-1)}}{\delta u} = -D^3\frac{\partial a^{-1}}{\partial u_3},$$
$$\sigma^{(2)} = -u_{t,5} = -(a^5u_5 + 5a^4a_3u_4^2),$$
$$\gamma^{(2)} = \frac{\delta\rho^{(1)}}{\delta u} = D^4\frac{\partial\rho^{(1)}}{\partial u_4} - D^3\frac{\partial\rho^{(1)}}{\partial u_3}, \quad \rho^{(1)} = \left(\frac{1}{a}a_3^2u_4^2\right).$$

$$\begin{aligned}
u_{t,5} &= a^5u_5 + 5a^4a_3u_4^2,\\
u_{t,7} &= a^7u_7 + 21a^6a_3u_6u_4 + 14a^6a_3u_5^2 + 245a^5a_3^2u_4^2u_5 + 455a^4a_3^3u_4^4,\\
u_{t,11} &= a^{11}u_{11} + 55a^{10}a_3u_{10}u_4 + 154a^{10}a_3u_9u_5 + 286a^{10}a_3u_8u_6 + 176a^{10}a_3u_7^2\\
&\quad + 1760a^9a_3^2u_9u_4^2 + 8844a^9a_3^2u_8u_5u_4 + 14\,014a^9a_3^2u_7u_6u_4 + 9482a^9a_3^2u_7u_5^2\\
&\quad + 12\,199a^9a_3^2u_6^2u_5 + 41\,140a^8a_3^3u_8u_4^3 + 268\,532a^8a_3^3u_7u_5u_4^2 + 173\,723a^8a_3^3u_6^2u_4^2\\
&\quad + 476\,850a^8a_3^3u_6u_5^2u_4 + 164\,560/3a^8a_3^3u_5^4 + 743\,325a^7a_3^4u_7u_4^4 + 5344\,460a^7a_3^4u_6u_5u_4^3\\
&\quad + 11\,133\,980/3a^7a_3^4u_5^3u_4^2 + 10\,343\,905a^6a_3^5u_6u_4^5 + 36\,171\,410a^6a_3^5u_5^2u_4^4\\
&\quad + 320\,101\,925/3a^5a_3^6u_5u_4^6 + 283\,758\,475/3a^4a_3^7u_48
\end{aligned}$$

6. Results and discussion

We obtained the 'top level' parts of seemingly new hierarchies up to their dependencies on the 'top order' derivative that is present in the separant. The dependencies on u, u_1, …, u_{b-1} could not be solved completely despite numerous attempts to attack this problem suggesting that one should use transformations to eliminate some arbitrary functions. Although the equations that we found are technically new, there are strong indications that they could be transformable to known equations. We indicate below the steps to map the top level parts to polynomial equations by a sequence of potentiations, generalized contact transformations and point transformations. The applicability of these transformation to lower levels is of course a serious problem that is not considered here. We recall that a potentiation is a special type of Miura map, defined by $v = u_1$. Then, if $u_t = F$, then $v_t = DF$. If F has no dependency on u, DF will be a local function of v and its derivatives. If F has a non-polynomial dependency on u_j, $j \leqslant b$, then v_t will be polynomial in $v_b = u_{b+1}$, hence, in the level grading terminology, the base level will decrease. This potentiation procedure can be continued until the base level is $b = 0$, that means the evolution equation is polynomial in u_j, $j > 0$ but has non-polynomial dependency on u only. This procedure was applied to non-quasilinear equations of orders 3 and 5 and corresponding equations in our list were obtained.

The next step in the reduction procedure is to apply the generalized contact transformation given by [7],

$$\mathrm{d}x' = \rho \mathrm{d}x + \sigma \mathrm{d}t, \quad u' = u, \quad t' = t, \quad \rho_t = D\sigma, \quad \rho \notin \mathrm{Im}(D),$$

that implies

$$u'_{t'} = u_t - (\sigma/\rho)\, u_1, \qquad u'_{k'} = (\rho^{-1}D)^k\, u.$$

If there is a non-trivial conserved density depending on u and u_1, this generalized contact transformation maps u_t to an equation where the separant is equal to 1. In our case, since we assumed that the conserved density $\rho^{(-1)}$ is non-trivial, we applied this transformation with $\rho^{(-1)}$ depending on u only to set the separant equal to 1 but at this stage we still had non-polynomial dependencies.

The final step in the sequence of transformations is a point transformation $\tilde{u} = \phi(u)$, aiming to eliminate the dependency on u_{m-1}. Since our evolutions are now of the form

$$u_t = u_m + B(u)u_1u_{m-1} + \dots,$$

it can be seen that

$$\tilde{u}_t - \tilde{u}_m = \phi_u(u_m + B(u)u_1u_{m-1} + \dots) - (\phi_u u_m + m\phi_{u,u}u_1u_{m-1} + \dots).$$

Thus, u_{m-1} dependency is eliminated by choosing

$$m\phi_{u,u} = B(u)\phi_u.$$

The existence of local solutions to this equation depends on the triviality of the canonical density $\rho^{(0)} = \frac{\partial F / \partial u_{m-1}}{\partial F / \partial u_m}$, that holds in our case. This transformation eliminates u_{m-1} and in our case, it reduces all equations to polynomial equations that are in fact independent of u also.

Based on these top level-top order classification and transformation results, we conjecture that all scalar evolution equations in 1 space dimension, integrable in the sense of admitting a formal symmetry, are symmetries of a polynomial equation of order 3 or 5.

Appendix A. Level homogeneous polynomials

A partition of the integer n given by the ith row of the matrix P_n corresponds to a monomial of level n above a base b as follows. Each non-zero value k_j at the (i, j) entry of the matrix corresponds to a term $u_{b+j}^{k_j}$. Since the sum of the values in the ith row is n, the product of the corresponding terms is a monomial of level n above the base b. As an example, we have the following correspondences for $n = 4$.

$$\begin{aligned}
1+1+1+1 &: u_{b+1}^4, \\
1+1+2 &: u_{b+1}^2 u_{b+2}, \\
2+2 &: u_{b+2}^2, \\
1+3 &: u_{b+1} u_{b+3}, \\
4 &: u_{b+4}.
\end{aligned}$$

We transfer the matrices P_n above the level homogeneous polynomials in REDUCE format as follows (i.e. $n = 4$).

```
P4 :=mat(
(4,     0,     0,     0),
(2,     1,     0,     0),
(0,     2,     0,     0),
(1,     0,     1,     0),
(0,     0,     0,     1));
t: = P4;ncol: = 4;nrow: = 5; % t = matrix of partions in REDUCE format
ubb:=mat((ubp1,ubp2,ubp3,ubp4)); % derivatives of order b + 1,b + 2,b + 3 b + 4
kat:=tp(mat((k01,k02,k03,k04,k05))); % coefficinets of the polynomial
term:=0;for i: = 1 step 1 until nrow do
≪ terma:=1;for j: = 1 step 1 until ncol do ≪terma:=terma*ubb(1,j)**t(i,j)≫;
term:=term + kat(i,1)*terma ≫ ; % result is below
utbmm04:=
k01*ubp1**4 + k02*ubp1**2*ubp2 + k03*ubp2**2 + k04*ubp1*ubp3 + k05*ubp4;
```

Some of the general form of the top level parts are given below as an example.

$$\begin{aligned}
m-b &= 1 : \{u_{b+1}\} \\
m-b &= 2 : \{u_{b+1}^2, u_{b+2}\} \\
m-b &= 3 : \{u_{b+1}^3, u_{b+1}u_{b+2}, u_{b+3}\} \\
m-b &= 4 : \{u_{b+1}^4, u_{b+1}^2 u_{b+2}, u_{b+2}^2, u_{b+1}u_{b+3}, u_{b+4}\}
\end{aligned}$$

Appendix B. Sample REDUCE program the commutativity of the flows

We show that: (a) the equations $u_{t,5}, u_{t,7}, u_{t,9}, u_{t,11}, u_{t,13}, u_{t,15}$ form a commuting flow with $u_{t,3}$. (b)the equations $u_{t,7}, u_{t,9}, u_{t,11}, u_{t,13}, u_{t,15}$ form a commuting flow with $u_{t,5}$. (c)the equations $u_{t,9}, u_{t,11}, u_{t,13}$ and $u_{t,15}$ form a commuting flow with $u_{t,7}$. (d)the equations $u_{t,11}, u_{t,13}$ and $u_{t,15}$ form a commuting flow with $u_{t,9}$. (e)the equations $u_{t,13}, u_{t,15}$ form a commuting flow with $u_{t,11}$. (f)the equations $u_{t,15}$ form a commuting flow with $u_{t,13}$. Below we give the program for part (a) for equations $u_{t,5}, u_{t,7}, u_{t,9}$ only.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Show that the flow is commuting
% Show that the equations ut5, ut7,ut9, are symmetries of ut3
% We need k times the total derivative of ut3
ff:=ut3$
ffx1:=tdf(ff)$
ffx2:=tdf(ffx1)$
...
ffx15:=tdf(ffx14)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sigma:=ut5$
denk: = df(sigma,u5)*ffx5 + df(sigma,u4)*ffx4 + df(sigma,u3)*ffx3
        -df(ff,u3)*tdf(tdf(tdf(sigma)));pause;
sigma: = ut7$
denk:=df(sigma,u7)*ffx7+df(sigma,u6)*ffx6
      +df(sigma,u5)*ffx5+df(sigma,u4)*ffx4+df(sigma,u3)*ffx3
      -df(ff,u3)*tdf(tdf(tdf(sigma)));pause;
sigma:=ut9$
denk: = df(sigma,u9)*ffx9 + df(sigma,u8)*ffx8
        + df(sigma,u7)*ffx7 + df(sigma,u6)*ffx6
        + df(sigma,u5)*ffx5 + df(sigma,u4)*ffx4 + df(sigma,u3)*ffx3
        -df(ff,u3)*tdf(tdf(tdf(sigma)));pause;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

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