

# An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions

Mansur I. Ismailov<sup>a,\*†</sup> and Fatma Kanca<sup>b</sup>

Communicated by A. Kirsch

In this paper, the inverse problem of finding the time-dependent coefficient of heat capacity together with the solution of heat equation with nonlocal boundary and overdetermination conditions is considered. The existence, uniqueness and continuous dependence upon the data are studied. Some considerations on the numerical solution for this inverse problem are presented with the examples. Copyright © 2011 John Wiley & Sons, Ltd.

**Keywords:** parabolic equation; inverse problem; nonorthogonal systems of functions; nonlocal boundary conditions; integral overdetermination condition

## 1. Introduction

In

$$Q_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$$

consider the equation

$$u_t = u_{xx} - a(t)u + F(x, t) \quad (1)$$

with the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1 \quad (2)$$

nonlocal boundary conditions

$$u(0, t) = u(1, t), \quad u_x(1, t) = 0, \quad 0 \leq t \leq T \quad (3)$$

and overdetermination condition

$$\int_0^1 u(x, t) dx = g(t), \quad 0 \leq t \leq T. \quad (4)$$

The problem of finding a pair  $\{a(t), u(x, t)\}$  in (1)–(4) will be called an inverse problem.

### Definition 1

The pair  $\{a(t), u(x, t)\}$  from the class  $C[0, T] \times C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$  for which conditions (1)–(4) are satisfied and  $a(t) \geq 0$  on the interval  $[0, T]$ , is called a classical solution of the inverse problem (1)–(4).

The problems of finding a coefficient  $a(t)$  together with the solution  $u(x, t)$  of heat equation (1) with the integral overdetermination condition (4) and different nonlocal boundary conditions are studied in [1, 2]. The interested reader can find different inverse problems for heat equations with nonlocal boundary and overdetermination conditions in [3, 4].

<sup>a</sup>Department of Mathematics, Gebze Institute of Technology, Gebze-Kocaeli 41400, Turkey

<sup>b</sup>Department of Mathematics, Kocaeli University, Kocaeli 41380, Turkey

\*Correspondence to: Mansur I. Ismailov, Department of Mathematics, Gebze Institute of Technology, Gebze-Kocaeli 41400, Turkey.

†E-mail: mismailov@gyte.edu.tr

These kinds of conditions such as (4) arise from many important applications in heat transfer, thermoelasticity, control theory, life sciences, etc. For example, in heat propagation in a thin rod in which the law of variation  $g(t)$  of the total quantity of heat in the rod is given in [5].

The paper is organized as follows. In Section 2, the nonorthogonal systems of functions, by using these systems it is possible to expand the generalized Fourier series, are introduced. In Section 3, the existence and uniqueness of the solution of inverse problem (1)–(4) is proved. In Section 4, the continuous dependence upon the data of the solution of the inverse problem is shown. Then, in Section 5, the numerical solution for the inverse problem is presented with the examples. Finally, some discussions related to causing difficulties in numerical solution of the inverse problems are given.

## 2. Some preliminary facts on the nonorthogonal systems of functions

Consider the following systems of functions on the interval  $[0,1]$ :

$$X_0(x) = 2, \quad X_{2k-1}(x) = 4 \cos 2\pi kx, \quad X_{2k}(x) = 4(1-x) \sin 2\pi kx, \quad k = 1, 2, \dots \quad (5)$$

$$Y_0(x) = x, \quad Y_{2k-1}(x) = x \cos 2\pi kx, \quad Y_{2k}(x) = \sin 2\pi kx, \quad k = 1, 2, \dots \quad (6)$$

Systems (5) and (6) arise in [5] for the solution of a nonlocal boundary value problem in heat conduction. For the systems of functions (5) and (6), the following lemmas hold.

### Lemma 1

The systems of functions (5) and (6) are biorthonormal on  $[0,1]$ .

The proof of this lemma is trivial.

### Lemma 2

The systems of functions (5) and (6) are complete in  $L_2[0,1]$ .

### Proof

Let  $f(x) \in L_2[0,1]$  be orthogonal to the functions of system (5).  $f(x)$  can be represented by the series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin 2\pi nx \quad (7)$$

that converges in  $L_2[0,1]$ . Since  $f(x)$  is orthogonal to (5),

$$\begin{aligned} 0 &= \int_0^1 f(x) 4(1-x) \sin 2\pi kx \, dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^1 4(1-x) \sin 2\pi nx \sin 2\pi kx \, dx = B_k, \quad k = 1, 2, \dots \end{aligned}$$

Thus  $B_k = 0, k = 1, 2, \dots$ , then  $f(x) = 0$ , from (7). The completeness of the system (6) is shown analogously.  $\square$

### Lemma 3

The systems of functions (5) and (6) are Riesz bases in  $L_2[0,1]$ .

### Proof

According to the results in book p. 310 [6], the system of functions (5) is Riesz basis in  $L_2[0,1]$  since it is complete in  $L_2[0,1]$  by Lemma 2 and the series

$$\begin{aligned} &4 \left( \int_0^1 f(x) \, dx \right)^2 + 16 \sum_{k=1}^{\infty} \left[ \left( \int_0^1 f(x) \cos 2\pi kx \, dx \right)^2 + \left( \int_0^1 f(x)(1-x) \sin 2\pi kx \, dx \right)^2 \right] \\ &\left( \int_0^1 xf(x) \, dx \right)^2 + \sum_{k=1}^{\infty} \left[ \left( \int_0^1 xf(x) \cos 2\pi kx \, dx \right)^2 + \left( \int_0^1 f(x) \sin 2\pi kx \, dx \right)^2 \right] \end{aligned}$$

are convergent for each  $f(x) \in L_2[0,1]$ . Similarly, it is shown that system (6) is Riesz basis in  $L_2[0,1]$ .  $\square$

## 3. Existence and uniqueness of the solution of the inverse problem

We have the following assumptions on  $\varphi, g$  and  $F$ .

- (A<sub>1</sub>)<sub>1</sub>  $\varphi(x) \in C^4[0, 1]$ ;  
 (A<sub>1</sub>)<sub>2</sub>  $\varphi(0) = \varphi(1), \varphi'(1) = 0, \varphi''(0) = \varphi''(1)$ ;  
 (A<sub>1</sub>)<sub>3</sub>  $\varphi_{2k} \leq 0, k = 1, 2, \dots$ ,  
 (A<sub>2</sub>)<sub>1</sub>  $g(t) \in C^1[0, T]$ ;  
 (A<sub>2</sub>)<sub>2</sub>  $g(0) = \int_0^1 \varphi(x) dx$ ;  
 (A<sub>2</sub>)<sub>3</sub>  $g(t) > 0, g'(t) \leq 0, \forall t \in [0, T]$ ,  
 (A<sub>3</sub>)<sub>1</sub>  $F(x, t) \in C(\overline{Q_T}); F(x, t) \in C^4[0, 1] \quad \forall t \in [0, T]$ ;  
 (A<sub>3</sub>)<sub>2</sub>  $F(0, t) = F(1, t), F_x(1, t) = 0, F_{xx}(0, t) = F_{xx}(1, t)$ ;  
 (A<sub>3</sub>)<sub>3</sub>  $F_0(t) \geq 0, F_{2k}(t) \geq 0 \quad \forall t \in [0, T], \min_{0 \leq t \leq T} F_{2k}(t) + [e^{-(2\pi k)^2 T} - 1] \max_{0 \leq t \leq T} F_{2k}(t) \geq 0, k = 1, 2, \dots$ ,

where  $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx, F_k(t) = \int_0^1 F(x, t) Y_k(x) dx, k = 0, 1, 2, \dots$ .

**Remark 1**

There are functions  $\varphi, g$  and  $F$  satisfying (A<sub>1</sub>)–(A<sub>3</sub>). For example

$$\begin{aligned} \varphi(x) &= 1 + \cos 2\pi x, \\ g(t) &= \exp(-(2\pi)^2 t), \\ F(x, t) &= (2\pi)^2 \cos 2\pi x \exp(-(2\pi)^2 t) + 2t(1 + \cos 2\pi x) \exp(-(2\pi)^2 t + 10t^2). \end{aligned}$$

The main result is presented as follows.

**Theorem 1**

Let (A<sub>1</sub>)–(A<sub>3</sub>) be satisfied. Then the inverse problem (1)–(4) has a unique solution for small  $T$ .

**Proof**

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (1)–(3) for arbitrary  $a(t) \in C[0, T]$ :

$$\begin{aligned} u(x, t) &= \left[ \varphi_0 e^{-\int_0^t a(s) ds} + \int_0^t F_0(\tau) e^{-\int_\tau^t a(s) ds} d\tau \right] X_0(x) + \sum_{k=1}^{\infty} \left[ \varphi_{2k} e^{-(2\pi k)^2 t - \int_0^t a(s) ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s) ds} d\tau \right] X_{2k}(x) \\ &+ \sum_{k=1}^{\infty} \left[ (\varphi_{2k-1} - 4\pi k \varphi_{2k}) e^{-(2\pi k)^2 t - \int_0^t a(s) ds} \right] X_{2k-1}(x) + \sum_{k=1}^{\infty} \left[ \int_0^t (F_{2k-1}(\tau) - 4\pi k F_{2k}(\tau)(t-\tau)) e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s) ds} d\tau \right] X_{2k-1}(x). \end{aligned} \tag{8}$$

Under conditions (A<sub>1</sub>)<sub>1</sub> and (A<sub>3</sub>)<sub>1</sub> the series (8) and  $\sum_{k=1}^{\infty} \partial / \partial x$  converge uniformly in  $\overline{Q_T}$  since their majorizing sums are absolutely convergent. Therefore, their sums  $u(x, t)$  and  $u_x(x, t)$  are continuous in  $\overline{Q_T}$ . In addition, the series  $\sum_{k=1}^{\infty} \partial / \partial t$  and  $\sum_{k=1}^{\infty} \partial^2 / \partial x^2$  are uniformly convergent for  $t \geq \varepsilon > 0$  ( $\varepsilon$  is an arbitrary positive number). Thus,  $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})$  and satisfies condition (1)–(3). In addition,  $u_t(x, t)$  is continuous in  $\overline{Q_T}$  because the majorizing sum of  $\sum_{k=1}^{\infty} \partial / \partial x$  is absolutely convergent under the condition  $\varphi''(0) = \varphi''(1)$  and  $F_{xx}(0, t) = F_{xx}(1, t)$  in  $\overline{Q_T}$ . Differentiating (4) under the condition (A<sub>2</sub>)<sub>1</sub>, we obtain

$$\int_0^1 u_t(x, t) dx = g'(t), \quad 0 \leq t \leq T. \tag{9}$$

(8) and (9) yield

$$a(t) = P[a(t)], \tag{10}$$

where

$$P[a(t)] = \frac{1}{g(t)} \left( -g'(t) + 2F_0(t) + \sum_{k=1}^{\infty} \left( \frac{2}{\pi k} F_{2k}(t) - 8\pi k \varphi_{2k} e^{-(2\pi k)^2 t - \int_0^t a(s) ds} \right) \right) - \frac{1}{g(t)} \sum_{k=1}^{\infty} 8\pi k \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_\tau^t a(s) ds} d\tau. \tag{11}$$

Let us denote

$$C^+[0, T] = \{a(t) \in C[0, T]: a(t) \geq 0\}.$$

It is easy to verify that under conditions  $(A_1)_3$ ,  $(A_2)_3$  and  $(A_3)_3$ ,

$$P: C^+[0, T] \rightarrow C^+[0, T].$$

Let us show that  $P$  is a contraction mapping in  $C^+[0, T]$ , for small  $T$ . Indeed, for  $\forall a(t), b(t) \in C^+[0, T]$

$$|P[a(t)] - P[b(t)]| \leq \frac{1}{|g(t)|} \sum_{k=1}^{\infty} 8\pi k |\varphi_{2k}| |e^{-\int_0^t a(s) ds} - e^{-\int_0^t b(s) ds}| + \frac{1}{|g(t)|} \int_0^T \sum_{k=1}^{\infty} 8\pi k |F_{2k}(\tau)| |e^{-\int_{\tau}^t a(s) ds} - e^{-\int_{\tau}^t b(s) ds}| d\tau.$$

Denote

$$\sum_{k=1}^{\infty} 8\pi k |\varphi_{2k}| = c_1, \quad \int_0^T \sum_{k=1}^{\infty} 8\pi k |F_{2k}(\tau)| d\tau = c_2, \quad \max_{0 \leq t \leq T} \frac{1}{|g(t)|} = c_3.$$

Since  $a(t) \geq 0$  and  $b(t) \geq 0$  the estimates

$$|e^{-\int_0^t a(s) ds} - e^{-\int_0^t b(s) ds}| \leq T \max_{0 \leq t \leq T} |a(t) - b(t)|, \quad |e^{-\int_{\tau}^t a(s) ds} - e^{-\int_{\tau}^t b(s) ds}| \leq T \max_{0 \leq t \leq T} |a(t) - b(t)|$$

are true by using the mean value theorem. From the last inequalities, we obtain

$$\max_{0 \leq t \leq T} |P[a(t)] - P[b(t)]| \leq \alpha \max_{0 \leq t \leq T} |a(t) - b(t)|,$$

where  $\alpha = c_3(c_1 + c_2)T$ . In the case  $\alpha < 1$ , Equation (10) has a unique solution  $a(t) \in C^+[0, T]$ , by the Banach fixed point theorem.

Now, let us show that the solution  $(a, u)$ , obtained for (1)–(4), is unique. Suppose that  $(b, v)$  is also a solution pair of (1)–(4). Then the uniqueness of the representation of the solution, we have

$$\begin{aligned} u(x, t) - v(x, t) &= [\varphi_0(e^{-\int_0^t a(s) ds} - e^{-\int_0^t b(s) ds})]X_0(x) \\ &+ \left[ \int_0^t F_0(\tau)(e^{-\int_{\tau}^t a(s) ds} - e^{-\int_{\tau}^t b(s) ds}) d\tau \right] X_0(x) + \sum_{k=1}^{\infty} \varphi_{2k} e^{-(2\pi k)^2 t} (e^{-\int_0^t a(s) ds} - e^{-\int_0^t b(s) ds}) X_{2k}(x) \\ &+ \sum_{k=1}^{\infty} \left( \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau)} (e^{-\int_{\tau}^t a(s) ds} - e^{-\int_{\tau}^t b(s) ds}) d\tau \right) X_{2k}(x) \\ &+ \sum_{k=1}^{\infty} (\varphi_{2k-1} - 4\pi k \varphi_{2k} t) e^{-(2\pi k)^2 t} (e^{-\int_0^t a(s) ds} - e^{-\int_0^t b(s) ds}) X_{2k-1}(x) \\ &+ \sum_{k=1}^{\infty} \left( \int_0^t (F_{2k-1}(\tau) - 4\pi k F_{2k}(\tau)(t-\tau)) e^{-(2\pi k)^2 (t-\tau)} (e^{-\int_{\tau}^t a(s) ds} - e^{-\int_{\tau}^t b(s) ds}) d\tau \right) X_{2k-1}(x), \end{aligned} \quad (12)$$

$$a(t) - b(t) = \frac{1}{g(t)} \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 t} (e^{-\int_0^t b(s) ds} - e^{-\int_0^t a(s) ds}) + \frac{1}{g(t)} \sum_{k=1}^{\infty} 8\pi k \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau)} (e^{-\int_{\tau}^t b(s) ds} - e^{-\int_{\tau}^t a(s) ds}) d\tau.$$

Following the same procedure leading to (11), we obtain

$$\|a - b\|_{C[0, T]} \leq \alpha \|a - b\|_{C[0, T]}$$

which implies that  $a = b$ . By substituting  $a = b$  into (12), we have  $u = v$ .

Theorem 1 has been proved. □

**Remark 2**

There are three types of conditions on the data of the inverse problem (1)–(4): the smoothness conditions  $((A_1)_1, (A_2)_1$  and  $(A_3)_1$ ), the consistency conditions  $((A_1)_2, (A_2)_2$  and  $(A_3)_2$ ) and the estimation conditions  $((A_1)_3, (A_2)_3$  and  $(A_3)_3$ ).

The smoothness and consistency types of conditions are well known in the theory of BVP (boundary value problems). It is known in Fourier analysis that some of these conditions are necessary but some of them are sufficient for existence of the classical solution. For example,  $\varphi(x) \in C^2[0, 1]$  with  $\varphi(0) = \varphi(1)$ ,  $\varphi'(1) = 0$  are necessary conditions; however,  $\varphi'''(x) \in C^1[0, 1]$  with  $\varphi''(0) = \varphi''(1)$  are sufficient for Theorem 1. It is useful to note that the condition  $\varphi^{(i, v)}(x) \in C[0, 1]$  can be changed with the weaker condition  $\varphi^{(i, v)}(x) \in L_2[0, 1]$ . Similar considerations are true for the conditions  $(A_3)_1$  and  $(A_3)_2$ . However, the condition that  $g(t) \in C[0, T]$  and  $(A_2)_2$  are necessary and  $g'(t) \in C[0, T]$  are sufficient and all of the conditions  $(A_1)_3, (A_2)_3$  and  $(A_3)_3$  are sufficient for the Theorem 1. Notice that such types of conditions are arisen in the inverse BVP for parabolic equations (see [3]).

**Remark 3**

The existence and uniqueness of the solution of the inverse problem (1)–(4) are obtained in  $Q_T$  for small  $T$ . The smallest value of  $T$  is sufficient for application of the Banach Fixed-point Theorem. Such types of conditions are also popular in the theory of inverse BVP. When the even numbered Fourier coefficients of the data  $\varphi(x)$  and  $F(x, t)$  are zero ( $\varphi_{2k} = 0, F_{2k}(t) = 0, k = 0, 1, \dots$ ), the conditions  $(A_1)_3$  and  $(A_3)_3$  vanish. In this case Theorem 1 is trivial and it is not necessary to apply a fixed-point theorem, therefore, the solution of the inverse problem (1)–(4) exists for not only small  $T > 0$ .

#### 4. Continuous dependence of $(a, u)$ upon the data

*Theorem 2*

Under assumption  $(A_1) - (A_3)$ , the solution  $(a, u)$  depends continuously upon the data.

*Proof*

Let  $\Phi = \{\varphi, g, F\}$  and  $\bar{\Phi} = \{\bar{\varphi}, \bar{g}, \bar{F}\}$  be two sets of data, which satisfy the conditions  $(A_1) - (A_3)$ . Let us denote  $\|\Phi\| = (\|g\|_{C^1[0,T]} + \|\varphi\|_{C^3[0,1]} + \|F\|_{C^{3,0}(\bar{Q}_T)})$ . Suppose that there exist positive constants  $M_i, i = 1, 2$  such that

$$0 < M_1 \leq |g|, \quad 0 < M_1 \leq |\bar{g}|, \quad \|\Phi\| \leq M_2 \quad \text{and} \quad \|\bar{\Phi}\| \leq M_2.$$

Let  $(a, u)$  and  $(\bar{a}, \bar{u})$  be the solutions of inverse problems (1)–(4) corresponding to the data  $\Phi$  and  $\bar{\Phi}$ , respectively. According to (10)

$$a(t) = \frac{1}{g(t)} \left( -g'(t) + 2F_0(t) + \sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \right) - \frac{1}{g(t)} \sum_{k=1}^{\infty} 8\pi k \left( \varphi_{2k} e^{-(2\pi k)^2 t - \int_0^t a(s) ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_{\tau}^t a(s) ds} d\tau \right),$$

$$\bar{a}(t) = \frac{1}{\bar{g}(t)} \left( -\bar{g}'(t) + 2\bar{F}_0(t) + \sum_{k=1}^{\infty} \frac{2}{\pi k} \bar{F}_{2k}(t) \right) - \frac{1}{\bar{g}(t)} \sum_{k=1}^{\infty} 8\pi k \left( \bar{\varphi}_{2k} e^{-(2\pi k)^2 t - \int_0^t \bar{a}(s) ds} + \int_0^t \bar{F}_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_{\tau}^t \bar{a}(s) ds} d\tau \right).$$

First, let us estimate the difference  $a - \bar{a}$ . It is easy to compute that

$$\left\| \frac{g'}{g} - \frac{\bar{g}'}{\bar{g}} \right\|_{C[0,T]} \leq M_3 \|g - \bar{g}\|_{C^1[0,T]},$$

$$\left\| \frac{F_0}{g} - \frac{\bar{F}_0}{\bar{g}} \right\|_{C[0,T]} \leq M_4 \|g - \bar{g}\|_{C^1[0,T]} + M_5 \|F - \bar{F}\|_{C^{3,0}(\bar{Q}_T)},$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \left\| \frac{F_{2k}}{g} - \frac{\bar{F}_{2k}}{\bar{g}} \right\|_{C[0,T]} \leq M_6 \|g - \bar{g}\|_{C^1[0,T]} + M_7 \|F - \bar{F}\|_{C^{3,0}(\bar{Q}_T)},$$

$$\left| \sum_{k=1}^{\infty} k \left( \frac{1}{g(t)} \varphi_{2k} e^{-(2\pi k)^2 t - \int_0^t a(s) ds} - \frac{1}{\bar{g}(t)} \bar{\varphi}_{2k} e^{-(2\pi k)^2 t - \int_0^t \bar{a}(s) ds} \right) \right| \leq M_8 \|g - \bar{g}\|_{C^1[0,T]} + TM_9 \|a - \bar{a}\|_{C[0,T]} + M_{10} \|\varphi - \bar{\varphi}\|_{C^3[0,1]},$$

$$\left| \sum_{k=1}^{\infty} k \left( \frac{1}{g(t)} \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_{\tau}^t a(s) ds} d\tau - \frac{1}{\bar{g}(t)} \int_0^t \bar{F}_{2k}(\tau) e^{-(2\pi k)^2 (t-\tau) - \int_{\tau}^t \bar{a}(s) ds} d\tau \right) \right|$$

$$\leq TM_{11} \|g - \bar{g}\|_{C^1[0,T]} + T^2 M_{12} \|a - \bar{a}\|_{C[0,T]} + TM_{13} \|F - \bar{F}\|_{C^{3,0}(\bar{Q}_T)},$$

where  $M_k, k = 3, 4, \dots, 13$  are constants that are determined by  $M_1$  and  $M_2$ . If we consider these estimates in  $a - \bar{a}$ , we obtain

$$(1 - M_{14}) \|a - \bar{a}\|_{C[0,T]} \leq M_{15} (\|g - \bar{g}\|_{C^1[0,T]} + \|\varphi - \bar{\varphi}\|_{C^3[0,1]} + \|F - \bar{F}\|_{C^{3,0}(\bar{Q}_T)}),$$

where  $M_{14} = 8\pi T(M_9 + TM_{12}), M_{15} = \max\{M_3 + 2M_4 + \frac{2}{\pi}M_6 + 8\pi M_8 + 8\pi TM_{11}, 8\pi M_{10}, 2M_5 + \frac{2}{\pi}M_7 + 8\pi TM_{13}\}$ . The inequality  $M_{14} < 1$  holds for small  $T$ . Finally, we obtain

$$\|a - \bar{a}\|_{C[0,T]} \leq M_{16} \|\Phi - \bar{\Phi}\|, \quad M_{16} = \frac{M_{15}}{(1 - M_{14})}.$$

The similar estimate is also obtained for the difference  $u - \bar{u}$  from (8):

$$\|u - \bar{u}\|_{C(\bar{Q}_T)} \leq M_{17} \|\Phi - \bar{\Phi}\|.$$

□

#### 5. Numerical method and examples

We will consider the examples of numerical solution of the inverse problem (1)–(4). For the convenience of discussion of the numerical method, we will rewrite (1)–(4) as follows:

$$v_t = v_{xx} + r(t)F(x, t), \quad (x, t) \in Q_T, \tag{13}$$

$$v(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \tag{14}$$

$$v(0, t) = v(1, t), \quad v_x(1, t) = 0, \quad 0 \leq t \leq T, \tag{15}$$

$$r(t)g(t) = \int_0^1 v(x, t) dx, \quad 0 \leq t \leq T \tag{16}$$

by transformations

$$r(t) = \exp\left(\int_0^t a(\tau) d\tau\right), \tag{17}$$

$$v(x, t) = r(t)u(x, t). \tag{18}$$

We subdivide the intervals  $[0,1]$  and  $[0,T]$  into  $M$  and  $N$  subintervals of equal lengths  $h = \frac{1}{M}$  and  $\tau = \frac{T}{N}$ , respectively. Then we add a line  $x = (M + 1)h$  to generate the fictitious point needed for the second boundary condition. We choose the Crank–Nicolson scheme. The scheme for (13)–(16) is as follows:

$$\frac{1}{\tau}(v_j^{n+1} - v_j^n) = \frac{1}{2} \left[ \frac{1}{h^2}(v_{j-1}^n - 2v_j^n + v_{j+1}^n) + (rF)_j^n \right] + \frac{1}{2} \left[ \frac{1}{h^2}(v_{j-1}^{n+1} - 2v_j^{n+1} + v_{j+1}^{n+1}) + (rF)_j^{n+1} \right], \tag{19}$$

$$v_j^1 = \phi_j, \tag{20}$$

$$v_0^n = v_M^n, \tag{21}$$

$$v_M^n = v_{M+1}^n, \tag{22}$$

where  $0 \leq j \leq M$  and  $1 \leq n \leq N$  are the indices for the spatial and time steps, respectively,  $v_j^n$  is the approximation to  $v(x_j, t_n)$ ,  $(rF)_j^n = r(t_n)F(x_j, t_n)$ ,  $\phi_j = \varphi(x_j)$ ,  $v_j^1 = \phi_j$ ,  $x_j = jh$ ,  $t_n = n\tau$ . At the  $t = 0$  level, adjustment should be made according to the initial condition and the compatibility requirements.

Now, we rewrite (16) as

$$r(t) = \frac{1}{g(t)} \int_0^1 v(x, t) dx \tag{23}$$

and approximate  $\int_0^1 v(x, t) dx$  formally by the trapezoidal formula

$$\int_0^1 v(x, t) dx = h \left( \frac{v_1}{2} + v_2 + \dots + v_{M-1} + \frac{v_M}{2} \right), \tag{24}$$

where  $v_j = v(x_j, t)$ ,  $0 \leq j \leq M$ .

Substituting (23), with  $\int_0^1 v(x, t) dx$  given by (24) into (13), and rewriting the resulting system into a matrix form, we obtain  $M \times M$  linear system of equations

$$\left( A + \frac{h^2}{g^{n+1}} \tilde{A} \right) v^{n+1} = \left( B + \frac{h^2}{g^n} \tilde{B} \right) v^n, \tag{25}$$

where

$$v^n = (v_1^n, v_2^n, \dots, v_M^n)^T, \quad g^n = g(t_n), \quad 1 \leq n \leq N,$$

$$A = \begin{bmatrix} -2\left(1 + \frac{h^2}{\tau}\right) & 1 & 0 & \dots & 0 & 1 \\ 1 & -2\left(1 + \frac{h^2}{\tau}\right) & 1 & 0 & \dots & 0 \\ 0 & 1 & -2\left(1 + \frac{h^2}{\tau}\right) & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & & \\ & & & & 0 & 1 & -2\left(1 + \frac{h^2}{\tau}\right) & 1 \\ & & & & 0 & 2 & -2\left(1 + \frac{h^2}{\tau}\right) \end{bmatrix},$$

$$B = \begin{bmatrix} 2\left(1 - \frac{h^2}{\tau}\right) & -1 & 0 & \dots & 0 & -1 \\ -1 & 2\left(1 - \frac{h^2}{\tau}\right) & -1 & 0 & \dots & 0 \\ 0 & -1 & 2\left(1 - \frac{h^2}{\tau}\right) & -1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & & \\ & & & & & 0 & -1 & 2\left(1 - \frac{h^2}{\tau}\right) & -1 \\ & & & & & 0 & -2 & 2\left(1 - \frac{h^2}{\tau}\right) \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} \frac{1}{2}c_1 & c_1 & \dots & c_1 & \frac{1}{2}c_1 \\ \frac{1}{2}c_2 & c_2 & c_2 & c_2 & \frac{1}{2}c_2 \\ \vdots & & & & \vdots \\ \frac{1}{2}c_M & c_M & c_M & c_M & \frac{1}{2}c_M \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \frac{1}{2}b_1 & b_1 & \dots & b_1 & \frac{1}{2}b_1 \\ \frac{1}{2}b_2 & b_2 & b_2 & b_2 & \frac{1}{2}b_2 \\ \vdots & & & & \vdots \\ \frac{1}{2}b_M & b_M & b_M & b_M & \frac{1}{2}b_M \end{bmatrix}$$

with

$$c_i = hF_i^{n+1}, \quad i = 1, \dots, M,$$

$$b_i = hF_i^n, \quad i = 1, \dots, M.$$

We can solve (25) by the Gauss elimination method. When  $v_j^{n+1}, j = 1, 2, \dots, M$ , have been obtained,  $r^{n+1}$  can be evaluated through (23) and (24).

Let us compare the solution  $v(x, t)$  of (13)–(15) and the solution  $v_j^n$  of the Crank–Nicolson scheme (19)–(22) for (13)–(16).

According to Theorem 1, problem (1)–(4) with the data satisfying the conditions  $(A_1)$ – $(A_3)$  has unique solution  $\{a(t), u(x, t)\}$  for some  $T$ . In this case, the function  $v(x, t) = r(t)u(x, t)$ ,  $r(t) = \exp(\int_0^t a(\tau) d\tau)$  satisfy (13)–(15). In addition, system (25) has unique solution that the matrices  $A + \frac{h^2}{g^{n+1}}\tilde{A}$ ,  $n = 1, 2, \dots, N$  are nonsingular.

In order to compare the solution  $v(x, t)$  of (13)–(15) and the solution  $v_j^n$  of the Crank–Nicolson scheme (19)–(22) for (13)–(16), let us evaluate the difference

$$z_j^n = v_j^n - v_j^n,$$

where  $v_j^n = v(x_j, t_n)$ . We proceed to the estimation of the order of approximation for scheme under the agreement that the solution  $v(x, t)$  of (13)–(15) possesses a necessary number of derivatives in  $x$  and  $t$ .

The following notations will be used on techniques in [7]:

$$v_j^n = v, v_{j+1}^n = \hat{v}, \quad v_t = \frac{\hat{v} - v}{\tau}, \quad \wedge v_j^n = \frac{v_{j-1}^n - 2v_j^n + v_{j+1}^n}{h^2}.$$

It is possible to set up the problem for  $z$ :

$$z_t = \frac{1}{2} \wedge (\hat{z} + z) + \Psi,$$

$$z(x, 0) = 0,$$

$$z(0, t) = z(1, t),$$

$$z_x(1, t) = 0,$$

where

$$\Psi = \frac{1}{2} \wedge (\widehat{V} + V) + \Gamma$$

is the error of approximation for the Crank–Nicolson scheme on the solution  $v(x, t)$  of (13)–(15), where  $\Gamma = \frac{1}{2}((rF)_j^n + (rF)_j^{n+1})$ . The Taylor series expansions for the function  $v(x, t)$  and  $r(t)F(x, t)$  about the node  $(x_j, t_{n+1/2})$  lead to the estimation:

$$\Psi = O(h^2 + \tau^2).$$

Knowing  $v(x, t)$ ,  $r(t)$  we can find the solution pair  $(u, a)$  through the inverse transformations of (17) and (18)

$$u(x, t) = \frac{v(x, t)}{r(t)},$$

$$a(t) = \frac{r'(t)}{r(t)}.$$

We can use numerical differentiation to compute the values of  $r'(t)$ .

Two examples are given below. In the first example, the illustration of the theoretical results on the convergence of the Crank–Nicolson scheme to exact solution is demonstrated. In the second one, the Crank–Nicolson scheme that ends with an unstable scheme for some T is demonstrated.

*Example 1*

Consider the inverse problem (1)–(4), with

$$F(x, t) = (2\pi)^2 \cos 2\pi x \exp(-(2\pi)^2 t) + 2t(1 + \cos 2\pi x) \exp(-(2\pi)^2 t + 10t^2),$$

$$\varphi(x) = 1 + \cos 2\pi x, \quad g(t) = \exp(-(2\pi)^2 t), \quad T = \frac{1}{2}.$$

It is easy to check that the exact solution is

$$\{a(t), u(x, t)\} = \{(2\pi)^2 + 2t \exp(10t^2), (1 + \cos 2\pi x) \exp(-(2\pi)^2 t)\}.$$

Problem (13)–(16) is given by

$$v_t = v_{xx} + r(t)((2\pi)^2 \cos 2\pi x \exp(-(2\pi)^2 t) + 2t(1 + \cos 2\pi x) \exp(-(2\pi)^2 t + 10t^2)),$$

$$0 < x < 1, \quad 0 < t \leq \frac{1}{2},$$

$$v(x, 0) = 1 + \cos 2\pi x, \quad 0 \leq x \leq 1,$$

$$v(0, t) = v(1, t), \quad 0 \leq t \leq \frac{1}{2},$$

$$v_x(1, t) = 0, \quad 0 \leq t \leq \frac{1}{2},$$

$$\int_0^1 v(x, t) dx = r(t) \exp(-(2\pi)^2 t), \quad 0 \leq t \leq \frac{1}{2},$$

where

$$r(t) = \exp(((2\pi)^2 t) + \frac{1}{10}(\exp(10t^2) - 1)).$$

We use the Crank–Nicolson scheme to solve it for the values of  $v$ , and then use (23) and (24) to approximate  $r(t)$ . As a result, we obtain Tables I, II and Figures 1, 2 for exact and approximate values of  $a(t)$  and  $u(x, t)$ . The step sizes are  $h = 0.005$  and  $\tau = \frac{h}{2}$ .

Table I. Some values of $a(t)$ .			
Exact	Approximate	Error	Relative error
39.5236	39.5887	0.065	0.0016
39.5390	39.6037	0.0647	0.0016
40.4125	40.4732	0.0606	0.0015
40.9542	41.0130	0.0588	0.0014
41.3475	41.4051	0.0576	0.0014
41.8613	41.9175	0.0561	0.0013
42.5389	42.5933	0.0544	0.0013
43.4408	43.4932	0.0524	0.0012
44.6529	44.7031	0.0503	0.0011
46.2969	46.3454	0.0485	0.0010
48.5483	48.5963	0.0480	0.0009



Table II. Some values of $u(x, t)$ for 70. mesh point of $t$ .		
Exact	Approximate	Error
2.0700	2.0621	0.0079
1.7673	1.7616	0.0057
1.5053	1.5019	0.0034
1.3552	1.3532	0.0020
0.9703	0.9717	0.0014
0.5365	0.5418	0.0053
0.4268	0.4330	0.0062
1.0353	1.0361	0.0004
2.0199	2.0119	0.0080
2.0700	2.0616	0.0084

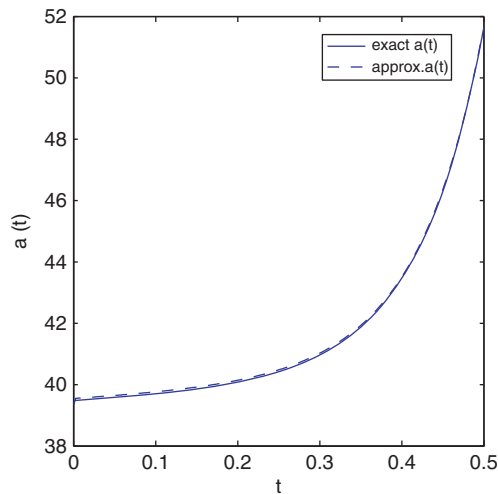


Figure 1. Exact and approximate  $a(t)$ .

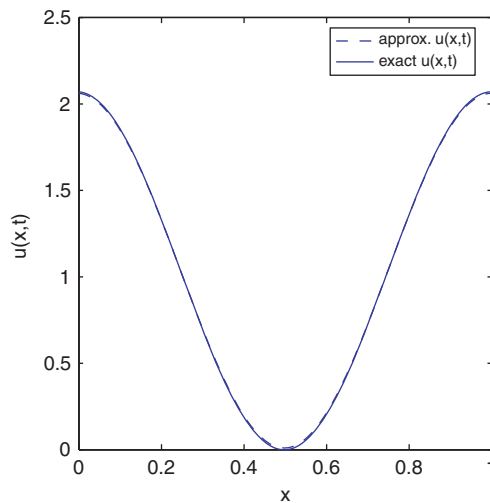


Figure 2. Exact and approximate solutions of  $u(x, t)$  for 70. Mesh point of  $t$ .

Example 2

Consider the problem with the equation, initial, boundary and overdetermination conditions as in Example 1 but for  $T = \frac{31}{40}$ .

Under the same step size as in Example 1 the Crank–Nicolson scheme is used to solve it for the values of  $v$ , and then (23) and (24) are used to approximate  $r(t)$ . As a result, Tables III, IV and Figures 3, 4 are obtained for exact and approximate values of  $a(t)$  and  $u(x, t)$ .

Table III. Some values of $a(t)$ .			
Exact	Approximate	Error	relative error
40.9542	41.013	0.0588	0.0014
46.2969	46.3454	0.0485	0.0010
83.3963	83.8099	0.4136	0.005
101.6186	102.5894	0.9707	0.0096
128.3653	130.7749	2.4096	0.0188
168.0331	174.3586	6.3255	0.0376
227.4841	245.2822	17.7981	0.0782
317.5319	372.8756	55.3437	0.1743
455.3868	662.0168	206.63	0.4537
668.7135	511.2322	157.4813	0.2355

Table IV. Some values of $u(x, t)$ for 280. mesh point of $t$ .			
Exact	Approximate	Error	Relative error
$7.5994 \times 10^5$	$1.1657 \times 10^6$	$4.0578 \times 10^5$	$2.3825 \times 10^{-8}$
$7.0830 \times 10^5$	$1.0809 \times 10^6$	$3.7264 \times 10^5$	$2.5562 \times 10^{-8}$
$6.2527 \times 10^5$	$9.4888 \times 10^5$	$3.2361 \times 10^5$	$2.8956 \times 10^{-8}$
$5.1899 \times 10^5$	$7.8248 \times 10^5$	$2.6349 \times 10^5$	$3.4886 \times 10^{-8}$
$3.9984 \times 10^5$	$5.9802 \times 10^5$	$1.9817 \times 10^5$	$4.5281 \times 10^{-8}$
$2.7951 \times 10^5$	$4.1356 \times 10^5$	$1.3405 \times 10^5$	$6.4775 \times 10^{-8}$
$1.6977 \times 10^5$	$2.4716 \times 10^5$	$7.7391 \times 10^4$	$1.0665 \times 10^{-7}$
$4.9582 \times 10^5$	$2.8248 \times 10^5$	$2.8665 \times 10^5$	$3.6516 \times 10^{-8}$
$7.5242 \times 10^5$	$1.1657 \times 10^6$	$4.1331 \times 10^5$	$2.4063 \times 10^{-8}$
$7.7514 \times 10^5$	$1.1949 \times 10^6$	$4.1979 \times 10^5$	$2.3357 \times 10^{-8}$

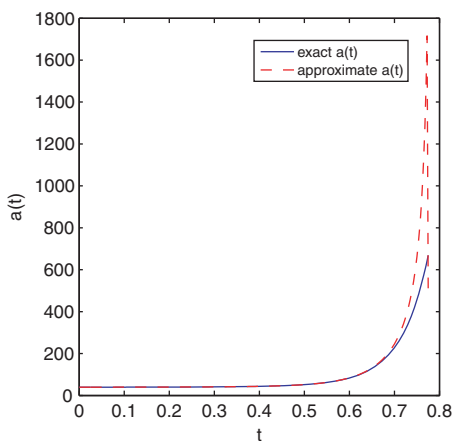


Figure 3. Exact and approximate  $a(t)$ .

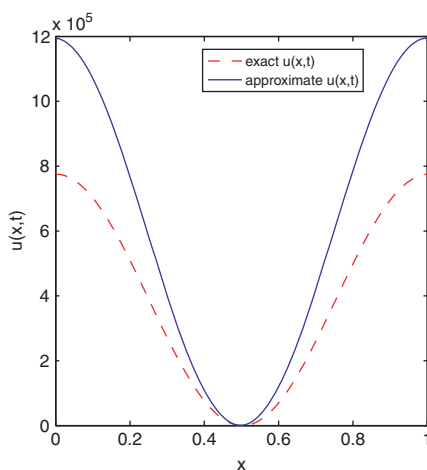


Figure 4. Exact and approximate solutions of  $u(x, t)$  for 280. Mesh point of  $t$ .

## 6. Some discussions

Numerical differentiation is used to compute the values of  $r'(t)$  in the formula  $a(t) = \frac{r'(t)}{r(t)}$ . It is well known that numerical differentiation is slightly ill-posed and it can cause some numerical difficulties. One can apply the natural cubic spline function technique [8] to still obtain decent accuracy.

The matrices  $A + \frac{h^2}{g^{n+1}} \tilde{A}$ ,  $n = 1, 2, \dots, N$  are dependent on the step sizes  $h$  and  $\tau$ . The condition number of the system (25) grows with  $N$  for fixed  $h$ , if the overdetermination data  $g(t)$  fast decreases in  $t$ . Therefore, it causes some numerical difficulties.

The condition number of the system (25) corresponding to BVP that is mentioned in the above examples, strongly grows in  $T > \frac{3}{4}$ , for the step size  $h = 0.005$ ,  $\tau = \frac{h}{2}$ . In this sense  $T \approx \frac{3}{4}$  is the critical upper bound of  $T$  for the step size  $h = 0.005$ ,  $\tau = \frac{h}{2}$ . The critical upper bound of  $T$  can change for the other step sizes  $h$  and  $\tau$ . For the problems that are mentioned in the above examples the critical upper bound of  $T$  is  $\frac{5}{8}$  in the case of  $h = \tau = 0.005$ , the critical upper bound of  $T$  is  $\frac{11}{16}$  in the case of  $h = 0.01$ ,  $\tau = 0.0069$ .

## References

1. Cannon JR, Lin Y, Wang S. Determination of a control parameter in a parabolic partial differential equation. *Journal of the Australian Mathematical Society, Series B* 1991; **33**:149–163.
2. Ivanchov MI, Pabyrivska NV. Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions. *Ukrainian Mathematical Journal* 2001; **53**(5):674–684.
3. Ivanchov MI. *Inverse Problems for Equations of Parabolic Type*. VNTL Publishers: Lviv, Ukraine, 2003.
4. Namazov GK. Definition of the unknown coefficient of a parabolic equation with nonlocal boundary and complementary conditions. *Transactions of Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences* 1999; **19**(5):113–117.
5. Ionkin NI. Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition. *Differential Equations* 1977; **13**:204–211.
6. Gohberg IC, Krein MG. *Introduction to the theory of linear nonselfadjoint operators*. American Mathematical Society, Providence, RI, 1969.
7. Samarskii AA. *The Theory of Difference Schemes*. Marcel Dekker, Inc.: New York, 2001.
8. Atkinson KE. *Elementary Numerical Analysis*. Wiley: New York, 1985.