Computational Modeling and Analysis of Seljukid Muqarnas in Kayseri

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As a historical and ornamental building element, muqarnas are widely found among the entrances of madrasas, mosques, and hans in Anatolian Seljuk architecture. In Kayseri (Turkey), muqarnas structures are characterized by symmetrical distribution of patterned geometric layers that presents computational rules for the design and construction of these ornamental structures. The presented research focuses on 12 unique muqarnas structures that are analyzed through a computational methodology combining photogrammetry, three-dimensional modeling, symmetry, and graph theory. The computational analysis shows that Seljukid muqarnas exhibit patterned branching of the symmetry axis between layers radiating from their geometric center. Using the modeled samples, the article analyzes inherent symmetry rules and growth patterns while offering a novel way of studying, modeling, and categorizing muqarnas.

CCS Concepts: • **Computing methodologies** → *Computer graphics;*

Additional Key Words and Phrases: Muqarnas, 3D modeling, symmetry, photogrammetry, graphs

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1 INTRODUCTION

Islamic architecture presents a vast array of geometric patterns and ornaments that can illuminate today's problems of modular design processes and constructions. Furthermore, geometric and computational study of these structures can provide insight for their maintenance and restoration in case these cultural heritage artifacts are damaged, either through natural or artificial causes [5]. A valuable artifact of Islamic architecture that is found in middle east is muqarnas that are mostly used in madrasas, mosques, tombs, and hans while offering rich solutions of modular construction and decoration [18].

The word "muqarnas" means stalactite vault that is made of stacked and packed elements on an architectural surface. In the late 20th century the geometric analysis of muqarnas has drawn considerable interest from

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scholars worldwide [2, 3, 8, 16, 23, 40]; however, these studies have not focused on examples found in Anatolia, particularly dating back to the 12th century Seljukid period [31]. Most of the structures that are of interest [36, 42], are still being used as they have retained cultural and architectural interest, but little consideration has been given to a comprehensive understanding of their morphology, structural construction, and geometric configurations.

A key feature of the Anatolian Seljukid muqarnas is the presence of repetitive geometric layers that radiate from a single center while infilling rectangular corners using modular stones. These stones often have triangular plans that make their geometric construction from plans possible. While contemporary studies on muqarnas focus on modular construction and analysis [14, 15, 41] as a manual labor, this article focuses on 12 historical structures found in Kayseri (Turkey) that present a common growth pattern. This aspect can illuminate both the overall morphological understanding of Seljukid muqarnas and the role of geometry and construction for their structural maintenance.

The novelty of the presented research is the overall methodology that combines computational scanning (photogrametry), digital modeling, and geometric analysis of an array of historical structures to gain a comprehensive understanding of design and construction principles of muqarnas [9, 29], The overall aim of the article is to combine cultural heritage and computational methods for the geometric analysis of an Islamic ornament while providing cumulative analysis techniques that can be used for restoration, design, production, visualization, and typological classification of muqarnas structures found among Anatolian Seljukid architecture.

2 SYMMETRY AND GEOMETRY

Symmetry is an immanent and widely used concept in many fields ranging from natural sciences to art. The origin of the term, based on the combination of the words "syn" (together) and "metron" (to measure) can be traced back to ancient Greek [35]. It refers to group of related parts that constitute a whole in a certain harmony. In Ancient Greek mathematics to "symmetrize" is considered as an intentional act of measuring through comparison [11] to achieve a the right balancing of parts in relation.

2.1 Symmetry in Mathematics

Historically, symmetry has been an attractive topic to many scholars for the analysis, encoding and decoding of natural forms. D'Arcy Thompson uses the terms "bilateral" and "radiate" in his studies that examine the growth and formation processes of living things in nature [34]. Johan Gielis examines circular and spherical forms in nature such as starfish, shells, and flowers to achieve a generalized Fourier series [11]. Hermann Weyl introduces a comprehensive overview of symmetry concepts and their relevant projections in inorganic and organic nature, apart from the artefacts of human-made environments [39]. Another contribution of Weyl is to define and approach symmetry as an automorphic transformation [39]. By doing so, it becomes possible to compute the form of a nautilus as a combination of rotational symmetry and translation operations, as well as to gain basic understanding of the Mysterium Cosmographicum model of Kepler [39]. Advances in computational geometry and computer vision led to the emergence of new possibilities and techniques to employ radial symmetry to get a better understanding of complex forms [11, 24, 27]. Radial symmetry, as a subset of symmetry, also used as rotational [39], radiate [34], and angular or n-fold symmetry [24], refers to how the identical parts are arranged around a central axis [37] with equal distances.

2.2 Ornament Geometry and Symmetry in Seljukid Architecture

Development of geometrical ornaments in Islamic architecture has been formed over the centuries [1], while the contribution of practical geometry and mathematical science can be traced back to the 9th century [4, 30]. In the Anatolian Seljuk period (1081–1307), there has been a remarkable contribution in the transformation of ornaments from the floral and figural toward pure geometrical patterns [1]. Geometric patterns in the Anatolian

Seljuk period can be considered as an experimental field that manifests itself in material, technique, and spatial levels, beyond merely functioning as ornament. Experiments on the raised-brick patterns, friezelike stone cuttings, wood carving, mosaic, and tiling [28] have resulted with complex arrangements of basic geometries and implementation of different symmetry groups. Particularly in the early 12th century, 5-point and 8-point star concepts have been extensively utilized in decorative elements, as well as in the integration of decorative and structural elements [1]. In addition, 6-, 8-, 10- and later 7-, 9-, 13-point geometrical points have also been used [1] in the Anatolian Seljuk period.

There is remarkable research on demystifying Islamic geometric patterns through the lenses of geometrical axioms and today's computer graphics [6, 19, 20, 25, 26]. Peter Cromwell searches for evidence of quasi-periodicity and employs both translational symmetry and radial symmetry to regenerate traditional geometric tilings [6]. Basic mathematical definitions of two-dimensional (2D) patterns where "n" represents the variable for number of divisions are utilized [20] and developed [19, 21, 25, 26]. Apart from the plane group (reflection) symmetries, N-fold point group (rotational) symmetries provided new insights to understand potentials of Islamic geometrical patterns, where it is possible to achieve complex geometrical organizations based on a limited spectrum of symmetry operations [28]. Jay Bonner highlights the need for a categorization for Islamic geometrical patterns, where it is possible to fill a plane with a set of initial units and symmetry-based repetitive rules [5]. In contrast to existing studies, the presented work focuses on approaching geometrical patterns through graph-based logic that relates symmetry to the recursive growth of geometric patterns [7, 12]. While providing an alternative method for the representation of the geometrical pattern itself, the graph-based parametric model enables both an interactive user intervention and a generative way of producing variations of the defined patterns.

2.3 Muqarnas in Anatolian Seljukid Architecture

Muqarnas as an architectural transition element from a square and / or rectangular plan to a structural dome originated from North Africa, Iran, and the Mesopotamian region. During the Anatolian Seljukid period (1081–1307), muqarnas were used in a wide spectrum of building typologies, such as caravanserai, madrasah, tomb, hammam, mosque, and public soup-kitchens. Most of the muqarnas structures are located at the entrance portals of buildings that characterize the exterior space. Furthermore, the diverse use of materials during this period such as rubble stone, marble, and brick contributed to the ornamental articulation [43], as well as understanding of muqarnas. Different from other geometrical ornamentation, muqarnas are mostly designed on the ground plan as two-dimensional projections [17]. During the Anatolian Seljukid period, projection technique from the two-dimensional ground plan continued, while the angle of the parts became less limited [17].

3 COMPUTATIONAL MODELS OF/FOR THE MUQARNAS GEOMETRY

This section will review some of the existing approaches for the computational modeling of architectural heritage. Some of these methods have also been applied to the geometric modeling and analysis of muqarnas found in different geographical locations. These are summarized under three distinct methods categorized by computational tools, methodology, and workflow.

3.1 Computer-aided Design (CAD) Models

As a volumetric infill, muqarnas structures can be studied through projected plans that are made of patterned distribution of geometric pieces. Using this principle, Yaghan developed a repertoire of muqarnas configurations that can be procedurally modeled by layering standardized three-dimensional muqarnas parts [41]. This early study enabled the identification of common geometric elements of muqarnas to develop possible configurations that can be correlated between plan drawings and three-dimensional models. In a later study, Yaghan presented a catalogue of muqarnas that can be developed from plans, layers, structure, and topology [40]. Sakkal [33] defined the elements of muqarnas as three dimensional (3D) solid model and presented the creation of muqarnas blocks with a series of geometric operations. These studies showed the potential parametric variability and richness of the muqarnas models while offering a way to analyze, categorize, and represent unique configurations found among various heritage structures. In a later study, Hamekasi and colleagues [15] showed that how the topological integrity of the muqarnas can be modeled interactively to develop new designs by manipulating the sections, layers, and parts [15]. Gorjian [38] introduced a rule-based modelling approach utilizing graph-based representation technique. Briefly, recent development of CAD models exhibit more parametric variability for the procedural modeling of architectural heritage.

3.2 Generative Models

A more recent approach for the study of variability in muqarnas is the development of generative algorithms that requires the identification of productive geometric rules for the design and construction of muqarnas. Harmsen introduced a novel vector-based plan analysis tool for the reconstruction of stalactite vaults found in Anatolian Seljuk muqarnas that propagates the edges of rhombic muqarnas parts through vertical layers as graphs [17]. Using this approach, various configurations of muqarnas are modeled by extracting subgraphs of muqarnas plans revealing common patterned development. This approach has been utilized to develop a cumulative database for the stylistic identification, generative modeling, representation, and restoration of muqarnas structures from retrieved plans [8]. In another study, the polygonal geometric parts of muqarnas have been analyzed according to technique of folding and that can be used to develop computational strategies for form-finding and analysis of vaulting structures [3]. Similar parametric approaches are also carried for the computational study of various Islamic pattern-based motifs. An example focuses on the geometric analysis of existing quasi-periodic tiles made of rhombic tiles that can be divided into right triangles to generate new parametric pattern designs [25].

3.3 Integrated Compound Models

This group represents the integration of multiple computational models into a workflow to provide data transfer between different software for various modeling, analysis, and simulation of heritage structures. In these models, data acquired from photogrammetry scans are often transformed into a 3D or **Building Information Model (BIM)** that can be used for different computational applications. Various examples of **integrated compound models (ICM)** are found in the applications developed for computational modeling of cultural heritage. Marek Miłosz and colleagues showed an integrated approach for the scanning, processing, and visualization of Timurid architecture and provided insight for the efficient documentation of historical monument structures that are open to visitors [29]. Using a computational approach to transform the 3D scanning data into a **historic building information modeling (hBIM)**, Raphael Rolin and colleagues developed finite element analysis and structural simulation for the spire of Cathedral of Senlis to provide insight on the physical condition of the heritage structure [32]. Rodriguez-Echavarria and Song showed a methodology for the integrated analysis of shape saliency through 3D scanning that can automate stylistic classification of historical artifacts [9]. While these studies exhibit computational methodology to integrate data developed in different software, an ICM model for the computational study of Anatolian Seljukid muqarnas is not found in current scholarship and a prototypical study will be presented in this article.

4 PRESENTATION OF THE STUDY

4.1 Scope and Constraints

During the Seljukid period, many cities in central Anatolia were connected with trading routes and numerous structures were erected in cities to support the activities and spatial requirements of visitors. Among these cities, Kayseri had a special importance due to its geographic location and centralized location to surrounding cities. This is also reflected with the amount of structures built in Kayseri during the 12th to 13th centuries that are characterized with tall entrances, all of which have a unique muqarnas structure. These structures are located in close proximity to the city center and most of them are still being used today. This made Kayseri a strategic focus

Table 1. General Information about the Seljukid Structures in Kayseri

Structure	Other Names	Construction Date	Developer	N.M.G.*
Gülük Mosque	Külük / Kölük Mosque	1134-35/ 1142-43	Melik Muhammed (not certain)	1
Alaca Kumbet	—	1184	Emir Sadreddin Ömer	1
Gevher Nesibe Madrasah	Şifahiye and Gıyasiye Medresah, Çifte Madrasah	1205–6	Sister of I.Gıyaseddin Keyhüsrey, Gevher Nesibe Sultan	1
Sultan Han	Tuzhisarı Sultan Han	1229–1236	Alaeddin Keykubay (not certain)	2
Cifte Kumbet	—	1247–1248	Adile Hatun	1
Karatay Han	—	1247–1255	Celalettin Karatay	1
Hacı Kılıc Mosque	Mevlana Tusî Madrasah	1249–50	Ebul Kasım Ibn Ali El Tusi	2
Sahabiye Madrasah	Sahip Ata Madrasah, Sahibiye Madrasah	1267–68	Seljukid Vizier Sahip Ata Fahreddin Ali	1
Doner Kumbet	—	1276	Shah Cihan Sultan	1
Şah Kutluğ Kümbet	—	1349	Şah Kutluğ Hatun	1

*Number of Muqarnas Gates.

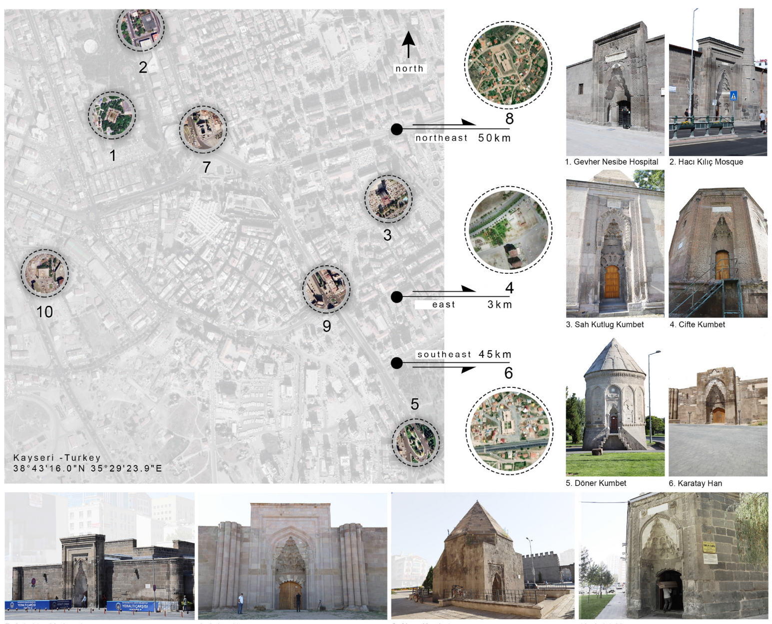

Fig. 1. Map of Kayseri showing the locations of selected Seljukid Structures.

for our research that presented temporal and physical advantages for field trips and extraction of digital models through photogrammetry. General information about the studied structures in Kayseri is listed in Table 1 below presenting their names, construction date, developer, and amount of entrances with muqarnas found.

Figure 1 presents the locations of structures in Kayseri that are selected as part of the field research. Among the chosen structures Sultan Han is located 50 km east of Kayseri toward Sivas, and Karatay Han is located 45 km on the toward Malatya, both of which were other Seljuk cities in the 13th century. All of the other structures are within 3-km radius of the city center marked by the Castle of Kayseri. Apart from the tombs (Kumbet), most of the structures are still being used today. All of the muqarnas that were studied are located at the entrances of the

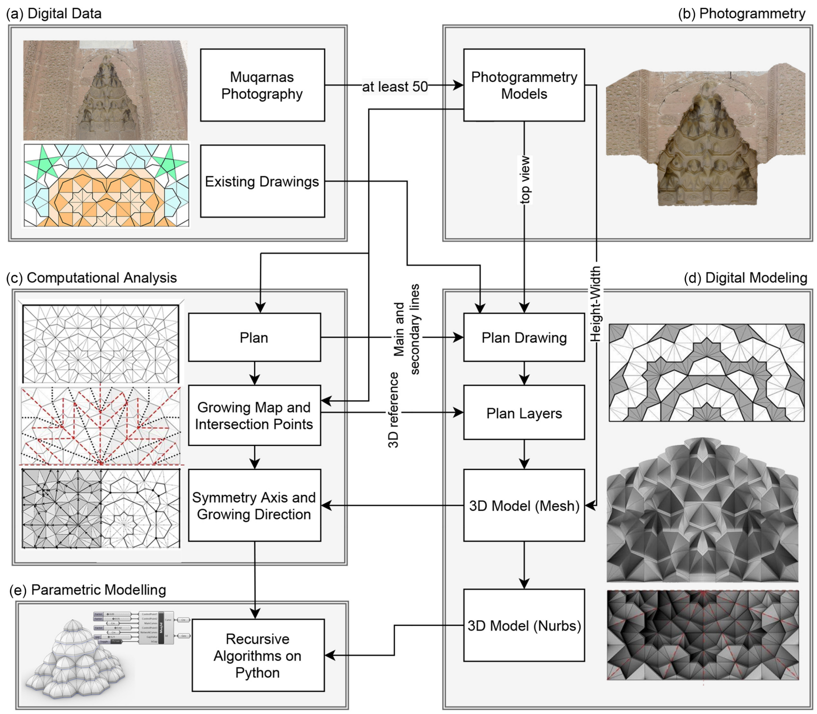

Fig. 2. Workflow for computational modeling of Seljukid muqarnas in Kayseri combining photogrammetry, 3D modeling, and computational analysis stages. In each stage, the produced model/images/drawings are presented next to the operation. Arrows designate the data transfer between programs.

selected structures in the Kayseri region.[1] For the collection of the muqarnas samples, each structure is visited and documented using a Canon EOSR camera and hundreds of pictures were taken to produce photogrammetric models and structural details are documented. Due to the construction material for the studied muqarnas being stone, some of the structures showed natural decay and erosion of muqarnas pieces. However, this natural transformation did not affect the carried methodology and computational modeling of the structures that is presented below.

4.2 Methodology

The goal of our project is to extract 3D models of existing structures, reproduce refined 3D models/plans and develop computational analysis of symmetry for geometric construction and classification of acquired samples. This workflow was achieved in four stages. In the first stage, all the available digital data of the structures were acquired through archival and online research (Figure 2(a)). Most of the structures in Kayseri exist in the online repository presented by Takahashi [36]; however, some of the plan drawings and photographs were also retrieved from physical archives [42].

The next phase of research required a field trip to Kayseri for photogrammetric documentation of the structures. For the heritage model extraction, hundreds of photos of each structure were taken and combined using

[1]The city center of Kayseri also presents other important Seljukid structures characterized with muqarnas entrances such as Hunad Hatun mosque complex, Hatuniye Madrasah, and Emir Şahap Tomb. Although these structures are also documented as part of the field research, they are omitted from this article, as they are produced with different geometric rules.

Agisoft software (Figure 2(b)). In the third phase, the extracted photogrammetry models were analyzed with reconstructed plans to determine structural layers and identical parts with symmetrical relationships (Figure 2(c)). This process was simultaneously developed with digital model reconstruction, as some of the vertical faces of the muqarnas were not identifiable from plan drawings. In the final stage, all the scanned muqarnas structures were digitally modelled by juxtaposing information gathered from photogrammetric models, archive drawings, and additional resources/photos (Figure 2(d)). These models were developed in four steps. First, muqarnas layers and symmetry groups within each layer were identified from photogrammetric models and plans. Since muqarnas are constructed with repetitive geometric parts, the identification of symmetry groups or mirror reflections of identical parts aided the reproduction and refinement of muqarnas plans. Second, the plan drawings were elevated creating a triangulated mesh model in Rhinoceros combining height data from photogrammetric models and geometrically reproduced plans according to specified angles and polygons. Most of the studied muqarnas presented similar geometric elements ranging from triangles and quadrilaterals, the latter of which can be further broken down into two triangulated pieces. Finally, the triangulated mesh models were transformed into lofted surfaces with profile curves to produce a refined model in Rhinoceros. The final models were organized with reproduced plans showing muqarnas layers, geometric parts, and symmetry groups that were later analyzed using computational diagrams.

Using the methodology described above, 12 different muqarnas extracted from 10 different structures were reproduced using photogrammetry (Agisoft) and 3D modeling (Rhinoceros) software. Figure 2 shows the reproduction stages of Gülük mosque muqarnas as a case study. The photogrammetric models produced from pictures are first transferred to Rhinoceros, then the plan drawings are prepared using the top view of scans by highlighting the muqarnas layers and geometric parts. In plan drawings, symmetrical relations between identical parts are examined and identical geometries located on the same levels are determined (Figure 2(c)). Unique geometry parts are modeled and multiplied using symmetrical relationships (Figure 2(d)). In the next stage, the heights of each layer are extracted by juxtaposing 3D models in Rhinoceros with photogrammetric models. In the last stage, the reproduced symmetrical 3D models are used to produce axonometric drawings and renderings. The overall methodology focused on the identification of symmetrical relationships between muqarnas pieces that aided the analysis, modeling, and classification of muqarnas.

4.3 Data Collection with the Use of Photogrammetry

Recent developments in computational scanning software have transformed the scope of study for architectural heritage, enabling detailed model extraction with surface texturing calculated from photo collections [9, 10, 22, 29]. A common method used for this study is photogrammetry that transforms a large collection of pictures into a three-dimensional surface model. Although these surface models cannot be used directly for additional computational methods, they can provide valuable input for the geometric analysis, modeling, and refinement of heritage structures. In Seljuk muqarnas, the application of photogrammetry scanning can provide valuable information for the extraction of geometric data that can be used for producing plan drawings and 3D models that can be used for possible restoration of these historic structures.

Compared to existing methods of photogrammetric documentation of architectural heritage, Seljuk muqarnas found at the entrances of heritage structures present numerous advantages for efficient documentation and computational model extraction. In plan, the muqarnas structure progressively stacks to a central origin point that can be used for photogrammetric calibration. Furthermore, in section muqarnas layers can be easily documented from the ground level due to the geometric alignment of its cascading faces. Thus, the muqarnas geometry presents centrality in plan and progressive stacking of layers in section, both of which align naturally with camera positions (Figure 3).

Due to the inherent morphological characteristics of muqarnas, the photogrammetry procedure requires targeting the vaults from various radial angles toward the pinnacle. Using this orientation, the 12 muqarnas

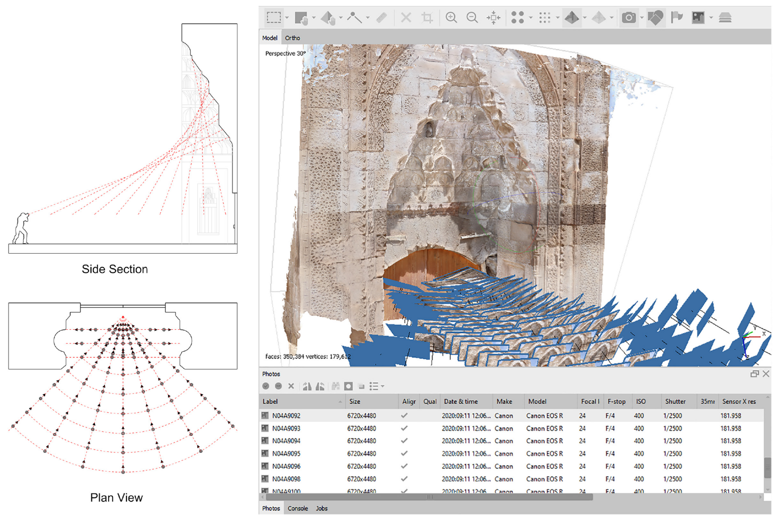

Fig. 3. Schematic explanation of the photogrammetric camera positions and targeting of muqarnas in plan and section (Left). Agisoft interface for the photogrammetric modeling of Karatay Han (Right).

Table 2. Photogrammetric Data of Scanned Muqarnas Structures in Kayseri

Muqarnas	Picture Count	Point Cloud	Model Faces
Gülük Mosque	142	38652	368462
Hacı Kılıc Mosque (Madrasah Gate)	151	35372	346868
Hacı Kılıc Mosque (Mosque Gate)	173	39530	549963
Gevher Nesibe Hospital	165	26436	250650
Döner Kümbet	164	69890	499642
Çifte Kümbet	205	55885	436297
Alaca Kümbet	112	38605	340057
Sultan Han (Main Gate)	100	49040	261618
Sultan Han (Side Wall)	29	32605	294011
Karatay Han	343	49588	430384
Şah Kutluğ Kümbet	55	20549	340269
Sahabiye Madrasa	146	33841	426759

structures found in Kayseri were documented using a Canon EOSR camera along virtual lines radiating from the geometric center of the muqarnas (Figure 4). The structures were captured multiple times moving along these axis while additional pictures were taken under the vaults to document details. The acquired photos were combined in Agisoft to produce 3D mesh models with textures extracted from pictures. This process is repeated for all the muqarnas structures. The details of the photogrammetric process for each muqarnas is listed in Table 2.

The extracted muqarnas models are presented in Figure 4. The overall quality of the models aided identification of stones, layers, details, and natural material decay for muqarnas structures. These models aided the reproduction and simplification of 3D models in the next phase of the project that required computational processing of scanned geometry and identifying symmetry relationships of the muqarnas geometry.

4.4 Symmetry Analysis

According to the data obtained from photogrammetric models, most of the muqarnas present asymmetry and geometric irregularities. This may have occurred either through natural causes through time or the construction

Fig. 4. The photogrammetric models of 12 muqarnas structures produced in Agisoft.

process could have created imperfect solutions for craftsmen. Despite geometric irregularities, a common feature among the analyzed structures is the continuity of a diagonal axis that connects the centralized highest tip point of the muqarnas to the square corner, fitting the radiating structure into an orthogonal plan. Among the muqarnas

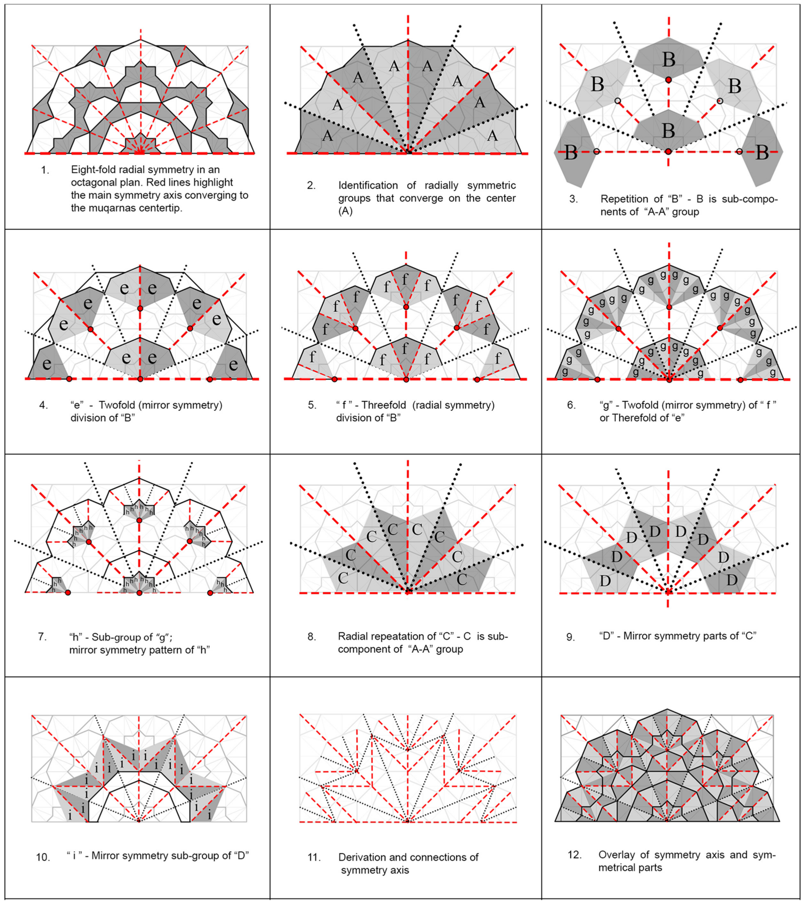

Fig. 5. Symmetry analysis and breakdown of Gülük mosque muqarnas.

that were analyzed, the muqarnas of the Gülük Mosque presented seven different layers that fit into a radial symmetry based on an octagonal plan. Because of its simplicity and radial symmetry [13] this muqarnas was initially analyzed to study the symmetrical relationships between parts to develop a strategy for the modeling and classification of the variations.

In Figure 5, the symmetry analysis of Gülük mosque muqarnas is presented. For this study, the plan and three-dimensional photogrammetric models are studied in a coordinated manner to find mirror and radial symmetries between parts. Muqarnas are formed by geometric outlines of stacked stones that expose unique vertical layer lines that coincide with construction levels. The preliminary step of this study is the identification of muqarnas layers marked with alternating white and gray groups (1) and main symmetry axis between groups (2). In the next step, symmetrical relationships between sub-parts (B, C, and D) within each layer are plotted (3–5) that can be related to upper symmetry groups. For instance, the radial symmetry within the B group pieces was identified by looking for further subdivision and symmetry relations in the group that revealed the parts e, f, g, h marked in the steps 6 to 9. Similarly, the group D specified in the step 5 showed a branching connection to the initial radial symmetry axis in step 2. When all the symmetry axis emerging from the fragmentation process were drawn, the muqarnas showed radial branching and growth patterns between vertical layers (10–12). In each layer, the

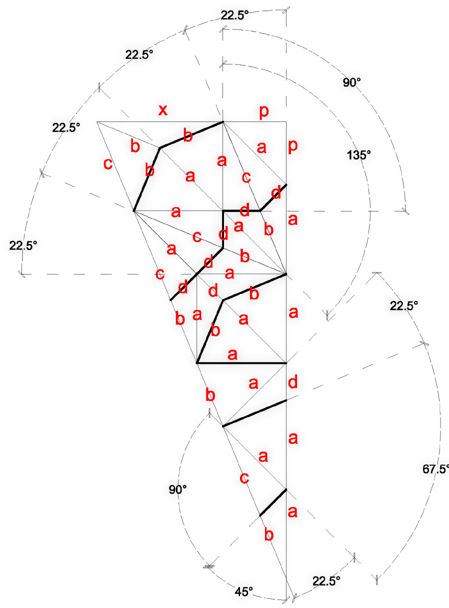

Fig. 6. The geometric analysis of lengths and angles for Gülük mosque muqarnas showing similar lengths distributed among subsequent layers. The layer lines are shown in thick black lines that correspond with specified angles.

boundary curve defining the outline of the pieces coincides with branching symmetry axes that converge on the tipping point of the muqarnas. In Gülük mosque, due to the octagonal symmetry between parts, the repeating radial segments are defined at 45°.

In the next step, the parts in the recurring radial denominator of Gülük muqarnas are analyzed over plan to define geometric relations, angles, and unique lengths between the parts. At this stage, it was observed that the edge lengths of the triangulated pieces of the muqarnas described four different lengths (a, b, c, and d) and these lengths produced a pattern with 22.5° and 67.5° (Figure 6). The geometric relationships of these lengths show that the initial angles and symmetry groups have a definitive role in the prescription of the lengths of pieces all of which could be trigonometrically derived. All of the muqarnas pieces that are found in the Gülük mosque sit on triangulated geometries in the plan. By observing the relations of these pieces to the symmetry axis, four different morphological tile types are identified (Figure 7). In muqarnas, triangular parts can be classified as divergent (expanding to a line) or convergent (contracting to a point) according to their geometric relations to the symmetry axis (Figure 7, T1 and T2). In contrast, quadrilateral parts maintain expanding relations but could be identified according to their diagonal connections within the quad that also define two triangle groups. These are defined as divergent-diagonal connection to secondary axis or convergent-diagonal connection to main symmetry axis (Figure 7, Q1 and Q2). In these diagrams, the black arrows designate the secondary axis that grows simultaneously with the main symmetry axis shown in red arrows. These two can often coincide when there is radial branching resulting in the freezing of the secondary axis. The two grey triangles show the mirror symmetry groups emerging along the red axis. Similar patterns are also identified among other muqarnas in Kayseri, which led to the analysis and classification of them according to growth graphs and symmetry groups.

4.5 Muqarnas Growth Diagrams

The analysis of Gülük mosque muqarnas shows multiple geometric layers that are connected through bifurcating symmetrical relationships. To further visualize this relationship, the symmetrical axes that are defined through

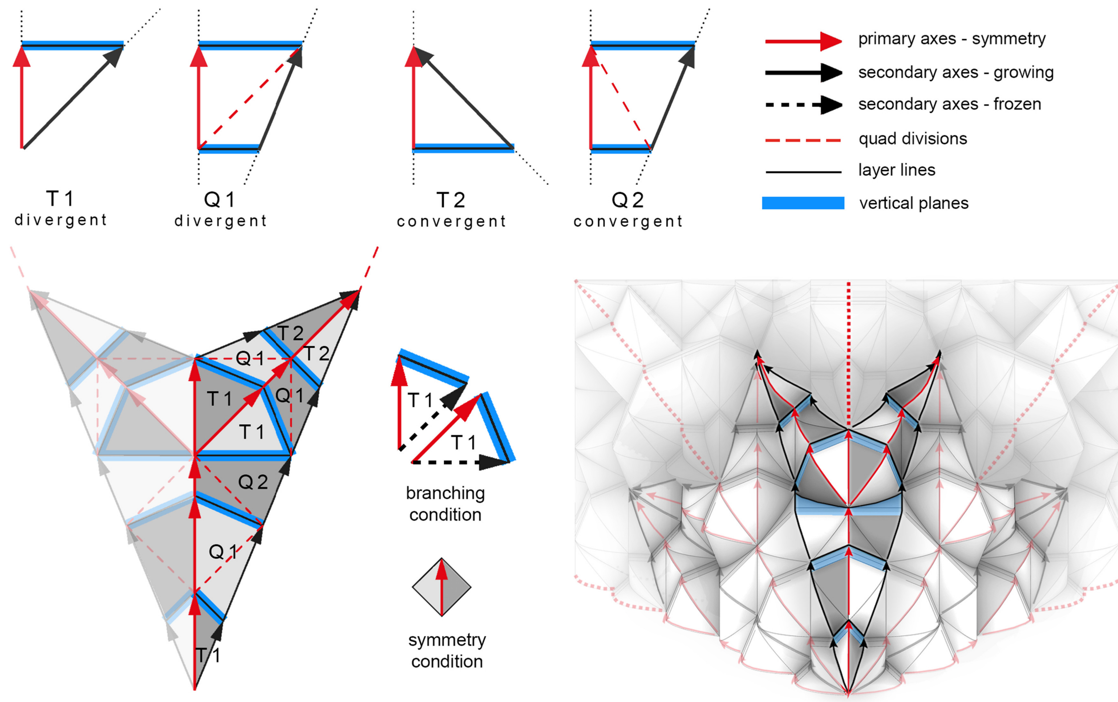

Fig. 7. The propagation of growth axis between layers in Gülük mosque muqarnas that coincide with symmetry groups as a branching graph. The triangles and quadrilateral pieces can be classified into four different types.

plan are visualized on the three-dimensional model as growth. The overall principle here is to define growth between layers as a flow diagram [8]. By starting from the tip of the muqarnas, each symmetry axis moves to the next layer by defining triangular pieces. This way the transformation of initial radial symmetry at the base is visualized where the muqarnas meets the square plan. The asymmetric distributions between parts in the lower layers where the radial symmetry is transformed to fit into a square can also be described by similar flow directions that correspond with the symmetry axis. In particular, the linear continuity of the diagonal axis of the muqarnas presents the main fragmentation logic of the layers that articulate radial and mirror symmetries between pieces.

Figure 8 presents the growth diagram of Gülük Mosque muqarnas showing the flow of the symmetry axis from the central tip point. This diagram is produced by juxtaposing the geometric analysis of plan layers and identification of symmetrical parts from 3D models. In the diagram, the muqarnas layers are visualized in letters (A, B, C, D, . . . , H) starting from the central tip point (O). The points that make up eight different layers are identified, and symmetrical relations between parts are identified using red arrows, while the flow of the secondary growth axis is visualized using black arrows. The axial continuity of these arrows coincide with the major symmetry axis and describe branching growth from the central point (O) toward the square plan. According to this movement, some points only propagate vertically between layers (i.e., C4 → D7 or F12 → G10) producing flat panels that are not visible from plan drawings and can only be visible in 3D models. However, in their planar analysis these panels do not disrupt the continuity of growth where other branches converge on symmetrically arranged channels (i.e., C4 → D7 and E6 → F4). These converging growth axis are shown using dotted arrows (Figure 8). The diagram shows that the first six layers of the muqarnas (A–F) show a predominant radial

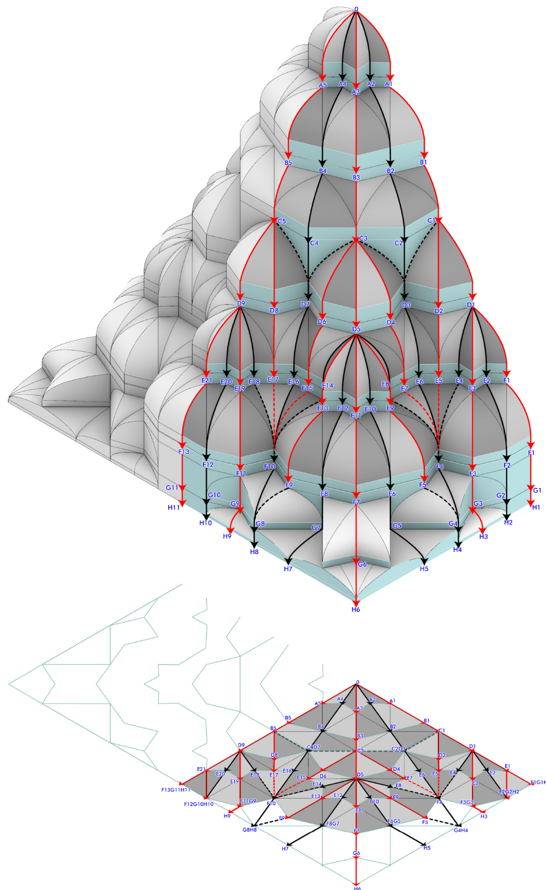

Fig. 8. Growth diagrams of Gülük Mosque muqarnas. By analyzing the plan and model of the muqarnas, a hierarchical branching scheme was created from the symmetry axes between the parts in each layer. Red arrows show main growth along symmetry axis while black arrows show growth of secondary axis.

symmetry fitting into an octagon (Figure 8, O–F). In the layers F and G, asymmetrical relations between parts are identified where the muqarnas meets the square plan. This diagram shows a hierarchical patterned growth between layers where muqarnas pieces are organized according to bifurcating symmetry axes. This principle also follows patterns observed in fractal growth encountered in nature. Based on this principle, other muqarnas are examined and growth charts showcasing geometric continuity and propagation between layers are identified.

4.6 Computational Modeling and Classification

Using the prototypical analysis on Gülük Mosque, the same methodology was followed for the modeling and analysis of a total of 12 muqarnas structures found in Kayseri. Due to their volumetric characteristic these structures are presented using skewed axonometric drawings and renderings showing the geometric articulation and diversity of muqarnas while facilitating the identification of symmetrical parts (Figure 9).

A common characteristic of the studied muqarnas is the branching angle between symmetry axes of 22.5° that also defines a half octagon in plan. This principle also reveals common proportions and patterned development among the muqarnas that is further analyzed by drafting the possible trigonometric relationships two

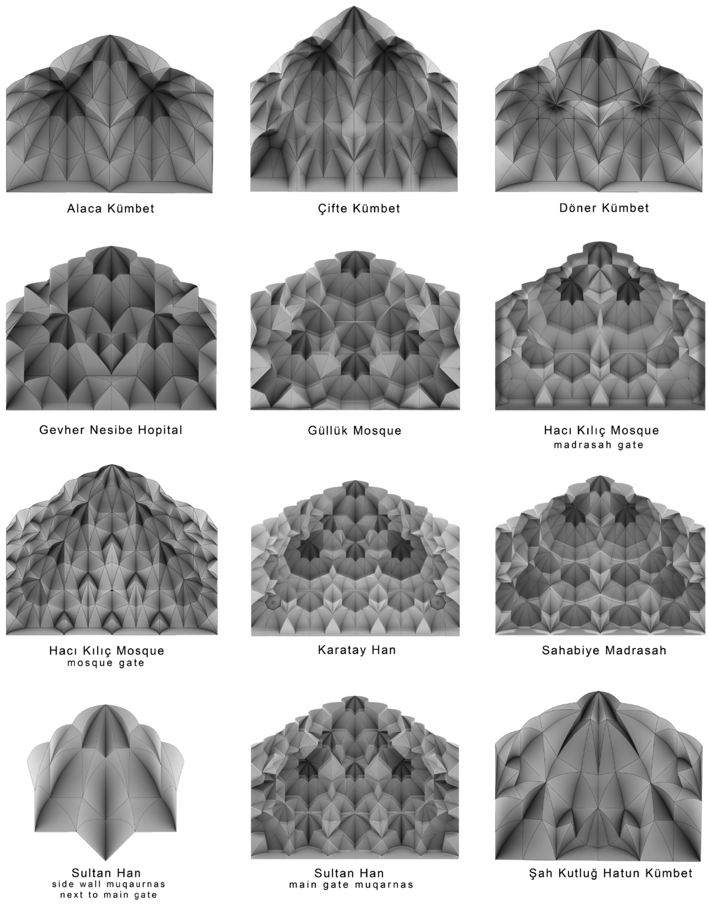

Fig. 9. The digital models of studied muqarnas in Kayseri.

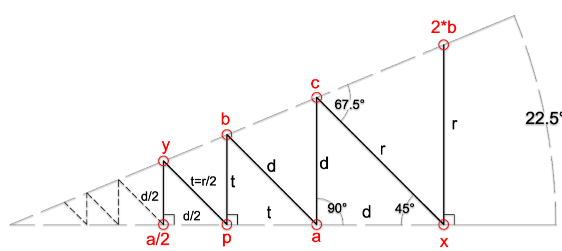

Fig. 10. The trigonometric relationships of the muqarnas units (a, b,c, d,. . . , etc.) discovered using 22.5°-67.5°-90° right triangles and hypothenus of 45°-45°-90° isosceles right triangles. This diagram shows harmonic progression among measured units of growth used in muqarnas plans. These discovered units are shown on plans in Figures 12 and 13.

types of right triangles (Figure 10). In this diagram a unit measurement is propagated by constructing 45°-45°-90° isosceles triangles and 22.5°-67.5°-90° right triangles. Of all the muqarnas that are studied, all of them presented proportional measurements found in the 22.5° trigonometric chart. Another aspect of this diagram is that the proportions appear to be harmonically related, resonating with the growth pattern found among symmetry axis.

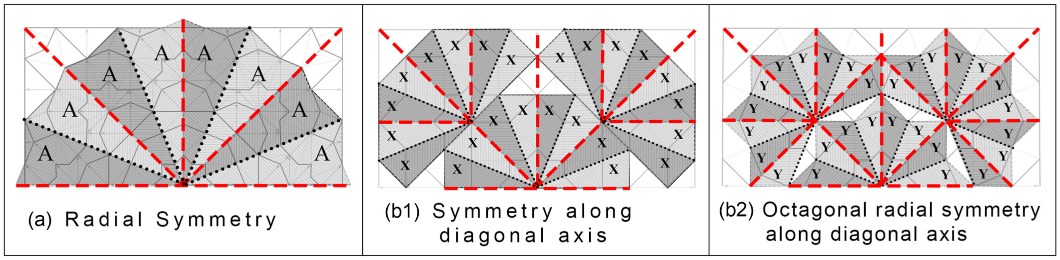

Fig. 11. Three types of muqarnas identified according to the types of symmetry found. Type A: Radial symmetry. Type B: Symmetry along diagonal axis (B1) and octagonal radial symmetry along diagonal axis (B2).

In the next phase, all the symmetry axis diagrams for 12 muqarnas are analyzed, and three different categories are identified (Figure 11). The first "radial symmetry" group follows the geometric principles observed in Gülük mosque where the growth of muqarnas in layers coincides between orthogonal and diagonal axis in an octagonal plan. In the second group, the predominant growth direction is defined as the diagonal axis where multiple bifurcations can occur. In this group, the development along the orthogonal axis are different to the diagonal axis. The last group presents a subgroup of the diagonal symmetry with the placement of an octagon along the diagonal axis placing radial symmetry toward the middle of the quadrant. With the identification of radial and mirror symmetries along the diagonal axis, the studied muqarnas can be categorized into three distinct groups shown below.

Figures 12 and 13 show the combination of computed plan drawings, branching growth diagrams, and morphological classifications that are constructed according to symmetry and proportional analysis for the 12 studied muqarnas in Kayseri. The first column shows the refined planimetric drawings that are extracted from photogrammetry scans showing muqarnas layers and triangulation of stones. The second column shows the growth diagrams that are constructed using symmetry lines between stones. For these, the primary and secondary symmetry lines are differentiated and asymmetry for the irregular pieces are associated with growth layers. The main principle behind these images is to highlight the inherent symmetry between parts and layers that characterize variations to a common archetype. With this perspective, the growth diagrams developed in the project describe hierarchies based on a tree graph starting from a single point and offer a mathematical model that facilitates comparison, categorization, modeling, and different muqarnas by considering the growth and dominant symmetry axes in terms of layers [16]. The third column shows the trigonometric analysis for plans showing common proportions used for the geometric design of muqarnas (Figures 12 and 13). The location of these partial plans are marked on the first column using red areas.

5 CONCLUSION

This study aims to present a systematic approach to document, analyse, and categorize a historical and ornamental building element entitled muqarnas. A novel ICM consisting of computational scanning, digital modeling and geometric analysis is introduced. In the scope of the study the proposed model is implemented into 12 muqarnas patterns extracted from 10 Anatolian Seljuk buildings located in Kayseri, Turkey. Geometrical analysis processes utilize the data obtained from the computational scanning (photogrammetry) models, three-dimensional mesh model of the muqarnas structures, geometric patterns of muqarnas projected onto two-dimensional planar surface, as well as the specific lenses of symmetry and graph-based diagrammatic representation. This study can be considered as a proof of concept of the proposed ICM. Adoption of symmetry and graph-based diagrammatic representation led to parametric growth models that have potential to offer a new understanding of cultural heritage and development of generative archetypes for studied models.

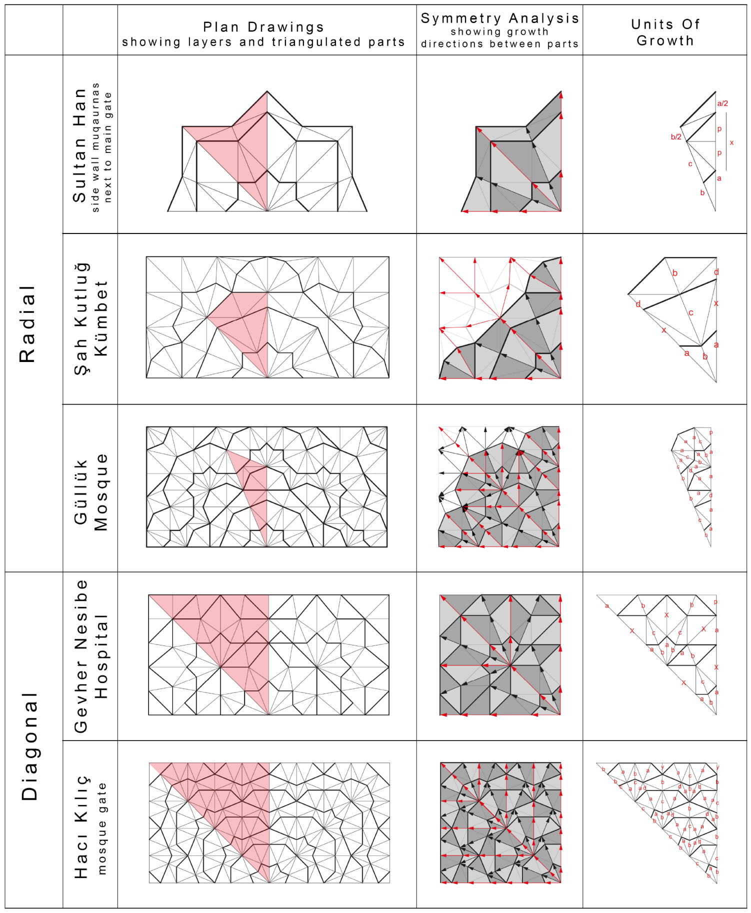

Fig. 12. The computational analysis of muqarnas in Kayseri showing radial symmetry from center and symmetry along diagonal growth axis.

During the research, the Gülük muqarnas provided a case study to analyze symmetry groups between stacked geometric layers. This investigation was extended toward a larger group of heritage structures in Kayseri to analyze common patterns and geometric relations of parts. This perspective led to the discovery of similar trigonometric angles and proportional relationships for the design of muqarnas in the Anatolian Seljuk period. These proportions were plotted using perpendicular and isosceles triangles formed at 22.5° showing harmonic distribution and progression between layers. While some of the muqarnas showed irregular development toward lower layers where the geometric parts deviated to other angles, the symmetry groups within pieces can be traced back to main growth axes. When the plans and models of the analyzed muqarnas were compared, similar growth patterns and hierarchical branching graphs were discovered. These were classified into three symmetry groups—radial, diagonal, and octagonal—all of which show common geometric types and patterned symmetry breaking mechanisms.

The presented work shows that muqarnas can be studied through notions of growth, branching patterns, and symmetry axes by considering recursive development of geometric layers inversely related to the physical construction and stacking of muqarnas pieces. This aspect is presented through the growth diagrams of the analyzed muqarnas structures, all of which utilize common proportions and branching angles from the tip (center),

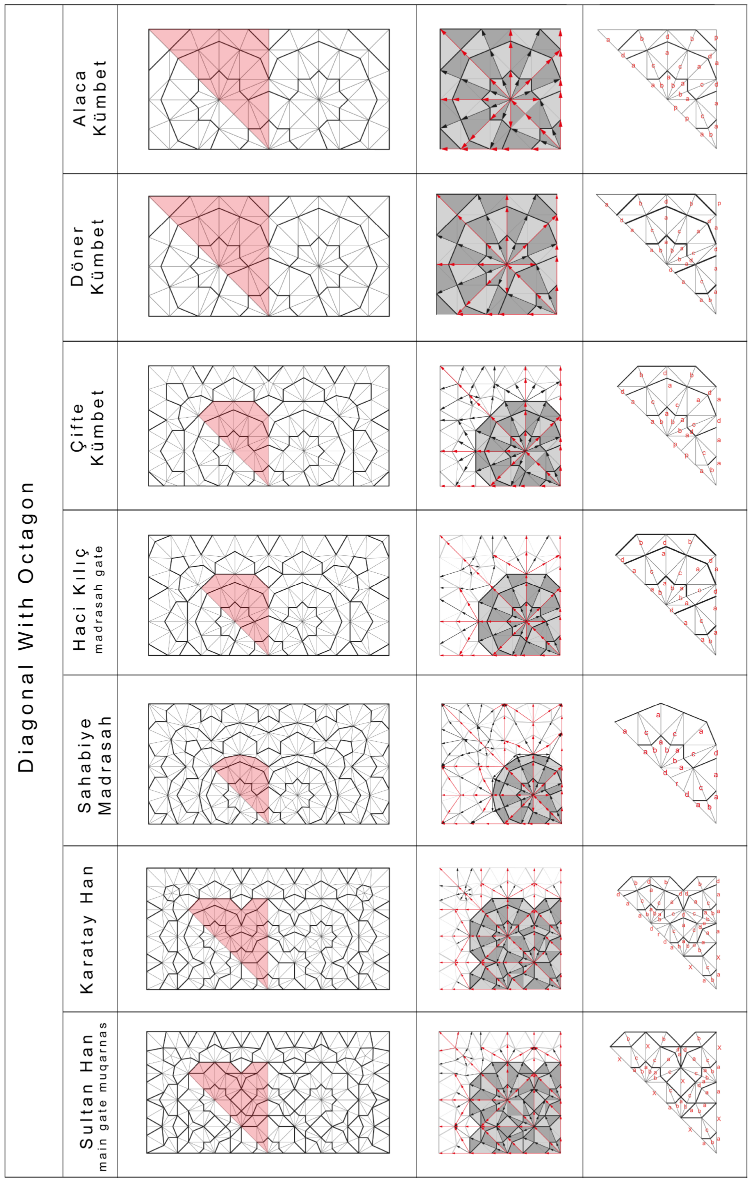

Fig. 13. The computational analysis of muqarnas in Kayseri showing octagonal radial symmetry along diagonal growth axis.

revealing a common developmental mechanism. The presented methodology combining computational modeling and analysis strategies can be extended toward other heritage sites to investigate parametric relationships between muqarnas structures. The provided directed tree graphs can inform potential application of recursive algorithms for parametric modeling and comparisons of the muqarnas archetype.

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