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Inverse coefficient problem for a second-order elliptic equation with nonlocal boundary conditions

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In this research article, the inverse problem of finding a time-dependent coefficient in a second-order elliptic equation is investigated. The existence and the uniqueness of the classical solution of the problem under consideration are established. Numerical tests using the finite-difference scheme combined with an iteration method are presented, and the sensitivity of this scheme with respect to noisy overdetermination data is illustrated. Copyright © 2015 John Wiley & Sons, Ltd.

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1. Introduction

The problem of identifying a coefficient in partial differential equation is an interesting problem for many scientists. For surveys on the subject, we refer the reader to [1–10] and the references therein. Recently, nonlocal boundary and overdetermination conditions have become a center of interest in the mathematical formulation [11–18].

Nonlocal problems are widely for mathematical modeling of various process of physics, chemistry, ecology, and industry. For example in [19], the authors considered a nonlocal elliptic problem appearing in the theory of plasma. Nonclassical boundary and initial-boundary value problems with integral and discrete nonlocal boundary conditions were studied for various equations. (see[19–21] and reference there in).

Various inverse problems for partial differential equations with nonlocal boundary conditions were studied in [16, 18, 20].

In the present research article, we consider the inverse problem for an elliptic equation with nonlocal boundary and integral overdetermination conditions. We obtain a uniqueness criterion and prove the existence of a solution of the inverse problem (1.1)-(1.4) by Fourier method. Also we construct the numerical procedure of this problem.

Let $T > 0$ be a fixed number and consider the inverse problem of finding a pair of functions $(r(t), u(x,t))$ satisfying the following elliptic equation

$$u_{tt} + u_{xx} = r(t)f(x,t),\ (x,t) \in Q_T = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}, \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = \varphi(x),\ u_t(x,T) = \psi(x),\ 0 \le x \le 1, \tag{1.2}$$

the nonlocal boundary conditions

$$u(0,t) = u(1,t),\ u_x(1,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

and the integral overdetermination condition

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$$\int_0^1 u(x,t)dx = E(t),\ 0 \le t \le T, \tag{1.4}$$

$f(x,t)$, $\varphi(x)$, $\psi(x)$ and $E(t)$ are given functions.

Definition 1
The pair $\{r(t),\ u(x,t)\}$ from the class $C[0,T] \times \left[C^2(Q_T) \cap C^1\left(\overline{Q}_T\right)\right]$ for which conditions (1.1)–(1.4) are satisfied is called the classical solution of the inverse problem (1.1)–(1.4).

The research article is organized as follows. In Section 2, the existence and the uniqueness of the solution of inverse problem (1.1)–(1.4) are proved by using the Fourier method. In Section 3, the numerical procedure for the solution of the inverse problem the finite-difference scheme combined with an iteration method is given. Finally, numerical experiments are presented and discussed in Section 4.

2. Existence and uniqueness of the solution of the inverse problem

The main result on existence and uniqueness of the solution of the inverse problem (1.1)–(1.4) is presented as follows.

Theorem 1
Suppose that the following conditions hold:

(A_1) $\varphi(x) \in C^3[0,1], \varphi(0) = \varphi(1), \varphi'(1) = 0, \varphi''(0) = \varphi''(1)$;
(A_2) $\psi(x) \in C^2[0,1], \psi(0) = \psi(1), \psi'(1) = 0$;
(A_3) $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T); f(0,t) = f(1,t), f_x(1,t) = 0, \int_0^1 f(x,t)dx \neq 0, 0 \le t \le T$;
(A_4) $E(t) \in C^2[0,T]$.

Then the inverse problem (1.1)–(1.4) has a unique solution for small T.

Proof
Consider the following systems of functions on the interval $[0,1]$:

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos(2\pi kx),\ X_{2k}(x) = 4(1-x)\sin(2\pi kx),\ k = 1,2,\ldots, \tag{2.1}$$

$$Y_0(x) = x,\ Y_{2k-1}(x) = x\cos(2\pi kx),\ Y_{2k}(x) = \sin(2\pi kx),\ k = 1,2,\ldots \tag{2.2}$$

The system of functions (2.1) and (2.2) arise in [22] for the solution of a nonlocal boundary value problem in heat conduction.
It is easy to verify that the system of functions (2.1) and (2.2) are biorthogonal on [0, 1]. They are also Riesz bases in $L_2[0,1]$ ([17]).
By applying the standard procedure of the Fourier method, we can write the solution of (1.1)–(1.3) in the following form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x),$$

where

$$u_k(t) = \int_0^1 u(x,t)Y_k(x)dx, k = 0,1,2,\ldots$$

is the solution of the following system:

$$u_0''(t) = r(t)f_0(t),\ 0 \le t \le T,$$

$$u_{2k}''(t) - \lambda_k^2 u_{2k}(t) = r(t)f_{2k}(t),\ 0 \le t \le T;\ k = 1,2,\ldots,$$

$$u_{2k-1}''(t) - \lambda_k^2 u_{2k-1}(t) = r(t)f_{2k-1}(t) + 2\lambda_k u_{2k}(t),\ 0 \le t \le T;\ k = 1,2,\ldots,$$

$$u_k(0) = \varphi_k, u_k'(T) = \psi_k,\ k = 0,1,\ldots,$$

where $\lambda_k = 2\pi k, .\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, f_k(t) = \int_0^1 f(x,t)Y_k(x)dx, \psi_k = \int_0^1 \psi(x)Y_k(x)dx, k = 0,1,2,\ldots.$

Then we obtain the solution of (1.1)–(1.3) in the following form for arbitrary $r(t) \in C[0,T]$[18]:

$$\begin{aligned} u(x,t) = {} & \left(\varphi_0 + \psi_0 t + \int_0^T r(\tau) G_0(t,\tau) f_0(\tau) d\tau\right) X_0(x) \\ & + \sum_{k=1}^{\infty} \left\{ \frac{\cosh(\lambda_k(T-t))}{\cosh(\lambda_k T)} \varphi_{2k} + \frac{\sinh(\lambda_k t)}{\lambda_k \cosh(\lambda_k T)} \psi_{2k} + \int_0^T r(\tau) G_k(t,\tau) f_{2k}(\tau) d\tau \right\} X_{2k}(x) \\ & + \sum_{k=1}^{\infty} \left\{ \frac{\cosh(\lambda_k(T-t))}{\cosh(\lambda_k T)} \varphi_{2k-1} + \frac{\sinh(\lambda_k t)}{\lambda_k \cosh(\lambda_k T)} \psi_{2k-1} + \int_0^T r(\tau) G_k(t,\tau) f_{2k-1}(\tau) d\tau \right. \\ & \quad + \frac{1}{\cosh^2(\lambda_k T)} \left[(T \sinh(\lambda_k t) + t \cosh(\lambda_k T) \sinh(\lambda_k (T-t))) \right. \varphi_{2k} \\ & \quad + \left[-T \sinh(\lambda_k T) \sinh(\lambda_k t) + t \cosh(\lambda_k T) \cosh(\lambda_k t) - \frac{1}{\lambda_k} \cosh(\lambda_k T) \sinh(\lambda_k t) \right] \left. \frac{1}{\lambda_k} \psi_{2k} \right] \\ & \quad \left. + \int_0^T G_k(t,\tau) \left(\int_0^T r(\xi) G_k(t,\xi) f_{2k}(\xi) d\xi \right) d\tau \right\} X_{2k-1}(x), \end{aligned} \tag{2.3}$$

where

$$G_0(t,\tau) = \begin{cases} -t, \ t \in [0,\tau], \\ -\tau, \ t \in [\tau,T], \end{cases} \tag{2.4}$$

$$G_k(t,\tau) = \begin{cases} \frac{-1}{2\lambda_k \cosh(\lambda_k T)} \left[\sinh(\lambda_k(T+t-\tau)) - \sinh(\lambda_k(T-(t+\tau)))\right], \ t \in [0,\tau], \\ \frac{1}{2\lambda_k \cosh(\lambda_k T)} \left[\sinh(\lambda_k(T-(t+\tau))) - \sinh(\lambda_k(T-(t-\tau)))\right], \ t \in [\tau,T]. \end{cases} \tag{2.5}$$

Under the conditions (A_1) –(A_3) the series (2.3), its x-partial derivative and its t-partial derivative are uniformly convergent in $\overline{Q}_T$ because their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$, $u_x(x,t)$, and $u_t(x,t)$ are continuous in $\overline{Q}_T$. In addition, the tt -partial derivative and the xx-second-order partial derivative series are uniformly convergent in Q_T. Thus, we have $u(x,t) \in C^2(Q_T) \cap C^1(\overline{Q}_T)$. In addition, $u_{tt}(x,t)$ is continuous in $\overline{Q}_T$. Differentiating (0.4) under the condition (A_4) , we obtain a Fredholm integral equation of the second kind as follows:

$$r(t) = F(t) + \int_0^T K(t,\tau) r(\tau) d\tau, \tag{2.6}$$

where

$$F(t) = \frac{1}{\int_0^1 f(x,t)dx} \left[E''(t) - 4 \sum_{k=1}^{\infty} \lambda_k \frac{\cosh(\lambda_k(T-t))}{\cosh(\lambda_k T)} \varphi_{2k} + \frac{\sinh(\lambda_k t)}{\lambda_k \cosh(\lambda_k T)} \psi_{2k} \right], \tag{2.7}$$

$$K(t,\tau) = \frac{\sum_{k=1}^{\infty} G_k(t,\tau) f_{2k}(\tau)}{\int_0^1 f(x,t)dx}. \tag{2.8}$$

Under the conditions $(A_1) - -(A_4)$,the series in the (2.7) and (2.8) are convergent. So, the right-hand side $F(t)$ and the kernel $K(t,\tau)$ are continuous functions in $[0,T]$ and $[0,T] \times [0,T]$, respectively. We therefore obtain a unique function $r(t)$, continuous on $[0,T]$, which, together with the solution of the problem (1.1)-(1.3) given by the Fourier series (2.3), form the unique solution of the inverse problem (1.1)–(1.4) for small T, where $T < 1/ \max\limits_{t,\tau \in [0,T]} (K(t,\tau))$ by using the theory of the existence and uniqueness of the solution of the Fredholm integral equation of the second kind. □

Theorem 1 has been proved.

3. Numerical method

We use the finite-difference method with an iteration to problem (1.1)–(1.4). We use MATLAB for the programming.

We subdivide the intervals $[0,1]$ and $[0,T]$ into subintervals M and N of equal lengths $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$, respectively.

The finite-difference scheme for (1.1)-(1.4) is as follows:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2} + \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} = r^j f_i^j, \tag{3.1}$$

$$u_i^0 = \phi_i,\ u_i^N = \tau\Psi_i + u_i^{N-1}, \tag{3.2}$$

$$u_0^j = u_M^j,\ u_{M-1}^j = u_{M+1}^j, \tag{3.3}$$

where $1 \le i \le M$ and $0 \le j \le N$ are the indices for the spatial and time steps, respectively, $u_i^j = u(x_i, t_j)$, $r^j = r(t_j)$, $\phi_i = \varphi(x_i)$, $\Psi_i = \psi(x_i)$, $f_i^j = f(x_i, t_j)$, $x_i = ih$, and $t_j = j\tau$.

From (3.1), we can write the following:

$$u_i^{j+1} = 2(1+R)u_i^j - u_i^{j-1} - R(u_{i+1}^j + u_{i-1}^j) + \tau^2 r^j f_i^j \tag{3.4}$$

where $R = \tau^2/h^2$.

Now let us construct the iteration. First, integrating the equation (1.1) with respect to x from 0 to 1 and using (1.3) and (1.4), we obtain the following:

$$r(t) = \frac{E''(t) - u_x(0,t)}{\int_0^1 f(x,t)dx}. \tag{3.5}$$

The finite-difference approximation of (3.5) is as follows:

$$r^j = \frac{\left[(E^{j+1} - 2E^j + E^{j-1})/\tau^2\right] - \left[\left(u_1^j - u_0^j\right)/h\right]}{(fin)^j} \tag{3.6}$$

where $E^j = E(t_j)$, and $\int_0^1 f(x,t_j)dx$ is approximated by trapezodial formula as $(fin)^j = \int_0^1 f(x,t_j)dx = h(\frac{f_1^j}{2} + f_2^j + f_3^j + \ldots + f_{M-1}^j + \frac{f_M^j}{2})$, $j = 0, 1, \ldots, N$.

We denote the values of r^j, u_i^j at the s-th iteration step $r^{j(s)}$, $u_i^{j(s)}$, respectively. In numerical computation, because the time step is very small, we can take $r^{j+1(0)} = r^j$, $u_i^{j+1(0)} = u_i^j$, $j = 0, 1, 2, ...N$, $i = 1, 2, \ldots, M$. At each $(s+1)$ -th iteration step, we first determine $r^{j(s+1)}$ from the formula as follows:

$$r^{j(s+1)} = \frac{\left[(E^{j+1} - 2E^j + E^{j-1})/\tau^2\right] - \left[\left(u_1^{j(s)} - u_0^{j(s)}\right)/h\right]}{(fin)^j}. \tag{3.7}$$

Then from (3.4), we determine

$$u_i^{j+1(s+1)} = 2(1+R)u_i^{j+1(s)} - u_i^{j(s)} - R(u_{i+1}^{j+1(s)} + u_{i-1}^{j+1(s)}) + \tau^2 r^{j(s+1)} f_i^j. \tag{3.8}$$

If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped, and we accept the corresponding values $r^{j+1(s+1)}$ and $u_i^{j+1(s+1)} (i = 1, 2, \ldots, M)$ as r^{j+1} and u_i^{j+1} $(i = 1, 2, \ldots, M, j = 0, 2, \ldots, N-1)$, respectively.

4. Numerical example

Example 1

Consider the inverse problem (1.1)–(1.4) with the following:

$$F(x,t) = (x^3 - 2x^2 + 7x + 1)\exp(t),$$
$$\varphi(x) = x^3 - 2x^2 + x + 5,\ \psi(x) = \left(x^3 - 2x^2 + x + 5\right)\exp(1)\ E(t) = \frac{61}{12}\exp(t).$$

It is easy to check that the exact solution is as follows:

$$\{r(t),\ u(x,t)\} = \left\{\exp(2t),\ \left(x^3 - 2x^2 + x + 5\right)\exp(t)\right\}.$$

We use the finite-difference scheme and the iteration, which are explained in the previous section. In result, we obtain Figures 1 and 2 for exact and approximate values of $r(t)$ and $u(x,t)$. The step sizes are $h = 0.02$ and $\tau = 0.02$. The prescribed tolerance of the iteration is $h/4$.

Next, we will illustrate the stability of the numerical solution with respect to the noisy overdetermination data (1.4), defined by the function as follows:

$$E_\gamma(t) = E(t)(1 + \gamma\theta), \tag{4.1}$$

where γ is the percentage of noise and θ are random variables generated from a uniform distribution in the interval $[-1, 1]$. These random variables are generated using the *rand* command in MATLAB.

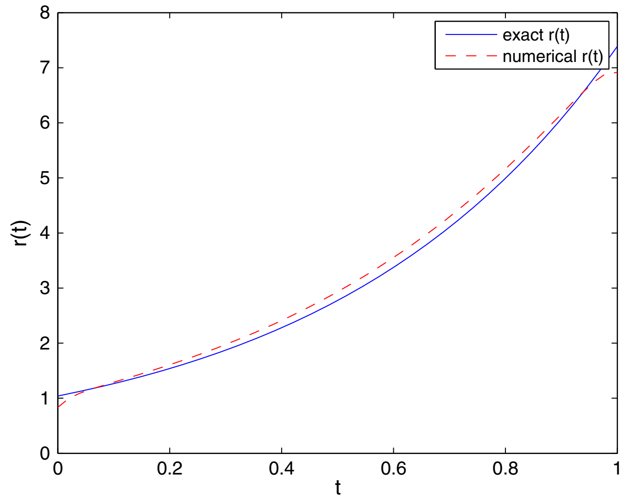

Figure 1. The exact and approximate solutions of r(t).

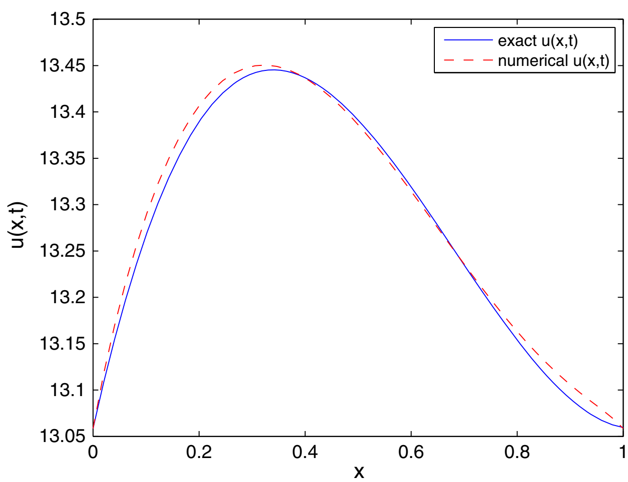

Figure 2. The exact and approximate solutions of u(x,1).

Figure 3 shows the exact and the numerical solutions of $r(t)$ when the input data (1.4) are contaminated by $\gamma = 3, 5$, and 10% noise. From these figures, it can be seen that the numerical solution becomes unstable as the input data are contaminated with noise. Under the random noisy input (4.1), its second derivative present in (3.5) is unstable if it is calculated using simply finite differences. In order to obtain a stable numerical derivative, we employ the mollification method with a Gaussian mollifier, see [23] , given by the following:

$$J_\delta(t) = \frac{1}{\delta\sqrt{\pi}} \exp(-t^2/\delta^2), \tag{4.2}$$

where $\delta > 0$ is the radius of mollification (or the regularization parameter) acting as an averaging filter. Its choice is based on standard methods for choosing the regularization parameter in ill-posed problems such as the generalized cross-validation criterion. The mollification of the noisy data (4.1) is performed through the convolution as follows:

$$J_\delta * E(t) = \int_{-\infty}^{\infty} J_\delta(\tau)E(t-\tau)d\tau. \tag{4.3}$$

We notice that the mollifier J_δ is always positive and becomes close to zero outside the interval centered at the origin and of radius 3δ. Good results for the derivative $E''(t)$ are therefore expected in the interval $[3\delta, T-3\delta]$. We remark that although $E_\gamma(t)$ given by (4.1) is non-smooth, its mollification $J_\delta * E_\gamma(t)$ is a C^∞ function, hence differentiable. The mollified derivative is then computed using that

$$(J_\delta * E_\gamma)'(t) = J_\delta * E'_\gamma(t) = J'_\delta * E_\gamma(t)$$

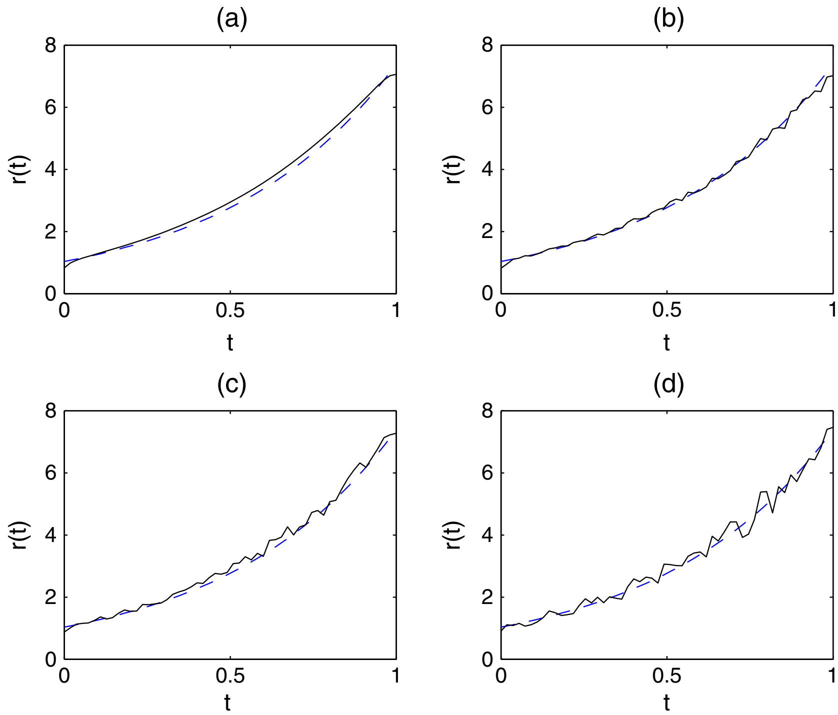

Figure 3. The exact and the approximate solutions of r(t),(a) for 0% noisy data, (b) for 3% noisy data, (c) for 5% noisy data, and (d) for 10% noisy data. In figure (a)–(d), the exact solution is shown with dashes line.

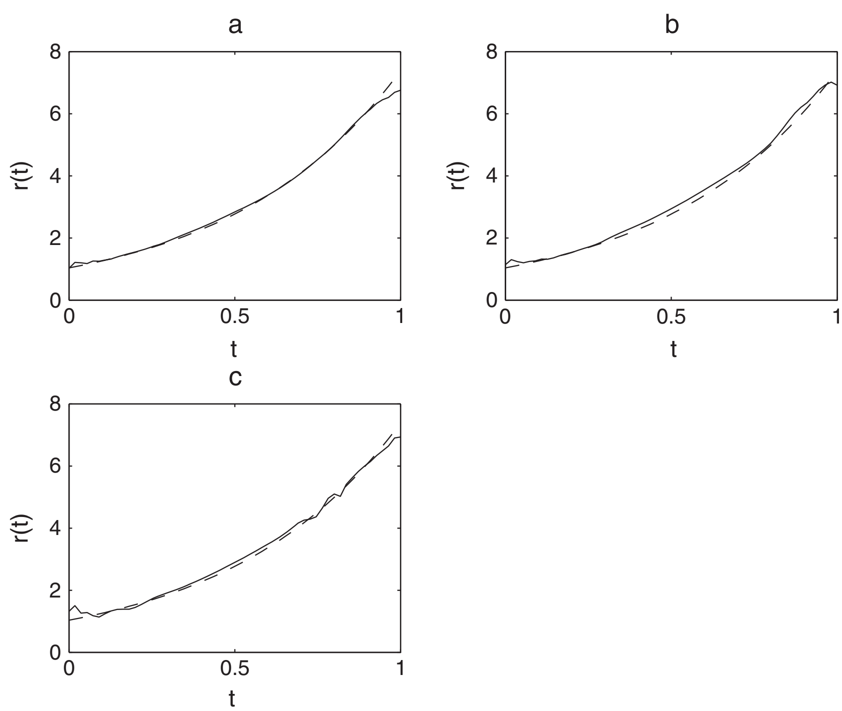

Figure 4. The exact and the approximate solutions of r(t), after mollification, (a) for 3% noisy data, (b) for 5% noisy data, and (c) for 10% noisy data. In figure (a)–(c), the exact solution is shown with dashes line.

We then use this mollified data to approximate (3.6), that is, we replace the finite-difference quotients $(E^{j+1} - 2E^j + E^{j-1})/\tau^2$ in (3.6)-(3.8) by $(J_\delta * E_\gamma)''\,(t_j)$ for $j = 0, 1, \dots, N$.

Figure 4 shows the exact and the numerical solutions of $r(t)$, obtained after mollification, when the input data (1.4) are contaminated by 3, 5, and 10% noise. From these figures, it can be seen that the application of the mollification to stabilise the derivative of the noisy function $E(t)$ produces stable numerical solution for $r(t)$, compare with the previously obtained unstable solutions in Figure 2. Also, from all the numerical results presented in this section, it can be seen that the numerical solutions become more accurate as the amount of noise included in the input data decreases.

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