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A max–min model of random variables in bivariate random sequences

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ABSTRACT

We introduce a max–min model to bivariate random sequences and applying bivariate binomial distribution in fourfold scheme derive the distributions of associated order statistics in a new model. Some examples for special cases are presented and applications of the results in reliability analysis and actuarial sciences are discussed.

1. Introduction

Let (X, Y) be a bivariate random vector given in probability space $\{\Omega, \mathcal{F}, P\}$ having joint distribution function $F(x, y) = P\{X \leq x, Y \leq y\}$ and marginal distribution functions $F_X(x)$ and $F_Y(y)$ of X and Y, respectively. Denote the joint survival function of X and Y by $\bar{F}(x, y)$, and the marginal survival functions by $\bar{F}_X(x)$ and $\bar{F}_Y(y)$, respectively. Let B be an event from $\mathcal{F}$, and B^c be the complement of B. Define a new random variable W as follows:

$$W(\omega) = \begin{cases} \max(X, Y), & \omega \in B \\ \min(X, Y), & \omega \in B^c \end{cases} \tag{1}$$

The random variable $W(\omega)$ can also be written as $W(\omega) = I_B(\omega)\max(X, Y) + I_B(\omega)\min(X, Y)$, where $I_B(\omega) = 1$ if $\omega \in B$ and $I_B(\omega) = 0$ if $\omega \in B^c$, is an indicator function of event B.

The motivation for studying the random variable $W(\omega)$ emerges from some models of reliability engineering and bivariate insurance claims in actuarial sciences.

In reliability engineering we often encounter systems with two subcomponents per component. Assume that the system may consist of two types of components: type I and type II components. Each type I component has parallel connected subcomponents and each type II component has series connected subcomponents. In other words, type I component is intact if at least one of the components is functioning, and type II component is intact if both of the components are working.

For example, if the lifetime of the subcomponents of the system are both less than given t, then we connect them with parallel structure, if not, with series structure. A practical example may be an electrical system of n components each consisting of two lamps (bulb, ampule, knocker) (subcomponents) of different quality. Assume that the lifetimes of some lumps are detected as to be less than t (for example $t = 2$ months) and the lifetime of others are greater than t. Then we connect the components with parallel or series structure depending on the quality of subcomponents (lamps).

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Therefore, the lifetime of the component will be modeled with the random variable W_t which is the maximum of lifetimes of lumps if both lifetimes are less than t, and minimum of lifetimes of subcomponents if at least one of the lifetimes is greater than t. To formalize this model mathematically we consider a probability space $\{\Omega, \mathcal{F}, P\}$ and the lifetimes of the subcomponents will be random variables defined in this probability space. The random variable W_t is actually a model for the lifetime of a system consisting of two dependent components with lifetimes X and Y. If event B occurs, then the components are connected with parallel structure, if B^c occurs then they are connected with series structure. It is not difficult to imagine that event B in general is connected with the random variables X and Y. For example, it may be $B = \{\omega : X(\omega) < Y(\omega)\}$ or $B = \{\omega : X(\omega) \leq t, Y(\omega) \leq t\}, t \geq 0$.

In another example, we may consider an insurance portfolio in which the main interest is the investigation of the random variable which represents the losses based on two types of claims. Let (X, Y) be a bivariate random vector of losses corresponding to two types of claims. This problem can also be modeled with random variable $W(\omega)$ defined by (1).

This paper investigates the distribution of order statistics $W_{r:n}, r = 1, 2, \ldots, n$ constructed from dependent random variables $W_1, W_2, \ldots, W_n$ in a max–min model. For evaluating the distribution of $W_{r:n}$ we use an approach to reduce the joint probabilities to fourfold scheme and bivariate binomial distribution. The paper is organized as follows. We consider the bivariate random sequence $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ and the random variables $W_i(\omega), i = 1, 2, \ldots, n$ defined as (1) and study the distribution of order statistics of $W_1, W_2, \ldots, W_n$ under condition that there are a total of m ($m \leq n$) occurrences of B. The results are applied to reliability analysis of coherent systems consisting of components each having two dependent subcomponents and to insurance models where the losses correspond to two types of claims. In Section 3 we provide some simple particular examples of random variable W, to understand the structure of the model and study the distribution of W for some special events B and different underlying bivariate distributions.

This model can be represented in more general form considering any random variables $\xi_1(\omega)$ and $\xi_2(\omega)$ defined in the same probability space instead of $\min(X, Y)$ and $\max(X, Y)$ as it is mentioned in Remark 2 of this paper.

2. A general model and order statistics

In this section we consider a model of the random variable (1) and derive the distribution of order statistics constructed from the sample of dependent random variables in this model using bivariate binomial distribution.

2.1. Auxiliary material. The bivariate binomial distribution

To derive the main result we need the short description of bivariate binomial model. The bivariate binomial model was first introduced in [1] and it assumes that in conducted experiment event A may occur either with B or B^c and also B may occur either with A or A^c. The corresponding probabilities are $p_{11} = P(AB)$, $p_{12} = P(AB^c)$, $p_{21} = P(A^cB)$ and $p_{22} = P(A^cB^c)$. Let ζ_1 and ζ_2 be the number of occurrences of A and B in n times repeating of the experiment, respectively. The fourfold scheme is:

$A \setminus B$	B	B^c
A	AB	AB^c
A^c	A^cB	A^cB^c

Then

$$P\{\zeta_1 = i, \zeta_2 = k\} = \sum_{j=\max(0,i+k-n)}^{\min(i,k)} \frac{n!}{j!(i-j)!(k-j)!(n-i-k+j)!} p_{11}^{j} p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-i-k+j} \tag{2}$$

This distribution introduced first by Aitken and Gonin [1] and its properties have been studied in [2–4]. Some modifications are considered in [5,6].

2.2. The distributions of order statistics

Let (X, Y) be a bivariate random vector given in probability space $\{\Omega, \mathcal{F}, P\}$ having a joint distribution function $F(x, y) = P\{X \leq x, Y \leq y\}$, where $F_X(x)$ and $F_Y(y)$ denote the marginal distribution functions of X and Y, respectively. Let B be any event in $\mathcal{F}$ and let B^c be a complement of B. Define a new random variable W as follows:

$$W(\omega) = \begin{cases} \max(X, Y), & \omega \in B \\ \min(X, Y), & \omega \in B^c. \end{cases}$$

Consider events $A = \{W \le x\}$ and B in fourfold bivariate binomial model. From the definition of W it can be easily observed that

$$\begin{aligned}
p_{11} &= P(AB) = P\{W \le x, B\} = P\{\max(X, Y) \le x, B\} \\
p_{12} &= P(AB^c) = P\{W \le x, B^c\} = P\{\min(X, Y\} \le x, B^c\} \\
p_{21} &= P(A^cB) = P\{W > x, B\} = P\{\max(X, Y) > x, B\} \\
p_{22} &= P(A^cB^c) = P\{W > x, B^c\} = P\{\min(X, Y) > x, B^c\} \qquad (3)
\end{aligned}$$

Equalities (3) hold, because if B occurs then $W = \max(X, Y)$ and if B^c occurs then $W = \min(X, Y)$.

Assume now that

$$W_i = \begin{cases} \max(X_i, Y_i), & \omega \in B \\ \min(X_i, Y_i), & \omega \in B^c \end{cases}, i = 1, 2, \ldots, n$$

and

$$\xi_i = \begin{cases} 1, & \omega \in B \\ 0, & \omega \in B^c \end{cases}, i = 1, 2, \ldots, n,$$

$$\eta_i = \begin{cases} 1, & \omega \in A \\ 0, & \omega \in A^c \end{cases}, i = 1, 2, \ldots, n.$$

i.e. $\xi_i = 1$ ($\eta_i = 1$) if event B (A) occurs in ith trial and $\xi_i = 0$ ($\eta_i = 0$) if event B^c (A^c) occurs in ith trial. Let $\zeta_2 = \sum_{i=1}^n \xi_i$ and $\zeta_1 = \sum_{i=1}^n \eta_i$ be the number of occurrences of events B and A, in n times repeating of the experiment, respectively. It is important to note that the random variables $W_1, W_2, \ldots, W_n$ are dependent. Let $W_{1:n} \le W_{2:n} \le \cdots \le W_{n:n}$ be the order statistics of $W_1, W_2, \ldots, W_n$. (For order statistics see [7]). Theorem 1 finds the distribution of order statistic $W_{r:n}$.

Theorem 1. *If $W_{1:n}, W_{2:n}, \ldots, W_{n:n}$ are order statistics of $W_1, W_2, \ldots, W_n$ then*

$$\begin{aligned}
&P\{W_{r:n} \le x \mid \zeta_2 = k\} \\
&= \frac{1}{\binom{n}{k}(P(B))^k(1 - P(B))^{n-k}} \sum_{i=r}^{n} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} \frac{n!}{j!(i-j)!(k-j)!(n-i-k+j)!} \\
&\times p_{11}^j p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-i-k+j} \qquad (4)
\end{aligned}$$

and the distribution of order statistic $W_{r:n}$, $1 \le r \le n$ is

$$\begin{aligned}
&P\{W_{r:n} \le x\} \\
&\quad = \sum_{k=0}^{n} \sum_{i=r}^{n} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j} p_{11}^j p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-i-k+j}, \qquad (5)
\end{aligned}$$

where $p_{11}, p_{12}, p_{21}, p_{22}$ are as in (3).

Proof. Follows from obvious interpretation of fourfold model and bivariate binomial distribution (2) for events $B \in \mathcal{F}$ and $A = \{\omega : W(\omega) \le x\} \in \mathcal{F}$. We have

$$\begin{aligned}
&P\{W_{r:n} \le x, \sum_{i=1}^{n} \xi_i = k\} \\
&\quad = \sum_{i=r}^{n} P\{\text{exactly } i \text{ of } W_1, W_2, \ldots., W_n \text{ are less than} \\
&\quad \text{or equal to } x \text{ and event } B \text{ occurs } k \text{ times }\} \\
&\quad = \sum_{i=r}^{n} P\{\zeta_1 = i, \zeta_2 = k\} \\
&\quad = \sum_{i=r}^{n} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j} p_{11}^j p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-i-k+j}.
\end{aligned}$$

Therefore, the conditional distribution of $W_{r:n}$ given $\zeta_2 = k$ is as in (4) and the distribution of $W_{r:n}$ is as in (5). ∎

2.2.1. Special case 1

Let $B=\{X\le t, Y\le t\}, t>0$. Consider first a special case $r=n$. Then from (4) we have

$$P\{W_{n:n}\le x \mid \sum_{i=1}^{n}\xi_i=k\}$$
$$=\frac{p_{11}^{k}p_{12}^{n-k}}{(P(B))^k(1-P(B))^{n-k}}=\frac{(P(AB))^k\,(P(AB^c))^{n-k}}{(P(B))^k(1-P(B))^{n-k}}.$$

Then

$$\begin{aligned}P(AB)&=P\{X\le x, Y\le y, X\le t, Y\le t\}\\&=P\{X\le \min(x,t), Y\le \min(x,t)\}\\&=F(\min(x,t),\min(x,t))\end{aligned}\tag{6}$$

and

$$\begin{aligned}P(AB^c)&=P\{\min(X,Y)\le x, B^c\}\\&=P(B^c)-P\{\min\{X,Y\}>x, B^c\}\\&=1-P(B)-[P\{\min(X,Y)>x\}-P\{\min(X,Y)>x,B\}]\\&=1-F(t,t)-\bar{F}(x,x)+P\{x<X\le t, x<Y\le t\}.\end{aligned}\tag{7}$$

Therefore,

$$P\{W_{n:n}\le x \mid \sum_{i=1}^{n}\xi_i=k\}$$
$$=\frac{(F(\min(x,t),\min(x,t)))^k\,(1-F(t,t)-\bar{F}(x,x)+P\{x<X\le t,x<Y\le t\})^{n-k}}{(F(t,t))^k(1-F(t,t))^{n-k}}$$

Hence, taking into account that $\bar{F}(x,x)=1-F_X(x)-F_Y(x)+F(x,x)$ and $P\{x<X<t, x<Y\le t\}=0$, if $x>t$, $P\{x<X\le t, x<Y\le t\}=F(x,x)-F(x,t)-F(t,x)+F(t,t)$, if $x\le t$, we have

$$P\{\max(W_1,W_2,\ldots,W_n)\le x \mid \sum_{i=1}^{n}\xi_i=k\}$$
$$=\frac{F^k(\min(t,x),\min(t,x))}{F^k(t,t)(1-F(t,t))^{n-k}}(F_X(x)+F_Y(x)$$
$$-F(t,t)-F(x,x)+P\{x<X<t,x<Y<t\})^{n-k}$$
$$=\begin{cases}\frac{(F_X(x)+F_Y(x)-F(t,t)-F(x,x))^{n-k}}{(1-F(t,t))^{n-k}}, & x>t\\ \frac{F^k(x,x)(F_X(x)+F_Y(x)-F(t,x)-F(x,t))^{n-k}}{F^k(t,t)(1-F(t,t))^{n-k}}, & x\le t.\end{cases}\tag{8}$$

It is clear that $\lim_{t\to\infty}F_{W_t}(x)=F(x,x)=P\{\max(X,Y)\le x\}$ and $\lim_{t\to 0}F_{W_t}(x)=1-\bar{F}(x,x)=P\{\min(X,Y)\le x\}$.

Example 1. Let X and Y be independent random variables having uniform (0,1) distribution (see Fig. 1). Then

$$P\{W_{n:n}\le x\mid \zeta_2=k\}$$
$$=P\{\max(W_1,W_2,\ldots,W_n)\le x\mid \sum_{i=1}^{n}\xi_i=k\}$$
$$=\begin{cases}\frac{(2x-t^2-x^2)^{n-k}}{(1-t^2)^{n-k}}, & x>t\\ \frac{x^{2k}(2x-2tx)^{n-k}}{t^{2k}(1-t^2)^{n-k}}, & x\le t.\end{cases}\tag{9}$$

For an illustration we provide a graph of (9) for $n=5, k=3, t=0.5$:

2.2.2. Special case 2

Let $B=\{X\le t, Y\le t\}, t>0,\ 1\le r\le n$

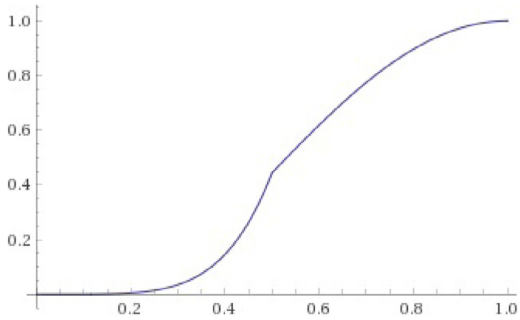

Fig. 1. The graph of $P\{W_{n:n} \le x \mid \zeta_1 = k\}$ for $n = 5, k = 3, t = 0.5$.

Then

$$P\{W_{r:n} \le x \mid \sum_{i=1}^{n} \xi_i = k\}$$
$$= \frac{\sum_{i=r}^{n} \sum_{j=\max(0,k+i-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j} p_{11}^{j} p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-k-i+j}}{\binom{n}{k} F^{k}(t,t)(1-F(t,t))^{n-k}}, \tag{10}$$

where

$$\begin{aligned} p_{11} &= P\{\max(X,Y) \le x, B\} \\ &= P\{X \le t, Y \le t, X \le x, Y \le x\} \\ &= F(\min((t,x), \min(t,x))) \\ &= \begin{cases} F(t,t), & x > t \\ F(x,x), & x \le t. \end{cases} \end{aligned} \tag{11}$$

$$\begin{aligned} p_{12} &= P\{\min(X,Y) \le x, B^c\} \\ &= 1 - \bar{F}(x,x) - F(t,t) + P\{x < X < t, x < Y < t\} \\ &= \begin{cases} 1 - \bar{F}(x,x) - F(t,t) & x > t \\ F_X(x) + F_Y(x) - F(t,x) - F(x,t), & x \le t \end{cases} \end{aligned} \tag{12}$$

$$\begin{aligned} p_{21} &= P\{\max(X,Y) > x, B\} \\ &= P\{(X,Y) \in B\} - P\{(X,Y) \in B, \max(X,Y) \le x\} \\ &= F(t,t) - F(\min(t,x), \min(t,x)) \\ &= \begin{cases} 0, & x > t \\ F(t,t) - F(x,x), & x \le t \end{cases} \end{aligned} \tag{13}$$

$$\begin{aligned} p_{22} &= P\{\min(X,Y) > x, B^c\} \\ &= P\{\min(X,Y) > x\} - P\{(X,Y) \in B, \min(X,Y) > x\} \\ &= \bar{F}(x,x) - P\{x < X \le t, x < Y \le t\} \\ &= \begin{cases} \bar{F}(x,x), & x > t \\ 1 - F_X(x) - F_Y(x) + F(x,t) + F(t,x) - F(t,t), & x \le t. \end{cases} \end{aligned} \tag{14}$$

Remark 1. It can be observed that if the random variable W would be defined as

$$W(\omega) = \begin{cases} \min(X,Y), & \omega \in B \\ \max(X,Y), & \omega \in B^c \end{cases}$$

then the conditional distribution and the correspondent probabilities (11)–(14) would be as follows:

$$P\{W_{r:n} \leq x \mid \sum_{i=1}^{n} \xi_i = k\}$$

$$= \frac{\sum_{i=r}^{n} \sum_{j=\max(0,k+i-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j}\pi_{11}^{j}\pi_{12}^{i-j}\pi_{21}^{k-j}\pi_{22}^{n-k-i+j}}{\binom{n}{k}F^{k}(t,t)(1-F(t,t))^{n-k}}, \tag{15}$$

$$\pi_{11} = P(AB^c) = P\{\max(X,Y) \leq x, B^c\} = P\{\max(X,Y) \leq x\} - p_{11}$$

$$\pi_{12} = P(AB) = P\{\min(X,Y) \leq x, B\} = P\{\min(X,Y) \leq x\} - p_{12}$$

$$\pi_{21} = P(A^cB) = P\{\max(X,Y) > x, B^c\} = P\{\max(X,Y) > x\} - p_{21}$$

$$\pi_{22} = P(A^cB) = P\{\min(X,Y) > x, B\} = P\{\min(X,Y) > x\} - p_{22} \tag{16}$$

Remark 2 (*More General Scheme*). In general, assume that $\xi_1(\omega)$ and $\xi_2(\omega)$, $\omega \in \Omega$ are two random variables defined in the same probability space $\{\Omega, \mathcal{F}, P\}$ and $G \in \mathcal{F}$.

$$M(\omega) = \begin{cases} \xi_1(\omega), & \omega \in G \\ \xi_2(\omega), & \omega \in G^c \end{cases}$$

and let $M_1(\omega), M_2(\omega), \dots, M_n(\omega)$ be the sample values of the random variable $M(\omega)$. Let $M_{r:n}(\omega)$, $1 \leq r \leq n$ be the rth order statistic of $M_1(\omega), M_2(\omega), \dots, M_n(\omega)$. Let $C = \{M(\omega) \leq x\}$, T_1 and T_2 be the number of occurrences of C and G, respectively. Considering fourfold scheme and bivariate binomial distribution with probabilities $q_{11} = P(C\,G) = P\{\xi_1(\omega) \leq x, G\}$, $q_{12} = P(CG^c) = P\{\xi_2(\omega) \leq x, G^c\}$, $q_{21} = P(C^cG) = P\{\xi_1(\omega) > x, G\}$, $q_{22} = P(C^cG^c) = P\{\xi_2(\omega) > x, G^c\}$. Then

$$P\{M_{r:n} \leq x\} =$$

$$= \sum_{k=0}^{n}\sum_{i=r}^{n}\sum_{j=\max(0,i+k-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j} q_{11}^{j} q_{12}^{i-j} q_{21}^{k-j} q_{22}^{n-i-k+j}.$$

Example 2. Assume that a technical system consists of n components and each component has two subcomponents. Therefore, the lifetime of ith component is defined by a random vector (X_i, Y_i), $i = 1, 2, \dots, n$, where X_i and Y_i are the lifetimes of first and second subcomponents of ith component, respectively. Assume that the subcomponents of each component may be connected by two ways, parallel or series ways, depending on whether the event B occurs or not. If the lifetimes of the components are $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$, where X_i and Y_i are the lifetimes of the first and second subcomponents of ith component, respectively. The lifetime of ith component will then be

$$W_i(\omega) = \begin{cases} \min(X_i, Y_i), & \omega \in B \\ \max(X_i, Y_i), & \omega \in B^c. \end{cases}$$

Assume that the system is a coherent system with $(n - r + 1)$—out-of-n structure, i.e. the lifetime of the system is $W_{r:n}$. Then the reliability of the system will be

$$P\{W_{r:n} > t\}$$

$$= 1 - \sum_{k=0}^{n}\sum_{i=r}^{n}\sum_{j=\max(0,i+k-n)}^{\min(i,k)} \binom{n}{j}\binom{n-j}{i-j}\binom{n-i}{k-j} p_{11}^{j} p_{12}^{i-j} p_{21}^{k-j} p_{22}^{n-i-k+j}.$$

We can use (10) to compute the system reliability in the case where $B = \{X \leq t, Y \leq t\}$, $t > 0$.

Example 3. Consider an insurance portfolio in which the random variable which represent the losses based on two types of claims is of interest. Let (X, Y) be a bivariate random vector of losses corresponding to two types of claims. We assume that these losses are associated. In health insurance we can consider the data that are the measured size of drug claims and other claims paid by the insurance company and the distribution of losses may depend on age, gender and other auxiliary variables. (see [8]). Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be the predefined losses corresponding to two types of claims and insurance company pays the amount $W_i(\omega) = I_B(\omega)\max(X_i, Y_i) + I_B(\omega)\min(X_i, Y_i)$ to ith insured. Then the right tail risk is the expected average of the $n - i$ largest claims, given by $\frac{1}{n-i}\sum_{j=i+1}^{n} E(W_{j:n})$. (see [9]). Since insureds may not claim both types of benefits the frequency probabilities are defined as $P\{X = 0, Y = 0\}$, $P\{x = 0, Y > 0\}$, $P\{X > 0, Y = 0\}$, $P\{X > 0, Y > 0\}$. We assume that $B = \{X \leq t, Y \leq t\}$, $t > 0$, i.e. B occurs if the amount of payment to the insured for drag claims X is less than t and the amount of payment for other claims Y is less than t. If B occurs the insurer's loss is $\max(X, Y)$, otherwise $\min(X, Y)$. For n portfolios the insurer maximum loss then will be $W_{n:n}$ and the probability of maximum loss given that B occurs, k times can be calculated as

$$P\{W_{n:n} > x \mid \sum_{i=1}^{n} \xi_i = k\}$$

$$= \frac{p_{11}^k p_{12}^{n-k}}{(\bar{F}(t,t))^k (1-\bar{F}(t,t))^{n-k}}$$

where $p_{11} = F(\min(x,t), \min(x,t))$, $p_{12} = 1 - F(t,t) - \bar{F}(x,x) + P\{x < X \le t, x < Y \le t\}$ as in (6) and (7).

3. Examples on distributions of the random variable W for some particular cases

In this section we provide some examples for distribution of the random variable W considering some special cases of underlying distribution $F(x, y)$ and events B.

Consider the random variable W defined as in (1).

Example 4. Let $B = \{X < Y\}$ and the joint pdf of (X, Y) is $f(x, y)$. Then,

$$W = \begin{cases} \max(X,Y), & X < Y \\ \min(X,Y), & X \ge Y \end{cases} = \begin{cases} Y, & X < Y \\ X, & X \ge Y. \end{cases}$$

The cdf of W can be found as follows:

$$F_W(t) \equiv P\{W \le t\} = P\{Y \le t, X < Y\} + P\{X \le t, X \ge Y\}$$

$$= \int_{-\infty}^{t} \int_{-\infty}^{y} f(x,y)dxdy + \int_{-\infty}^{t} \int_{-\infty}^{x} f(x,y)dydx.$$

For a particular choice of joint distribution function of X and Y as

$$F(x,y) = xy\{1 + \alpha(1-x)(1-y)\}, -1 \le \alpha \le 1$$

which is a classical bivariate Farlie–Gumbel–Morgenstern (FGM) joint distribution function with uniform(0,1) marginals and joint pdf $f(x, y) = 1 + \alpha(1 - 2x)(1 - 2y)$, $0 \le x, y \le 1$, then the cdf of W is

$$F_W(t) = P\{W \le t\}$$

$$= \alpha t^4 - 2\alpha t^3 + (1+\alpha)t^2, 0 \le t \le 1,$$

and the pdf of W is

$$f_W(t) = 4\alpha t^3 - 6\alpha t^2 + 2(1+\alpha)t, 0 \le t \le 1.$$

Example 5. Let $t > 0$ and $B = \{\omega \in \Omega : X \le t, Y \le t\}$ and let B^c be a complement of B. Then

$$W_t(\omega) \equiv W(\omega) = \begin{cases} \max(X,Y), & X \le t, Y \le t \\ \min(X,Y), & otherwise \end{cases}$$

If there is no need to point out that W_t depends on t we will use just W instead of W_t.

The distribution function of W can be found as follows.

We have

$$F_W(x) \equiv P\{W_t \le x\} = P\{\max(X,Y) \le x, B\} + P\{\min(X,Y) \le x, B^c\}$$

$$= P\{X \le x, Y \le x, X \le t, Y \le t\} + P(B^c) - P\{\min\{X,Y\} > x, B^c\}$$

$$= P\{X \le \min(x,t)\} + 1 - P(B) - [P\{\min(X,Y) > x\} - P\{\min(X,Y) > x, B\}]$$

$$= F(\min(x,t), \min(x,t)) + 1 - F(t,t) - \bar{F}(x,x) + P\{x < X \le t, x < Y \le t\}. \tag{17}$$

Therefore, taking into account that $\bar{F}(x,x) = 1 - F_X(x) - F_Y(x) + F(x,x)$ and $P\{x < X \le t, x < Y \le t\} = 0$, if $x > t$, $P\{x < X \le t, x < Y \le t\} = F(x,x) - F(x,t) - F(t,x) + F(t,t)$, if $x \le t$. We have

$$F_W(x) \equiv P\{W \le x\}$$

$$= \begin{cases} F_X(x) + F_Y(x) + F(x,x) - F(x,t) - F(t,x), & x \le t \\ F_X(x) + F_Y(x) - F(x,x), & x > t. \end{cases} \tag{18}$$

It is clear that $\lim_{t\to\infty} F_{W_t}(x) = F(x,x) = P\{\max(X,Y) \le x\}$ and $\lim_{t\to 0} F_{W_t}(x) = 1 - \bar{F}(x,x) = P\{\min(X,Y) \le x\}$.

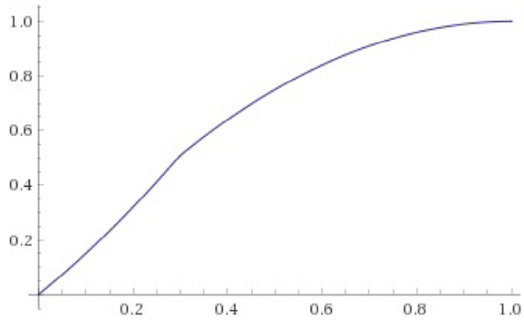

Fig. 2. The graph of $F_{W_t}(x)$ given in (19) for $t = 0.3$.

Hereafter we assume that X and Y are independent random variables. Let us write $F_W(x)$ for some special marginal distributions.

Example 5A (*Uniform(0,1) Distribution*). Let X and Y be independent and $F_X(x) = x, F_Y(x) = x, 0 < x < 1$. Then for $0 < t < 1$, from (18) we have

$$F_{W_t}(x) \equiv P\{W \le x\}$$

$$= \begin{cases} 0, & x < 0 \\ 2x + x^2 - 2xt, & 0 \le x \le t \\ 2x - x^2, & t < x \le 1 \\ 1 & x > t. \end{cases} \tag{19}$$

The graph of the function $F_W(x)$ in (19) for $t = 0, 3$ (see Fig. 2).
The pdf of W is

$$f_W(x) \equiv \frac{d}{dx} F_W(x)$$

$$= \begin{cases} 0, & x < 0 \text{ or } x > t \\ 2 + 2x - 2t, & 0 \le x \le t \\ 2 - 2x, & t < x \le 1. \end{cases} \tag{20}$$

The mean residual life function of W can be found as follows:

$$\Psi_{W_t}(s) \equiv E\{W - s \mid W > s\}$$

$$= \frac{1}{\bar{F}_W(s)} \int_s^1 x f_W(x) dx - s$$

$$= \begin{cases} \frac{1}{1-(2s+s^2-2st)} \int_s^t x(2+2x-2t)dx \\ + \frac{1}{1-(2s-s^2)} \int_s^1 x(2-2x)dx - s, & s < t \\ \frac{1}{1-(2s-s^2)} \int_s^1 x(2-2x)dx - s & s \ge t \end{cases}$$

$$= \begin{cases} \frac{t^3-3t^2+3ts^2-1+3s-3s^3}{3(1-2s-s^2+2ts)} & s < t \\ \frac{1}{3} - \frac{s}{3} & s \ge t. \end{cases}$$

Below for $t = 0.4$ (left) and $t = 0.8$ (right) we provide comparative graphs of MRL functions $\Psi_{F_1}(s) \equiv$ MRL1, $\Psi_{F_2}(s) =$MRL2 and $\Psi_{W_t}(s) =$ MRL3 of the lifetime distributions $F_1(x) = 1 - (1 - x)^2$, $F_2(x) = x^2$, and $F_W(x)$, $0 \le x \le 1$, respectively. Note that $F_1(x)$ is a cdf of $\min(X, Y)$, $F_2(x)$ is a cdf of $\max(X, Y)$ and $F_W(x)$ is a cdf of W (see Fig. 3).
For a definition and further results on MRL functions see e.g. [10–13].

Example 5B (*Exponential Distribution*). If $F_X(x) = 1 - e^{-\lambda x}, x \ge 0, \lambda > 0$, then we have

$$F_W(x) \equiv P\{W \le x\}$$

$$= \begin{cases} 0, & x < 0 \\ 2 - 2e^{-\lambda x} + (1 - e^{-\lambda x})^2 - \\ \quad -2(1 - e^{-\lambda x})(1 - e^{-\lambda t}) & 0 \le x \le t \\ 2 - 2e^{-\lambda x} - (1 - e^{-\lambda x})^2, & x > t. \end{cases} \tag{21}$$

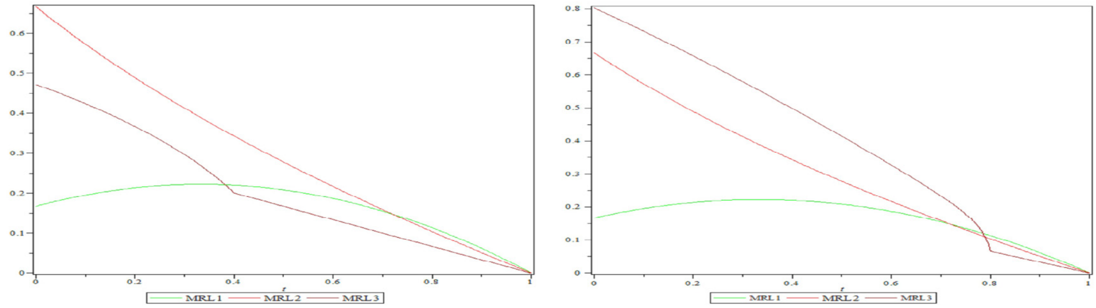

Fig. 3. The graph of MRL functions for $F_1(x)$, $F_2(x)$ and. $F_W(x)$ $\Psi_{F_1}(s) = MRL1$, $\Psi_{F_2}(s) = MRL2$ and $\Psi_{W_t}(s) = MRL3$ for $t = 0.4$ and $t = 0.8$.

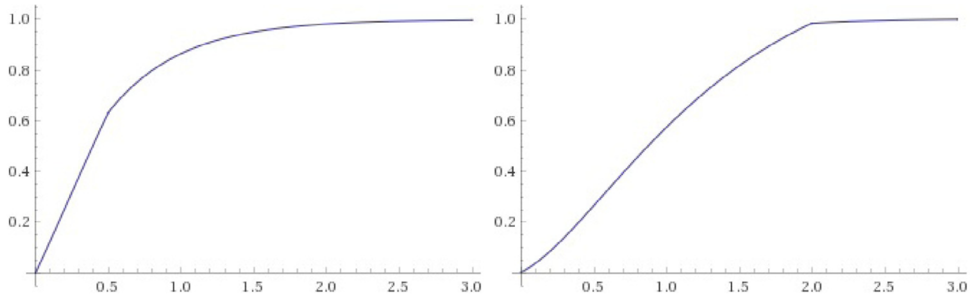

Fig. 4. The graphs of $F_W(x)$ given in (21) for $t = 0.3$ (left) and $t = 0.8$ (right)

The graphs of (21) for $\lambda = 0.5$ and for $t = 0.3$, $t = 0.8$ are given in Fig. 4.
The pdf of (21) is

$$f_W(x) = \begin{cases} 0, & x < 0 \\ 2\lambda e^{-\lambda x}(1 - e^{-\lambda x} + e^{-\lambda t}) & 0 \le x \le t \\ 2\lambda e^{-2\lambda x}, & x > t. \end{cases}$$

4. Conclusion

We consider a sequence of bivariate random vectors $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ defined in probability space $\{\Omega, \mathcal{F}, P\}$ and an event $B \in \mathcal{F}$. Depending on occurrence of B, we consider the model of the sequence of random variables as $W_i(\omega) = I_B(\omega)\max(X_i, Y_i) + I_B(\omega)\min(X_i, Y_i)$, $i = 1, 2, \ldots, n$, where $I_B(\omega) = 1$ if $\omega \in B$ and $I_B(\omega) = 0$ if $\omega \in B^c$, is an indicator function of event B. Then we study distributions of order statistics $W_{r:n}$, $1 \le r \le n$ constructed from the sequence of dependent random variables $W_1, W_2, \ldots, W_n$. To derive the distribution of $W_{r:n}$ we use bivariate binomial distribution. Some particular cases and distributions are considered, examples are provided. We also provide some examples for distribution of random variable W in special cases. The results can be applied to reliability analysis of the systems having n components, with two subcomponents per component. The model can also find applications in actuarial sciences.

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