Journal of Computational and Applied Mathematics 247 (2013) 68–83

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Bivariate Pseudo–Gompertz distribution and concomitants of its order statistics

Serap Yörübulut [a], Omer L. Gebizlioglu [b,*]

[a] Kırıkkale University, Faculty of Science and Letters, Department of Statistics, 71450 Kırıkkale, Turkey

[b] Kadir Has University, Faculty of Economics, Administrative and Social Sciences, Cibali, 34083 Istanbul, Turkey

ARTICLE INFO

Article history:
Received 24 July 2012
Received in revised form 10 January 2013

Keywords:
Bivariate Gompertz distribution
Pseudo-distribution
Order statistics
Concomitants
Survival function
Insurance and reliability

ABSTRACT

This paper presents a new bivariate Pseudo–Gompertz distribution that sprouts from the classical Gompertz distribution and possesses the features of pseudo-distribution functions. In addition to some standard properties of the proposed distribution, distributions of order statistics and their concomitants for samples drawn from the new distribution are obtained. The survival and hazard functions of the concomitants are shown and their values are tabled. Interpretations of the results are given in connection with risk events and risk management.

1. Introduction

The Gompertz distribution is a widely used probability distribution in the modeling of lifetimes of components of physical systems and organisms of biological populations [1]. Life tables for human beings, in particular, are constructed on the basis of the distributions like the Gompertz distribution. Specifically, it is an empirically veritable parametric probability model that is used to express age specific probabilities of lifetime and mortality or hazard rates for lives in human populations. Detailed examples of such important uses of the Gompertz distribution are given by [2–4].

A bivariate distribution $F(x, y)$ for a pair of random variables (X, Y) expresses the dependence between X and Y as embedded in its functional form and parameters. When X or Y is related with each other through a real-valued function $\phi(\cdot)$, then the distribution that emerges from $F(x, y)$ is a pseudo-distribution with $\phi(\cdot)$ included in its set of parameters. It is obvious that $\phi(x)$ must satisfy the condition that the pseudo distribution has all the properties to be a probability distribution. A pseudo-distribution that has a stationary distribution property is introduced by [5], way back in sixties, by redefining several parameters of a probability function. The Wishart distribution under singularity is discussed in [6] where Pseudo–Wishart distributions are obtained by tackling with the causes of singularity. The work in [7] concentrates on a bivariate Gompertz distribution and finds a Gompertz-type distribution with the use of some suitable functions that tie some concerned random variables with each other. A generalization of the Gompertz distribution is proposed in [8] by some parametrizations that enables the application of some survival models with empirically identifiable mortality concepts. Thus, the interest in theory and methods about the Gompertz distribution is progressive.

* Correspondence to: Kadir Has University, Faculty of Economics, Administrative and Social Sciences, Department of International Trade and Finance, Cibali, 34083 Istanbul, Turkey. Tel.: +90 532 799 9719.

E-mail addresses: omer.gebizlioglu@khas.edu.tr, gebizli@ankara.edu.tr (O.L. Gebizlioglu).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2013.01.006

Recently, a class of pseudo-distributions is introduced by [9] as the probability distribution models for several linear combinations of random variables in the stochastic modeling attempts. Along the lines that combinations of random variables are of concern, our paper introduces a bivariate Pseudo–Gompertz distribution arising from the bivariate Gompertz distribution.

Concomitants are useful accompaniments in statistical modeling that may stand as random variables of interest in connection with order statistics of random samples. For such a situation, order statistics and their concomitants are taken under consideration in the current paper for the Pseudo–Gompertz distribution.

In the following sections; order statistics and concomitants are mentioned first and then the new bivariate Pseudo–Gompertz distribution is presented. A next step is the construction of the distribution of the concomitants of order statistics for the presented Pseudo–Gompertz distribution and the derivation of the survival and hazard functions for them. Some implications of the results are provided for reliability and risk modeling throughout the sections.

2. Order statistics and concomitants

A random sample of $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ from a bivariate distribution with bivariate distribution function $F(x, y)$ yields order statistics of the first coordinate as $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ such that $X_{i:n} \le X_{j:n}$ for $i < j$. If the pairs (X_i, Y_i), $i = 1, 2, \ldots, n$, are ordered by their X variates according to $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$, then the Y variate associated with the r-th order statistic $X_{r:n}$ of X, denoted by $Y_{[r:n]}$, $1 \le r \le n$, is called the concomitant of the r-th order statistic. For a detailed overview of concomitants, we refer to [10,11].

Concomitants are used in many applied areas where a population characteristic Y is investigated with respect to another characteristic X of the same population. Such applications are expedient in the risk management fields like system reliability, finance and actuarial sciences where losses due to frailties or defaults are of ultimate concern. Recently, tolerance intervals for bivariate quantiles of two dependent random variables are obtained in [12] by using the concomitants of order statistics. Another recent work by [13] considers the concomitants of the generalized order statistics as a unified model for ordered random variables.

An observed sample from a population helps to determine the underlying probability distribution model $F(x, y)$ for the (X, Y) pair and their marginal probability distribution functions $F(x)$ and $F(y)$. By the use of $F(x, y)$ as the parent distribution, the probability distribution models for the r-th order statistic $X_{r:n}$ of X, $1 \le r \le n$, and the distribution and density functions for the concomitants $Y_{[r:n]}$ of $X_{r:n}$ can be obtained. The essential expressions for such functions are clearly derived and shown by [14,10,11]. Following them, the general definitions of the probability distribution and density functions for $Y_{[r:n]}$ are shown below as standard definitions for a random variable in general:

$$F_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} F(y \mid x) f_{X_{r:n}}(x)dx \tag{1}$$

and

$$f_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{X_{r:n}}(x)dx \tag{2}$$

where $F(y \mid x)$ and $f(y \mid x)$ are the conditional distribution and density functions of Y given X. The density function of an r-th order statistic is defined as

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r)!} f(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}. \tag{3}$$

The joint distribution of two order statistics $X_{r:n} \le X_{s:n}$ is expressed as;

$$f_{X_{r:n},X_{s:n}}(x_1, x_2) = \frac{n!}{(r-1)!\,(s-r)!\,(n-s)!} f(x_1) f(x_2)\,[F(x_1)]^{r-1}\,[F(x_2)-F(x_1)]^{s-r-1}\,[1-F(x_2)]^{n-s}. \tag{4}$$

And, the joint distribution of two concomitants is given by

$$f_{Y_{[r:n]},Y_{[s:n]}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f(y_1 \mid x_1) f(y_2 \mid x_2) f_{X_{r:n},X_{s:n}}(x_1, x_2)dx_1 dx_2. \tag{5}$$

The dependence structure of the concomitants implicit in the function above is discussed by [11]. The general expressions given here are used in the following sections where a new bivariate Pseudo–Gompertz distribution is presented.

3. The bivariate Pseudo–Gompertz distribution

A new class of pseudo-distributions for linear combinations of random variables is introduced by [15] for the statistical applications where an actual distribution cannot be used easily. Some other pseudo-distributions have been introduced afterwards in a similar way. Following the pseudo-distributions obtained by [16,17,9], among them, we obtain a bivariate-Gompertz distribution as presented below.

The Gompertz distribution with parameters λ and μ_1 for a random variable X has the following density function

$$f_X\,(x; \lambda, \mu_1) = \lambda e^{\mu_1 x} \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right], \quad \mu_1 > 0,\ \lambda > 0,\ x > 0. \tag{6}$$

Suppose that another random variable Y also has a Gompertz distribution with parameters $\phi(x)$ and μ_2, where $\phi(x)$ is a real valued function of the random variable X. Then, the density function of Y is given by

$$f_{Y|X=x}\,(y; \phi(x), \mu_2 \mid x) = \phi(x) e^{\mu_2 y} \exp\left[-\frac{\phi(x)}{\mu_2}\left(e^{\mu_2 y} - 1\right)\right], \quad \mu_2 > 0,\ \phi(x) > 0,\ y > 0. \tag{7}$$

Using the marginal densities defined in Eqs. (6) and (7) above, the bivariate Pseudo–Gompertz distribution is obtained as the compound distribution of X and Y with density function;

$$\begin{aligned} f\,(x, y) &= \lambda \phi(x) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right) - \frac{\phi(x)}{\mu_2}\left(e^{\mu_2 y} - 1\right)\right], \\ &\mu_1 > 0,\ \mu_2 > 0,\ \lambda > 0, \phi(x) > 0,\ y > 0,\ x > 0, \end{aligned} \tag{8}$$

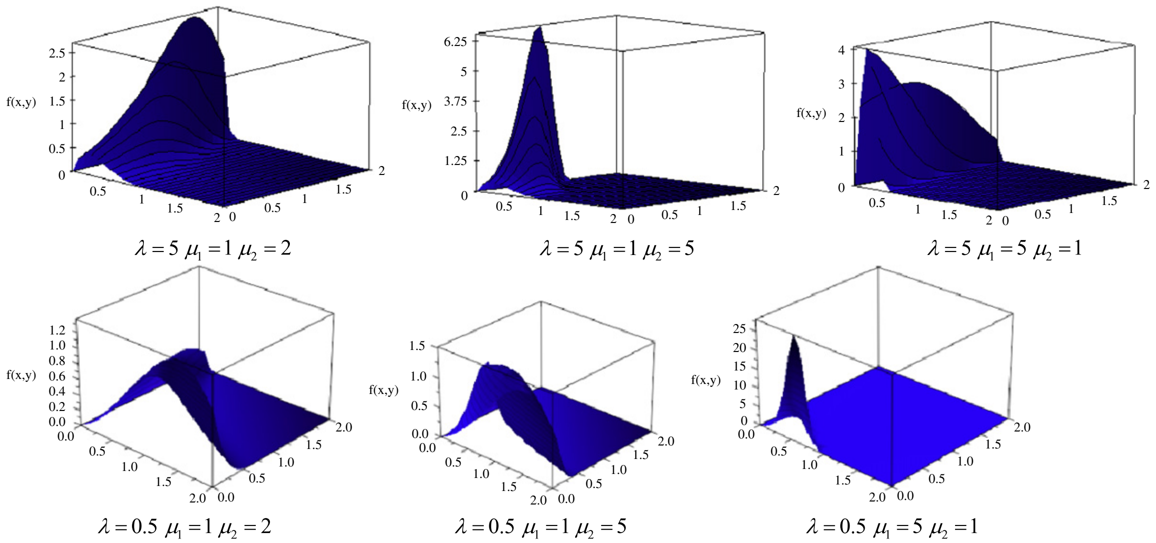
which follows from $f\,(x, y) = f_X\,(x; \lambda, \mu_1)\, f_{Y|X=x}\,(y; \phi(x), \mu_2 \mid x)$.

Depending upon various choices of function $\phi(x)$, several distributions can be generated from this general density function. Adopting $\phi(x) = e^{\mu_1 x} - 1$, the following form of bivariate Pseudo–Gompertz distribution is obtained:

$$f\,(x, y) = \lambda\left(e^{\mu_1 x} - 1\right) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-\left(e^{\mu_1 x} - 1\right)\left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)\right], \quad \mu_1,\ \mu_2,\ \lambda,\ y,\ x > 0. \tag{9}$$

The $\phi(x)$ function can be determined by the users in accordance to their needs of modeling. Here, the essential condition is that $F\,(x, y) = \int_x \int_y f\,(x, y)\, dydx$ must satisfy all the properties to be a probability distribution function. It can easily be checked that $f\,(x, y)$ above is a bivariate density function as proved in Appendix A of the paper.

The plots of the density function expressed in (9) are displayed below. From the plots, it is seen that the joint density has a long right tail as compared to its left tail.



The marginal distributions of X and Y are derived from Eq. (9) as

$$f(x) = \lambda e^{\mu_1 x} \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right] \quad \text{and} \quad f(y) = e^{\mu_2 y} \frac{\lambda}{\mu_1}\left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)^{-2}, \quad \text{respectively.}$$

The joint distribution function corresponding to $f(x, y)$ in (9) is

$$F(x,y) = \int_0^y \int_0^x \lambda \left(e^{\mu_1 x} - 1\right) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-\left(e^{\mu_1 x} - 1\right)\left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)\right] dxdy$$
$$= \frac{\lambda}{\mu_1} \frac{\left[\exp\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right) - 1\right]}{\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} + \frac{\lambda}{\mu_1}\right)} + \left(1 - \exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right)\right). \tag{10}$$

So, the joint survival function, that follows from (10) above, is

$$S(x,y) = 1 - F_1(x) - F_2(y) + F(x,y)$$

where $F_1(x)$ ve $F_2(y)$ are the marginal distribution functions of X and Y, respectively, and they are obtained for the bivariate Pseudo–Gompertz distribution, using Eq. (10), as follows:

$$F_1(x) = \operatorname*{Lim}_{y\to\infty} F(x,y) = 1 - \exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right)$$

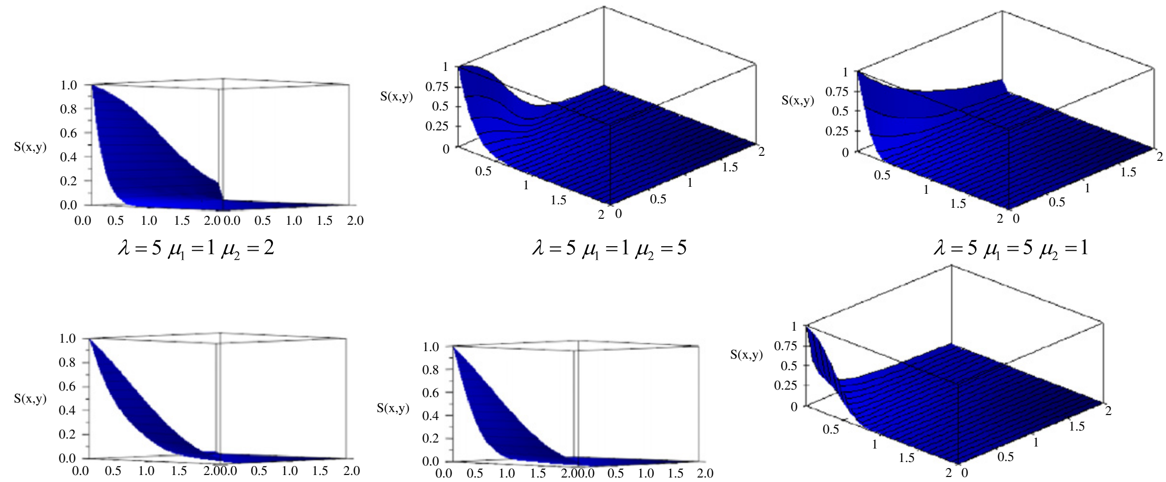
and

$$F_2(y) = \operatorname*{Lim}_{x\to\infty} F(x,y) = 1 - \frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y} - 1\right) + \mu_2\lambda}.$$

From the joint and marginal distribution functions above, the corresponding joint survival function is found as

$$S(x,y) = \frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y} - 1\right) + \mu_2\lambda} \exp\left[\frac{(e^{\mu_2 y} - 1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y} - 1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right].$$

The plots of the joint survival function are displayed below. The plots indicate that the survival function decreases faster in value as the values of λ and μ_1 increase.



$\lambda = 5\ \mu_1 = 1\ \mu_2 = 2$ $\lambda = 5\ \mu_1 = 1\ \mu_2 = 5$ $\lambda = 5\ \mu_1 = 5\ \mu_2 = 1$

$\lambda = 0.5\ \mu_1 = 1\ \mu_2 = 2$ $\lambda = 0.5\ \mu_1 = 1\ \mu_2 = 5$ $\lambda = 0.5\ \mu_1 = 5\ \mu_2 = 1$

4. Distribution of the concomitants

In this section, we obtain the distribution of the concomitant of the r-th order statistics for the bivariate Pseudo–Gompertz distribution, given in Eq. (10). This distribution is derived by placing in Eq. (2) the density function of the r-th order statistics for the random variable X,

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r)!}\lambda e^{\mu_1 x}\left[1 - \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right]\right]^{r-1}\left[\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right]\right]^{n-r+1}$$
$$= \frac{\lambda e^{\mu_1 x} n!}{(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h} e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+h+1)}, \quad \mu_1,\ \lambda,\ x > 0. \tag{11}$$

The conditional density function $f(y \mid x)$, shown below as derived from (9), is also needed for deriving the density function of the concomitant:

$$f(y \mid x) = \left(e^{\mu_1 x} - 1\right) e^{\mu_2 y} \exp\left[-\frac{(e^{\mu_1 x} - 1)}{\mu_2}\left(e^{\mu_2 y} - 1\right)\right], \quad \mu_1,\ \mu_2,\ x,\ y > 0. \tag{12}$$

Placing Eqs. (11) and (12) in Eq. (2), the density function of the concomitant of the r-th order statistic $X_{r:n}$ from the bivariate Pseudo–Gompertz distribution is found as:

$$\begin{aligned} f_{Y_{[r:n]}}(y) &= \frac{e^{\mu_2 y}\lambda n!}{(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h} \\ &\quad \times \int_0^\infty e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+h+1)}\left(e^{\mu_1 x}-1\right)e^{\mu_1 x}\exp\left[-\frac{(e^{\mu_1 x}-1)}{\mu_2}\left(e^{\mu_2 y}-1\right)\right]dx \\ &= \frac{\mu_1 e^{\mu_2 y} n!}{\lambda\,(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\left(h+n-r+1+\frac{\mu_1\,(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)^{-2}. \end{aligned} \tag{13}$$

Here, the random variable X may be taken as the lifetime of a unit or live at age x and Y may be the lifetime of another unit or live at age y whose relation with X is built by $\phi(x)$ in their joint distribution function. Accordingly, the $(X_{r:n}, Y_{[r:n]})$ pair stands for the r-th order lifetime and the lifetime of the r-th order concomitant in a random sample of n observations. The distribution obtained above can then be used for risk modeling purposes on the basis of the life contingencies of the pairs of ranked lifetimes and their concomitants.

A more useful form of the density function given above is derived below by using the results given in [18,19].

Consider Eq. (13) and let $\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda} + n - r + 1 = a$,

$$\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\left(h+n-r+1+\frac{\mu_1\,(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)^{-2} = \sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{1}{(h+a)^2} \tag{14}$$

and

$$\begin{aligned} \sum_{h=0}^{r-1}\binom{r-1}{h}\frac{(-1)^h}{h+a} &= \frac{(r-1)!}{a(a+1)\cdots(a+r-1)} = g(a) \\ &= a^{-1}\binom{r-1+a}{r-1}^{-1}, \quad a \notin (0, -1, \ldots, -(r-1)). \end{aligned} \tag{15}$$

Note that there is a connection between $g(a)$ defined above and the Gamma function. Since

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt, \quad a > 0$$

$$\Gamma(a) = \frac{\Gamma(a+1)}{a} = \frac{\Gamma(a+2)}{a\,(a+1)} = \cdots = \frac{\Gamma(a+r)}{a\,(a+1)\cdots(a+r-1)}, \tag{16}$$

we can re-expresses Eq. (15), as shown in [19], by using expression (16);

$$g(a) = \sum_{h=0}^{r-1}\binom{r-1}{h}\frac{(-1)^h}{h+a} = \frac{(r-1)!}{a\,(a+1)\cdots(a+r-1)} = \frac{(r-1)!\Gamma(a)}{\Gamma\,(a+r)}.$$

Differentiating $g(a)$ with respect to its argument, we get

$$g'(a) = -\sum_{h=0}^{r-1}\binom{r-1}{h}\frac{(-1)^h}{(h+a)^2} = g(a)\left\{\psi(a) - \psi\,(a+r)\right\}, \tag{17}$$

where the digamma function $\psi(a)$ is the logarithmic derivative of $\Gamma(a)$ such that

$$\psi(a) = \frac{d}{da}\log\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}.$$

It is known that a harmonic number of order n is

$$H_{(n)} = \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}\frac{1}{k} \quad \text{for } k = 1, 2, \ldots, n.$$

The connection between the harmonic number and the digamma function is,

$$\{\psi\,(a+r)-\psi(a)\}=\sum_{h=a}^{a+r-1}\frac{1}{h}=H_{(a+r-1)}-H_{(a-1)}. \tag{18}$$

Therefore, using the Eqs. (17) and (18), we re-express Eq. (14);

$$\begin{aligned}\sum_{h=0}^{r-1}\binom{r-1}{h}\frac{(-1)^h}{(h+a)^2}&=g(a)\,\{\psi\,(a+r)-\psi(a)\}\\&=\frac{(r-1)!\Gamma(a)}{\Gamma\,(a+r)}\left(H_{(a+r-1)}-H_{(a-1)}\right).\end{aligned} \tag{19}$$

Consequently, the density function for the concomitant $Y_{[r:n]}$ in (13) with $\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+n-r+1=a$ can be re-written as

$$f_{Y_{[r:n]}}(y)=\frac{\mu_1 e^{\mu_2 y}\Gamma\,(n+1)\,\Gamma\left(n-r+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right)\left[H_{\left(n+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)}-H_{\left(n-r+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)}\right]}{\lambda\Gamma\,(n-r+1)\,\Gamma\left(n+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right)}. \tag{20}$$

This function is used in the following section for deriving the relevant survival and hazard functions.

5. Survival and hazard functions

When the random vector (X, Y) is defined as the lifetimes of two units or lives at a certain time point with observable ages, the remaining lifetime probabilities can be computed by using the general definition of a survival function, $S(t)=1-F(t)$, for each of (X, Y).

In this section, we derive the survival and hazard functions for $Y_{[r:n]}$. These functions are useful for the survival analysis with regard to any parent probability distribution. Using expression (13), the distribution function $F_{Y_{[r:n]}}(y)$ can be written as

$$\begin{aligned}F_{Y_{[r:n]}}(y)&=\int_0^y f_{Y_{[r:n]}}(t)dt\\&=\int_0^y\frac{\lambda n!}{(r-1)!\,(n-r)!\mu_1}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}e^{\mu_2 t}\left(\frac{\lambda}{\mu_1}\,(n-r+h+1)+\frac{\left(e^{\mu_2 t}-1\right)}{\mu_2}\right)^{-2}dt\\&=\frac{n!}{(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{1}{(h+n-r+1)}\left(\frac{\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}}{\left((h+n-r+1)+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)}\right).\end{aligned} \tag{21}$$

A more useful form of this distribution function is constructed below. Using

$$\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}\frac{1}{(h+c)}\left(\frac{b}{(h+c+b)}\right)=\frac{\Gamma(c)\Gamma(r)}{\Gamma\,(c+r)}-\frac{\Gamma\,(c+b)\,\Gamma(r)}{\Gamma\,(c+b+r)} \tag{22}$$

and letting $n-r+1=c$, $\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}=b$ in the distribution function (21), we obtain

$$\begin{aligned}F_{Y_{[r:n]}}(y)&=\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{\Gamma\,(n-r+1)\,\Gamma(r)}{\Gamma\,(n-r+1+r)}-\frac{\Gamma\left(n-r+1+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}\right)\Gamma(r)}{\Gamma\left(n-r+1+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+r\right)}\right]\\&=1-\frac{\Gamma\,(n+1)\,\Gamma\left(n-r+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right)}{\Gamma\,(n-r+1)\,\Gamma\left(n+\frac{\mu_1(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right)}.\end{aligned} \tag{23}$$

Using Eq. (23), the survival function for the concomitant is found as

$$S_{Y_{[r:n]}}(y)=\Gamma(n+1)\Gamma\left(n-r+\frac{\mu_1\,(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right)/\Gamma(n-r+1)\Gamma\left(n+\frac{\mu_1\,(e^{\mu_2 y}-1)}{\mu_2\lambda}+1\right). \tag{24}$$

Furthermore, the hazard or mortality rate function, in the general form of $h(y) = f(y)\,[S(y)]^{-1}$, for the concomitant of the r-th order statistics from the bivariate pseudo-Gompertz distribution is obtained as

$$h_{Y_{[r:n]}}(y) = \frac{f_{Y_{[r:n]}}(y)}{S_{Y_{[r:n]}}(y)} = \frac{\mu_1 e^{\mu_2 y}}{\lambda}\left[H_{\left(n+\frac{\mu_1\left(e^{\mu_2 y}-1\right)}{\mu_2\lambda}\right)} - H_{\left(n-r+\frac{\mu_1\left(e^{\mu_2 y}-1\right)}{\mu_2\lambda}\right)}\right]. \tag{25}$$

It is seen that the survival and hazard functions for $Y_{[r:n]}$ are functions of n, r, μ_1, μ_2 and λ, given $\phi(x)$. The random variable X does not appear in expression (25), but its order of magnitude r and the sample size n do. So does the observable value y of $Y_{[r:n]}$.

Considering a system consisting of a number of components each with a pair of units whose ages are observable, and concentrating on these components as a matter of reliability investigation, the reliability of the system can be assessed by the survival and hazard functions. (X, Y) being the lifetimes of the first and the second units of a component, respectively, a sample of n observations $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, at a given time, on the system components yields the order statistics and their concomitants $\left\{\left(X_{1:n}, Y_{[1:n]}\right), \ldots, \left(X_{r:n}, Y_{[r:n]}\right), \ldots, \left(X_{n:n}, Y_{[n:n]}\right)\right\}$. Assuming that the proper running of the system relies on the lifetimes of these components, the matter of the system reliability reduces to the survival of at least one of the units in the components. In this regard, the survival function $S_{Y_{[r:n]}}(y)$ in Eq. (24) states the probability of the random event $\left(Y_{[r:n]} \geq y\right)$ for the r-th concomitant as the second unit of a component.

The probability of the hazardous event that the second unit of a component fails within the small interval $(y, y + dy)$ is $P\left(y < Y_{[r:n]} < y + dy \mid Y_{[r:n]} > y\right) = h_{Y_{[r:n]}}(y)dy$, given that $S_{Y_{[r:n]}}(y) \neq 0$, which is nothing but the hazard rate or mortality rate expressed in Eq. (25), above.

A loss may exceed some endurable limits for a system due to the hazardous events as described above. Therefore, it is imperative to analyze and manage the technical and economic risks of the systems through the system reliability measures like the survival and hazard functions shown here.

The survival and hazard functions, under the bivariate Pseudo–Gompertz distribution model, for $X_{r:n}$ and for the $\{X_{r:n}, Y_{[r:n]}\}$ can be similarly derived. The derivations are elaborated in Appendix B. All these functions will be used in another application example presented below.

In the areas of finance and insurance, the lifetimes and mortality rates of human beings are the most essential elements in risk modeling. For instance, liabilities inherent in credit loans or insurance policies bear financial loss risks due to high costs of non-performance cases or contingent claims situations that arise in connection with the lifetime durations or the deaths of the individuals. The death of loan borrowers is a financial default for a loan lender that results in the loss of a planned cash flow of dept payments. Similarly, the death of the owner of a life insurance with a death benefit contract may cause an extra loss to an insurer if the death occurs earlier than an actuarially expected future time, Or, if the benefit amount exceeds an allocated reserve amount. In such situations, measures can be taken against the risks of excessive losses by using risk preventing or loss reducing tools like hedging and reinsurance. The books by [20,21], among many others, lucidly present the concepts of risk management for the system reliability, finance and insurance areas.

Assume that there exists a sample of observations $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, at a given time, on the lifetimes of pairwise insureds from an insurance portfolio. Consider the order statistic and concomitant pairs $\{(X_{1:n}, Y_{[1:n]}), \ldots, (X_{r:n}, Y_{[r:n]}), \ldots, (X_{n:n}, Y_{[n:n]})\}$ from this sample. Let u stand for the age of live with random lifetime X and v stand for the age of live with random lifetime Y under concern. It is realistic to assume that the future lifetime of each live of the pair gets shorter as their ages get larger. Therefore, the choice of the rank r for the order statistic $X_{r:n}$ is a critical issue for the risk assessments, with the loss prevention aims, as an indicator of the age level in the analysis. Suppose that the type of insurance under concern is a whole life insurance with benefits payable at the moment of death. Here, the life insurance contracts can be in two general categories known as the contracts for general and contingent life insurances. The contingent insurance contracts provide benefits upon the order of death of a designated live and no benefit is paid if this does not happen. On the other hand, under the general life insurance contracts, there are two major insurance status types known as the joint-life status and the last-survivor status. The concepts and models about these contract categories and status types are well known in the literature and we refer to [22, pp. 444–473,488–491] and [23, pp. 83–92] for their details.

The termination of the joint-life status is realized when the first death of the pair (X, Y) occurs. Whereas, the last-survivor status terminates upon the last death of the pair of lives with lifetimes (X, Y). So, the lifetime of the joint-life status is "$\min\{T(u), T(v)\}$" and the lifetime of the last-survivor status is "$\max\{T(u), T(v)\}$" where $T(u)$ and $T(v)$ denote the random future lifetimes of the pair who are currently, that is at the time of the sample, at the ages of u and v, respectively. Then, the death of either one or both of the two lives is a cause of benefit payment for a life insurer in the future time durations of "$u + T(u)$" and "$v + T(v)$". It is clear that not only the termination of the life insurance statuses but also the times until the deaths are the matters of concern for the insurer who bears the liability of death benefits payments.

We first consider the last-survivor status type with respect to future lifetimes $T(u)$ and $T(v)$ of the pair who are at ages of u and v, respectively. Following [22,23] and their notation, the actuarial present value of the benefit payments, at the time that the last-survivor status ends, is expressed as

$$\overline{A}_{\overline{uv}} = \overline{A}_u + \overline{A}_v - \overline{A}_{uv}, \tag{26}$$

where "uv" denotes the joint-life status, "$\overline{uv}$" denotes the last-survivor status, and the separate lives statuses are denoted by "u and "v". $\bar{A}_{\overline{uv}}$, $\bar{A}_{uv}$, $\bar{A}_u$ and $\bar{A}_v$ are actuarial present values, or the net single premium amounts, of the same insurance applied to the respective statuses.

In order to express the components of the sum in Eq. (26), one needs to compute the future lifetime probabilities for each of the defined statuses. Concerning the second component of the sum, there follows the computation of the probability that the future lifetime of the concomitant, $T(v)$, is at least t time amount;

$$P\left(T(v) > t\right) = P\left(Y_{[r:n]} > v + t \mid Y_{[r:n]} > v\right)$$
$$= \frac{P\left(Y_{[r:n]} > v + t\right)}{P\left(Y_{[r:n]} > v\right)} = \frac{S_{Y_{[r:n]}}\left(v + t\right)}{S_{Y_{[r:n]}}(v)} = {}_t p_v \tag{27}$$

where $S_{Y_{[r:n]}}(\cdot)$ functions are as defined in Eq. (24). The benefit payment to the concomitant becomes due at time just the moment immediately after $T(v) + v$ if the death of the insured occurs then. Let the present value of the benefit payment, say w, which may be a level or a varying function of time t, be $\theta^t w$ where θ^t is the discount factor for a given force of interest. The actuarial present value of this benefit payment is then calculated as

$$\bar{A}_v = \int_0^{\infty} \left(\theta^t w\right) h_{Y_{[r:n]}}\left(v + t\right) {}_t p_v dt \tag{28}$$

where the hazard function (or mortality rate function) is as defined in Eq. (25) with y replaced by $v + t$, and ${}_t p_v$ is as defined above in expression (27). $\bar{A}_v$ is the basis of the age specific valuation of the cost that the insurer incurs, and it is a function of the survival and hazard functions for the concomitant lifetime $Y_{[r:n]}$. Note that, the larger the future lifetime and the age of the concomitant live the smaller is the value of the survival function.

The actuarial presented value $\bar{A}_u$ in Eq. (26) is computed as

$$\bar{A}_u = \int_0^{\infty} \left(\theta^t w\right) h_{X_{r:n}}\left(u + t\right) {}_t p_u dt \tag{29}$$

where $h_{X_{r:n}}(\cdot)$ is expressed in Eq. (B.6) of Appendix B, and ${}_t p_u$ is the probability that the first live in (X, Y) at the age of u will live for a time duration of $u + T(u)$, at least, where $T(u) = t$. Here, ${}_t p_u$ can be calculated by using the survival function for $X_{r:n}$, which is derived in the part (a) of Appendix B.

The value of $\bar{A}_{uv}$ is calculated similarly as follows:

$$\bar{A}_{uv} = \int_0^{\infty} \left(\theta^t w\right) h_{X_{r:n}, Y_{[r:n]}}\left(u + t, v + t\right) {}_t p_{uv} dt \tag{30}$$

with $h_{X_{r:n}, Y_{[r:n]}}(., .)$ standing as the joint hazard function, and ${}_t p_{uv}$ standing as the joint survival function for $X_{r:n}$ and $Y_{[r:n]}$. Both functions are derived and clearly shown in part (b) of Appendix B.

Under the contingent life insurance contracts category, assume that the concomitant live with lifetime Y is the designated live, and the benefit is payable only if this live dies second. Following [22,23], again, the actuarial present value of the death benefit for this case is expressed and calculated as

$$\bar{A}_{uv^2} = \bar{A}_v - \bar{A}_{uv^1}, \tag{31}$$

where $\bar{A}_v$ is already expressed in Eq. (28), and the second component is the actuarial present value of the death benefit payable on the event that the concomitant live dies first at the age of $v + t$:

$$\bar{A}_{uv^1} = \int_0^{\infty} \left(\theta^t w\right) {}_t p_{uv} h_{Y_{[r:n]}}\left(v + t\right) dt. \tag{32}$$

The characteristic behaviors of the survival and hazard functions for the concomitant $Y_{[r:n]}$ are Tabled below for $n = 10$, $r = 1, \ldots, 10$, and $0.1 \leq y \leq 1$ for the selected values of the parameters μ_1, μ_2 and λ of the bivariate Pseudo–Gompertz distribution.

Table 1 contains the values of the survival function (24). Looking at these tables one can see that the survival probability of the concomitant increases as the value of λ increases, holding y, r, μ_1 and μ_2 at a fixed level. For fixed λ, y, μ_1, μ_2; the survival probability decreases while the rank r of the order statistic $X_{r:n}$ increases. The table shows also that for the fixed λ, r, μ_1, μ_2; values, the survival probability decreases as the value y of the concomitant $Y_{[r:n]}$ increases.

Table 2 contains the values of the hazard function in Eq. (25). It is seen that the hazard rate of the concomitant decreases as the value of λ is increases, while holding y, r, μ_1 and μ_2 at fixed values. Further, for the fixed values of λ, y, μ_1, μ_2, the hazard function increases as r increases, and it declines as y gets larger.

The behavior of the survival function of the concomitant live is presented in the graphics below. Fig. 1(a) displays that the survival function of the concomitant $Y_{[r:n]}$ for $y = 0.1$, 0.5 and 1. It is seen that the survival function declines faster as

Table 1
Survival function for the concomitant $Y_{[r:n]}$.

y	r									
	1	2	3	4	5	6	7	8	9	10
$\lambda = 0.02$										
0.1	0.9901	0.9792	0.9671	0.9535	0.9378	0.9194	0.8970	0.8681	0.8267	0.7515
0.2	0.9804	0.9590	0.9356	0.9096	0.8802	0.8464	0.8060	0.7556	0.6868	0.5723
0.3	0.9708	0.9395	0.9055	0.8682	0.8268	0.7799	0.7254	0.6594	0.5733	0.4408
0.4	0.9615	0.9205	0.8766	0.8291	0.7772	0.7195	0.6540	0.5769	0.4806	0.3431
0.5	0.9523	0.9020	0.8488	0.7921	0.7310	0.6644	0.5904	0.5059	0.4045	0.2695
0.6	0.9432	0.8841	0.8223	0.7572	0.6881	0.6142	0.5339	0.4447	0.3418	0.2134
0.7	0.9344	0.8667	0.7968	0.7241	0.6482	0.5684	0.4835	0.3917	0.2899	0.1703
0.8	0.9257	0.8498	0.7723	0.6928	0.6110	0.5264	0.4384	0.3458	0.2467	0.1368
0.9	0.9171	0.8334	0.7488	0.6631	0.5763	0.4880	0.3981	0.3059	0.2107	0.1106
1.0	0.9087	0.8174	0.7262	0.6350	0.5439	0.4529	0.3619	0.2711	0.1804	0.0900
$\lambda = 0.05$										
0.1	0.9960	0.9916	0.9867	0.9811	0.9746	0.9668	0.9572	0.9446	0.9261	0.8905
0.2	0.9921	0.9833	0.9736	0.9625	0.9499	0.9349	0.9165	0.8927	0.8584	0.7947
0.3	0.9881	0.9751	0.9607	0.9445	0.9259	0.9042	0.8778	0.8440	0.7962	0.7107
0.4	0.9842	0.9670	0.9480	0.9268	0.9027	0.8746	0.8409	0.7982	0.7390	0.6369
0.5	0.9803	0.9590	0.9355	0.9095	0.8801	0.8461	0.8058	0.7553	0.6865	0.5718
0.6	0.9765	0.9511	0.9233	0.8926	0.8582	0.8187	0.7723	0.7149	0.6381	0.5143
0.7	0.9727	0.9432	0.9112	0.8761	0.8369	0.7923	0.7403	0.6769	0.5935	0.4633
0.8	0.9689	0.9355	0.8994	0.8599	0.8162	0.7669	0.7099	0.6412	0.5525	0.4181
0.9	0.9651	0.9278	0.8877	0.8441	0.7961	0.7424	0.6809	0.6076	0.5146	0.3779
1.0	0.9614	0.9202	0.8762	0.8286	0.7766	0.7188	0.6532	0.5760	0.4796	0.3421
$\lambda = 0.075$										
0.1	0.9973	0.9944	0.9911	0.9873	0.9830	0.9777	0.9713	0.9627	0.9500	0.9253
0.2	0.9947	0.9888	0.9823	0.9748	0.9662	0.9560	0.9434	0.9269	0.9028	0.8571
0.3	0.9921	0.9833	0.9735	0.9625	0.9498	0.9349	0.9165	0.8927	0.8583	0.7946
0.4	0.9894	0.9778	0.9649	0.9504	0.9338	0.9142	0.8904	0.8598	0.8162	0.7374
0.5	0.9868	0.9724	0.9564	0.9385	0.9180	0.8941	0.8652	0.8283	0.7764	0.6849
0.6	0.9842	0.9670	0.9479	0.9267	0.9026	0.8745	0.8408	0.7981	0.7388	0.6366
0.7	0.9816	0.9616	0.9396	0.9151	0.8874	0.8554	0.8171	0.7691	0.7032	0.5923
0.8	0.9790	0.9563	0.9313	0.9037	0.8725	0.8367	0.7942	0.7412	0.6695	0.5514
0.9	0.9765	0.9510	0.9232	0.8924	0.8580	0.8185	0.7720	0.7145	0.6377	0.5138
1.0	0.9739	0.9457	0.9151	0.8813	0.8437	0.8007	0.7505	0.6889	0.6075	0.4791
$\lambda = 0.1$										
0.1	0.9980	0.9958	0.9933	0.9905	0.9872	0.9832	0.9784	0.9719	0.9622	0.9434
0.2	0.9960	0.9916	0.9867	0.9811	0.9745	0.9668	0.9572	0.9446	0.9261	0.8904
0.3	0.9940	0.9874	0.9801	0.9717	0.9621	0.9507	0.9366	0.9182	0.8914	0.8409
0.4	0.9920	0.9833	0.9735	0.9625	0.9498	0.9348	0.9165	0.8926	0.8582	0.7945
0.5	0.9901	0.9792	0.9670	0.9534	0.9377	0.9193	0.8968	0.8678	0.8264	0.7511
0.6	0.9881	0.9751	0.9606	0.9444	0.9258	0.9040	0.8776	0.8438	0.7959	0.7104
0.7	0.9861	0.9710	0.9542	0.9355	0.9141	0.8891	0.8589	0.8205	0.7666	0.6722
0.8	0.9842	0.9669	0.9479	0.9266	0.9025	0.8744	0.8406	0.7979	0.7386	0.6363
0.9	0.9822	0.9629	0.9416	0.9179	0.8911	0.8600	0.8228	0.7760	0.7117	0.6027
1.0	0.9803	0.9589	0.9354	0.9093	0.8798	0.8458	0.8053	0.7548	0.6858	0.5710

λ increases when order rank value r increases. The rate of decrease in the survival function gets higher for the increasing values of the observable lifetime value y of the concomitant.

Fig. 1(b) displays that the survival function for the ranks of order $r = 1, 5$ and 10. As seen in the figure, the survival function declines for the increasing values of lifetime y, and the rate of the decline becomes faster as r and λ values get larger.

The behavior of the hazard function of the concomitant $Y_{[r:n]}$ are depicted in Fig. 2(a) and (b). As noticed from Fig. 2(a), the hazard function tends to increase as r increases and as λ gets smaller. The rate of increase is slightly faster for the smaller values of y. Fig. 2(b) presents that the hazard function has a declining shape for the increasing y values. It is also presented in the figure that the level of the hazard function values gets higher as r, the order of the magnitude of the lifetime X, grows large.

6. Conclusion

The bivariate Pseudo–Gompertz distribution that this paper introduces has many potential uses. The distribution suits extremely well to the applications of lifetime modeling in reliability and survival analysis, financial risk modeling and

Table 2
Hazard function for the concomitant $Y_{[r:n]}$.

y	r									
	1	2	3	4	5	6	7	8	9	10
$\lambda = 0.02$										
0.1	0.0991	0.2091	0.3327	0.4737	0.6378	0.8340	1.0782	1.4011	1.8777	2.7877
0.2	0.0982	0.2071	0.3293	0.4685	0.6301	0.8228	1.0614	1.3745	1.8299	2.6647
0.3	0.0974	0.2052	0.3261	0.463	0.6226	0.8119	1.0451	1.3490	1.7850	2.5563
0.4	0.0965	0.2033	0.3228	0.4585	0.6154	0.8013	1.0294	1.3246	1.7428	2.4596
0.5	0.0957	0.2015	0.3197	0.4537	0.6083	0.7910	1.0142	1.3013	1.7031	2.3725
0.6	0.0949	0.1997	0.3166	0.4490	0.6013	0.7809	0.9995	1.2789	1.6655	2.2936
0.7	0.0941	0.1979	0.3136	0.4443	0.5946	0.7712	0.9853	1.2573	1.6300	2.2215
0.8	0.0933	0.1961	0.3106	0.4398	0.5880	0.7617	0.9716	1.2366	1.5962	2.1552
0.9	0.0925	0.1944	0.3077	0.4354	0.5816	0.7525	0.9582	1.2167	1.5641	2.0941
1.0	0.0918	0.1927	0.3049	0.4311	0.5753	0.7435	0.9453	1.1975	1.5336	2.0374
$\lambda = 0.05$										
0.1	0.0399	0.0842	0.1340	0.1908	0.2571	0.3366	0.4357	0.5674	0.7637	1.1487
0.2	0.0398	0.0839	0.1335	0.1901	0.2560	0.3349	0.4332	0.5633	0.7560	1.1271
0.3	0.0396	0.0836	0.1330	0.1894	0.2549	0.3333	0.4307	0.5593	0.7485	1.1066
0.4	0.0395	0.0834	0.1326	0.1887	0.2539	0.3317	0.4282	0.5553	0.7412	1.0873
0.5	0.0394	0.0831	0.1321	0.1880	0.2528	0.3301	0.4258	0.5514	0.7341	1.0690
0.6	0.0393	0.0828	0.1317	0.1872	0.2517	0.3285	0.4234	0.5476	0.7272	1.0515
0.7	0.0392	0.0826	0.1312	0.1865	0.2507	0.3270	0.4210	0.5438	0.7204	1.0349
0.8	0.0391	0.0823	0.1308	0.1859	0.2496	0.3254	0.4187	0.5401	0.7138	1.0190
0.9	0.0390	0.0821	0.1303	0.1852	0.2486	0.3239	0.4164	0.5365	0.7074	1.0038
1.0	0.0388	0.0818	0.1340	0.1845	0.2476	0.3224	0.4142	0.5329	0.7011	0.9893
$\lambda = 0.075$										
0.1	0.0266	0.0562	0.0894	0.1274	0.1717	0.2248	0.2911	0.3793	0.5110	0.7710
0.2	0.0266	0.0561	0.0893	0.1272	0.1713	0.2242	0.2901	0.3776	0.5077	0.7614
0.3	0.0265	0.0560	0.0891	0.1269	0.1709	0.2235	0.2891	0.3759	0.5045	0.7521
0.4	0.0265	0.0559	0.0889	0.1266	0.1704	0.2229	0.2880	0.3742	0.5013	0.7432
0.5	0.0264	0.0558	0.0887	0.1263	0.1700	0.2222	0.2870	0.3726	0.4982	0.7346
0.6	0.0264	0.0557	0.0886	0.1260	0.1696	0.2216	0.2860	0.3709	0.4951	0.7263
0.7	0.0264	0.0556	0.0884	0.1258	0.1692	0.2209	0.2851	0.3693	0.4921	0.7183
0.8	0.0263	0.0555	0.0882	0.1255	0.1687	0.2203	0.2841	0.3677	0.4891	0.7105
0.9	0.0263	0.0554	0.0880	0.1252	0.1683	0.2197	0.2831	0.3661	0.4862	0.7030
1.0	0.0262	0.0553	0.0879	0.1249	0.1679	0.2190	0.2821	0.3646	0.4833	0.6957
$\lambda = 0.1$										
0.1	0.0200	0.0422	0.0671	0.0957	0.1289	0.1688	0.2186	0.2849	0.3840	0.5803
0.2	0.0200	0.0421	0.0671	0.0955	0.1287	0.1685	0.2181	0.2840	0.3822	0.5749
0.3	0.0199	0.0421	0.0670	0.0954	0.1285	0.1681	0.2175	0.2831	0.3805	0.5697
0.4	0.0199	0.0420	0.0669	0.0952	0.1283	0.1678	0.2170	0.2822	0.3787	0.5646
0.5	0.0199	0.0420	0.0668	0.0951	0.1281	0.1675	0.2165	0.2813	0.3770	0.5597
0.6	0.0199	0.0419	0.0667	0.0950	0.1279	0.1671	0.2160	0.2805	0.3754	0.5549
0.7	0.0199	0.0419	0.0666	0.0948	0.1276	0.1668	0.2155	0.2796	0.3737	0.5503
0.8	0.0198	0.0418	0.0666	0.0947	0.1274	0.1665	0.2150	0.2787	0.3721	0.5458
0.9	0.0198	0.0418	0.0665	0.0946	0.1272	0.1662	0.2144	0.2779	0.3704	0.5413
1.0	0.0198	0.0418	0.0664	0.0944	0.1270	0.1659	0.2139	0.2771	0.3688	0.5370

insurance product valuations as stressed in the manuscript. The combination of the variates in the random lifetimes vector (X, Y) by a real valued parameter function $\phi(x)$ is an essential feature of the model. This function must be a meaningful function in the survival analysis and risk modeling attempts that parametrically relates a random lifetime Y with another random lifetime X. The function must also satisfy that the joint distribution of (X, Y) carries all the properties to be a distribution function. The form of $\phi(\cdot)$ function, that is presented in the paper, is an example of its kind and it can be modified according to the needs of the users.

The distribution, survival and hazard functions for the lifetimes with a Pseudo–Gompertz distribution model are presented here in practical and useful forms that are achieved by utilizing Harmonic numbers and Gamma functions. An exemplification of the survival and hazard functions are provided in the tables of the paper for some selected parameters and values of the concerned variables. Similar tables can be constructed in a wider perspective for multiple lifetimes. In particular, life tables for the general and contingent multiple-life insurances can be constructed with actuarial considerations for the practical uses in finance and insurance areas. Similarly, by employing the results of this paper, more detailed survival and hazard tables can be computed for the reliability analysis of the physical systems with pairs of units in their components where each unit functions as reserve unit for the other.

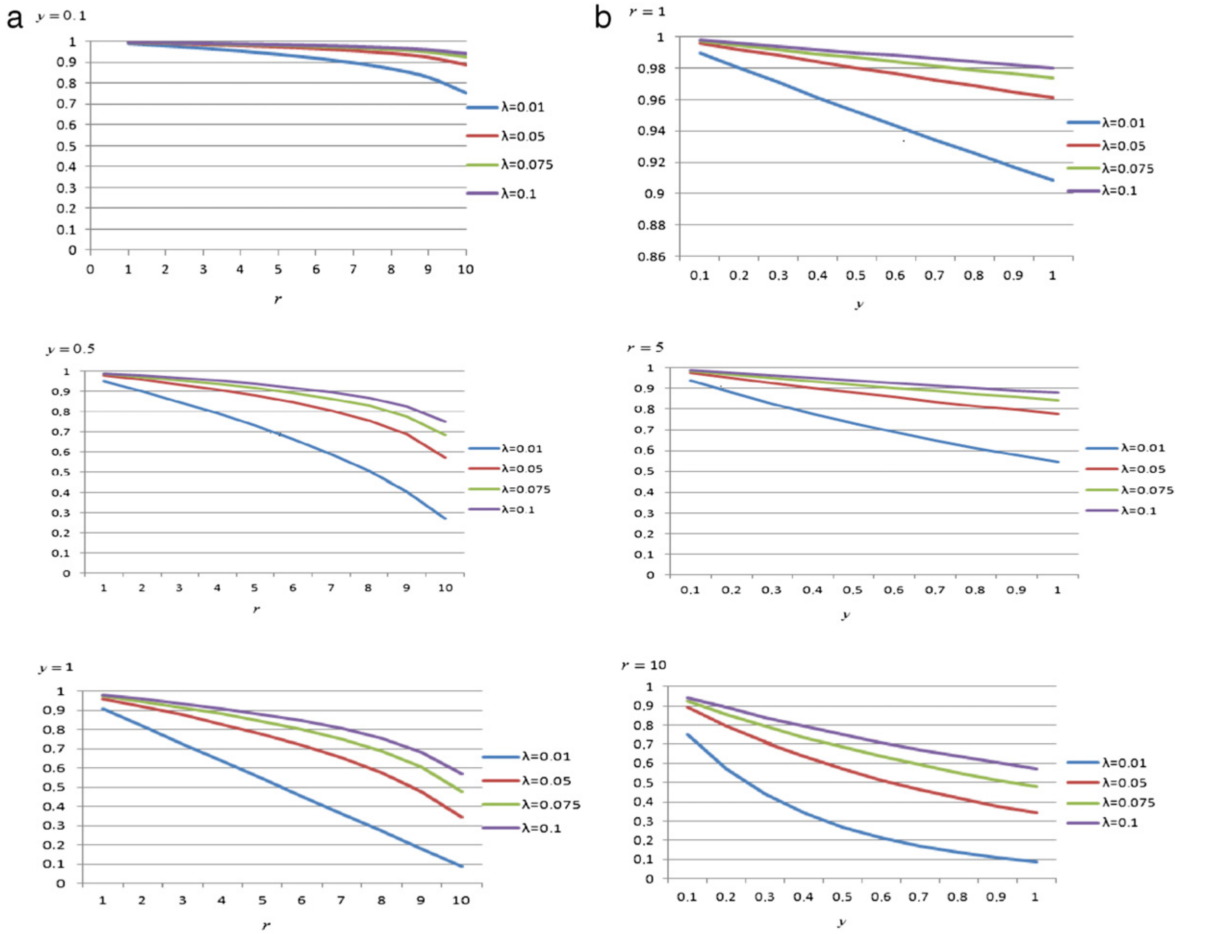


Fig. 1. (a) Survival function for the concomitant $Y_{[r:n]}$ values of $y = 0.1$, $y = 0.5$, $y = 1$. (b) Survival function for the concomitant $Y_{[r:n]}$ values of $r = 1$, $r = 5$, $r = 10$.

Appendix A

The computations below show that $f(x, y)$ in expression (9) is the density function of a bivariate Pseudo–Gompertz distribution:

$$\begin{aligned} F(x,y) &= \int_0^x \int_0^y f(x,y)dxdy \\ &= \int_0^y \int_0^x \lambda(e^{\mu_1 x}-1)e^{\mu_1 x}e^{\mu_2 x}\exp\left[-(e^{\mu_1 x}-1)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y}-1)}{\mu_2}\right)\right]dxdy \\ &= 1-\exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right)+\frac{\left(\exp\left(-\frac{(e^{\mu_1 x}-1)((e^{\mu_2 y}-1)\mu_1+\lambda\mu_2)}{\mu_1\mu_2}\right)-1\right)\lambda\mu_2}{(e^{\mu_2 y}-1)\,\mu_1+\lambda\mu_2} \end{aligned}$$

$$\operatorname*{Lim}_{x,y\to\infty} F(x,y) = 1$$

$$\operatorname*{Lim}_{x\to 0} F(x,y) = \operatorname*{Lim}_{y\to 0} F(x,y) = \operatorname*{Lim}_{x,y\to 0} F(x,y) = 0$$

$$F_1(x) = \operatorname*{Lim}_{y\to\infty} F(x,y) = 1-\exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right)$$

$$F_2(y) = \operatorname*{Lim}_{x\to\infty} F(x,y) = 1-\frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}.$$

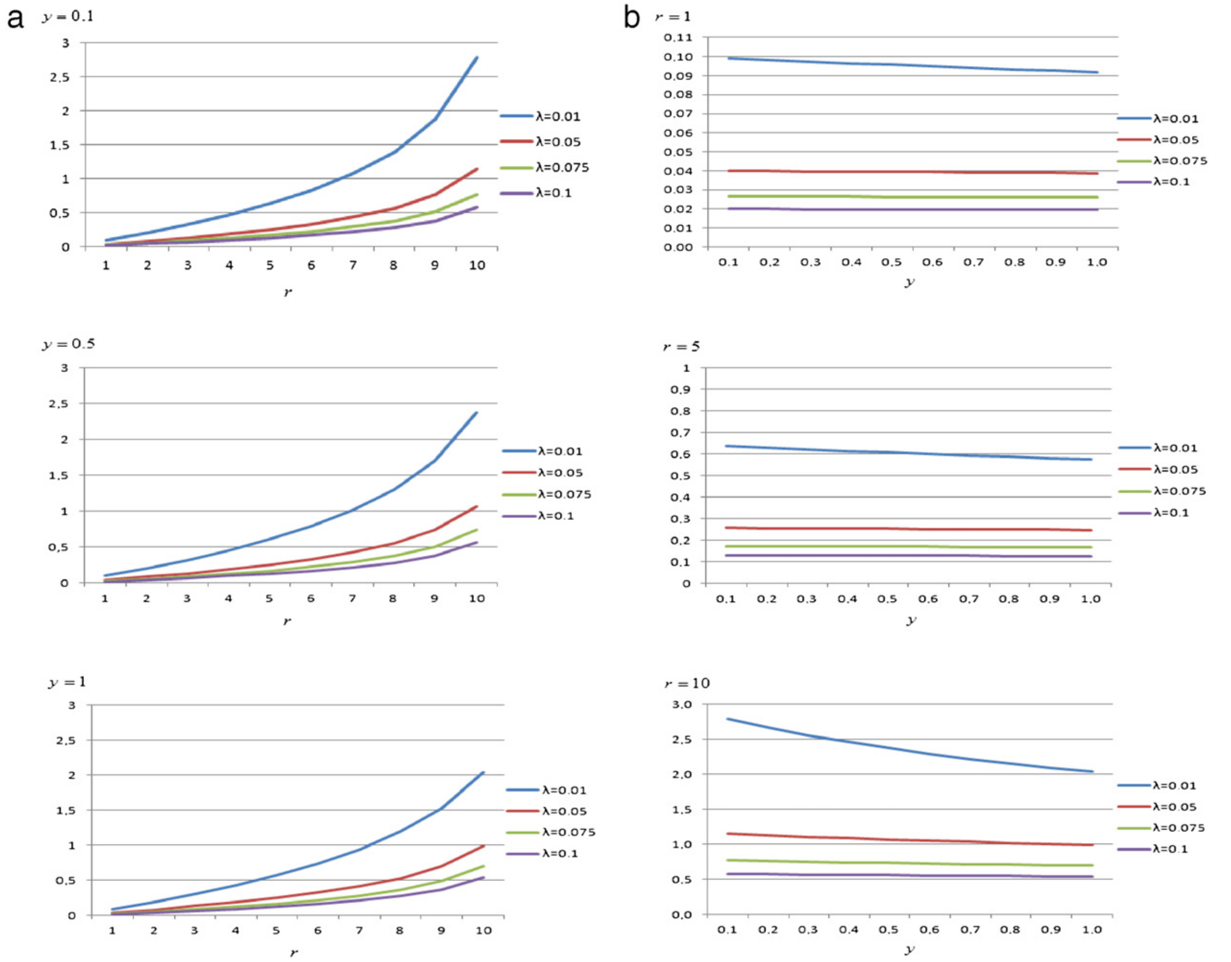


Fig. 2. (a) Hazard function for the concomitant $Y_{[r:n]}$ values of $y = 0.1, y = 0.5, y = 1$. (b) Hazard function for the concomitant $Y_{[r:n]}$ values of $r = 1$, $r = 5, r = 10$.

And, under the transformation $(e^{\mu_1 x} - 1) = u$, we can write

$$\begin{aligned} F(x,y) &= \int_0^\infty \int_0^\infty f(x,y)\,dxdy \\ &= \int_0^\infty \int_0^\infty \lambda\left(e^{\mu_1 x}-1\right)e^{\mu_1 x}e^{\mu_2 y}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y}-1)}{\mu_2}\right)\right]\,dxdy \\ &= \int_0^\infty \frac{\lambda}{\mu_1}e^{\mu_2 y}\int_0^\infty u\exp\left[-u\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y}-1)}{\mu_2}\right)\right]\,dudy \\ &= \int_0^\infty \frac{\lambda}{\mu_1}e^{\mu_2 y}\frac{1}{\left(\frac{\lambda}{\mu_1}+\frac{(e^{\mu_2 y}-1)}{\mu_2}\right)^2}dy \end{aligned}$$

which, under the transformation $(e^{\mu_2 y} - 1) = v$, turns out to be

$$\begin{aligned} F(x,y) &= \frac{\lambda}{\mu_1\mu_2}\int_0^\infty \frac{1}{\left(\frac{\lambda}{\mu_1}+\frac{v}{\mu_2}\right)^2}dv \\ &= \frac{\lambda}{\mu_1\mu_2}\frac{\mu_1\mu_2}{\lambda} = 1. \end{aligned}$$

Appendix B

(a) The survival and hazard functions for $X_{r:n}$, under the Pseudo–Gompertz distribution, are derived and displayed below: The probability density and distribution functions for $X_{r:n}$ are

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!\,(n-r)!}\lambda e^{\mu_1 x}\left[1-\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right]\right]^{r-1}\left[\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right]\right]^{n-r+1} \\ &= \frac{\lambda e^{\mu_1 x} n!}{(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+h+1)}, \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} F_{X_{r:n}}(x) &= \int_0^x f_{X_{[r:n]}}(x)dx \\ &= \int_0^x \frac{n!}{(r-1)!\,(n-r)!}\lambda e^{\mu_1 x}\left[1-\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right]\right]^{r-1}\left[\exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)\right]\right]^{n-r+1} dx \end{aligned} \tag{B.2}$$

which, after the transformation

$$\left(e^{\mu_1 x}-1\right)=u, \qquad \mu_1 e^{\mu_1 x}dx = du,$$

turns out to be

$$= \frac{n!}{(r-1)!\,(n-r)!}\frac{\lambda}{\mu_1}\int_0^{e^{\mu_1 x}-1}\left[1-\exp\left(-\frac{\lambda}{\mu_1}u\right)\right]^{r-1}\left[\exp\left(-\frac{\lambda}{\mu_1}u\right)\right]^{n-r+1}du.$$

Since the last component here can be expressed as

$$\left[1-\exp\left(-\frac{\lambda}{\mu_1}u\right)\right]^{r-1} = \sum_{h=0}^{r-1}\binom{r-1}{h}(-1)^h\left[\exp\left(-\frac{\lambda}{\mu_1}u\right)\right]^h,$$

Eq. (B.2) reduces to

$$\begin{aligned} &= \frac{n!}{(r-1)!\,(n-r)!}\frac{\lambda}{\mu_1}\int_0^{e^{\mu_1 x}-1}\sum_{h=0}^{r-1}\binom{r-1}{h}(-1)^h\left[\exp\left(-\frac{\lambda}{\mu_1}u\right)\right]^{n-r+h+1}du \\ &= \frac{n!}{(r-1)!\,(n-r)!}\sum_{h=0}^{r-1}\binom{r-1}{h}(-1)^h\frac{1}{(n-r+h+1)}\left(1-\exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+h+1)\right)\right). \end{aligned} \tag{B.3}$$

This can be re-expressed by making use of the identity

$$\begin{aligned} &\sum_{h=0}^{r-1}\binom{r-1}{h}(-1)^h\frac{1}{(a+h)}\left(1-\exp\left(-b\left(a+h\right)\right)\right) \\ &\quad = e^{-ab}\frac{\left(ae^{ab}\left(a-1\right)!(r-1)!-(a+r-1)!\,{}_2F_1\left(a,r-1,a+1,e^{-b}\right)\right)}{a\left(a+r-1\right)!} \end{aligned}$$

where ${}_2F_1\left(a,b;c;z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$ is the Gauss Hypergeometric function as given in [18]. This function will be used also in the rest of the expressions of Appendix B.

A simpler expression for (B.3) is given below by denoting $n-r+1=a$ and $\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)=b$;

$$\begin{aligned} &= \frac{n!}{(r-1)!(n-r)!}e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}\frac{1}{(n-r+1)(n-r+1+r-1)!} \\ &\quad \times\left((n-r+1)e^{\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}(n-r+1-1)!(r-1)!-(n-r+1+r-1)!\,{}_2F_1\right. \\ &\quad \times\left.\left(n-r+1,1-r,1+n-r+1,e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)}\right)\right) \\ &= \frac{n!}{(r-1)!(n-r)!}e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}\frac{1}{(n-r+1)n!} \\ &\quad \times\left((n-r+1)e^{\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}(n-r)!(r-1)!-n!\,{}_2F_1\left(n-r+1,1-r,1+n-r+1,e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right)\right) \\ &= \frac{n!}{(r-1)!(n-r)!}e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}\frac{1}{(n-r+1)n!}(n-r+1)e^{\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)}(n-r)!(r-1)! \end{aligned}$$

$$- \frac{n!}{(r-1)!(n-r)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} \frac{1}{(n-r+1)n!} n!\,{}_2F_1\left(n-r+1, 1-r, 1+n-r+1, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right)$$
$$= 1 - \frac{n!}{(r-1)!(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right).$$

Therefore the distribution function for $X_{r:n}$ is

$$F_{X_{r:n}}(x) = 1 - \frac{n!}{(r-1)!\,(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right). \tag{B.4}$$

By its definition, the survival function of $S_{X_{r:n}}(x) = 1 - F_{X_{r:n}}(x)$ of $X_{r:n}$ is

$$S_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right). \tag{B.5}$$

The hazard function for $X_{r:n}$ is derived from the definition $h_{X_{r:n}}(x) = \frac{f_{X_{r:n}}(x)}{S_{X_{r:n}}(x)}$ and shown below:

$$h_{X_{r:n}}(x) = \frac{\frac{\lambda e^{\mu_1 x} n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+h+1)}}{\frac{n!}{(r-1)!(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right)}. \tag{B.6}$$

(b) The derivations of the joint survival and hazard functions for the $\{X_{r:n}, Y_{[r:n]}\}$ pair are shown in this part. It is known that, by denoting $f_{Y_{[r:n]}}(y \mid X_{r:n} = x) = f(y \mid x)$, the joint density function for the random vector $\{X_{r:n}, Y_{[r:n]}\}$ is expressed by $f_{X_{r:n},Y_{[r:n]}}(x, y) = f(y \mid x) f_{X_{r:n}}(x)$ as shown in [24].

The joint survival and hazard functions have the forms of

$$S_{X_{r:n},Y_{[r:n]}}(x, y) = S(y \mid x) S_{X_{r:n}}(x) \quad \text{and} \tag{B.7}$$
$$h_{X_{r:n},Y_{[r:n]}}(x, y) = h(y \mid x) h_{X_{r:n}}(x), \tag{B.8}$$

respectively. Then, using the joint density function of the Pseudo–Gompertz distribution

$$f(x, y) = \lambda\left(e^{\mu_1 x} - 1\right) e^{\mu_1 x} e^{\mu_2 y} \exp\left[-\left(e^{\mu_1 x} - 1\right)\left(\frac{\lambda}{\mu_1} + \frac{(e^{\mu_2 y} - 1)}{\mu_2}\right)\right], \quad \mu_1,\ \mu_2,\ \lambda,\ y,\ x > 0,$$

we obtain the joint survival function as follows:

Given the definitions $S_{X_{r:n},Y_{[r:n]}}(x, y) = S(y \mid x) S_{X_{r:n}}(x)$ and $S(y \mid x) = \frac{S(x,y)}{S(x)}$, where

$$S(x, y) = \frac{\lambda\mu_2}{\mu_1(e^{\mu_2 y} - 1) + \mu_2\lambda} \exp\left[\frac{(e^{\mu_2 y} - 1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y} - 1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right] \quad \text{and} \tag{B.9}$$
$$S(x) = \exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right), \tag{B.10}$$

there follows the conditional survival function

$$S(y \mid x) = \frac{\frac{\lambda\mu_2}{\mu_1(e^{\mu_2 y}-1)+\mu_2\lambda} \exp\left[\frac{(e^{\mu_2 y}-1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right]}{\exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right)}. \tag{B.11}$$

Placing in (B.7) the expression in (B.11) and

$$S_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right) \tag{B.12}$$

we obtain

$$\begin{aligned} S_{X_{r:n},Y_{[r:n]}}(x, y) &= S(y \mid x) S_{X_{r:n}}(x) \\ &= \frac{\frac{\lambda\mu_2}{\mu_1(e^{\mu_2 y}-1)+\mu_2\lambda} \exp\left[\frac{(e^{\mu_2 y}-1)}{\mu_2} - e^{\mu_1 x}\left(\frac{(e^{\mu_2 y}-1)}{\mu_2} + \frac{\lambda}{\mu_1}\right) + \frac{\lambda}{\mu_1}\right]}{\exp\left(-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right)} \\ &\quad \times \frac{n!}{(r-1)!\,(n-r+1)!} e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)(n-r+1)} {}_2F_1\left(n-r+1, 1-r, n-r+2, e^{-\frac{\lambda}{\mu_1}(e^{\mu_1 x}-1)}\right) \end{aligned}$$

$$\begin{aligned} S_{X_{r:n},Y_{[r:n]}}(x,y) &= \frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}+\frac{\lambda}{\mu_1}(n-r+1)\right)\right] \\ &\quad\times\frac{n!}{(r-1)!\,(n-r+1)!}\,{}_2F_1\left(n-r+1,1-r,n-r+2,e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)}\right). \end{aligned} \tag{B.13}$$

Then, using these results, the probability that the two lives, currently at the ages of u and v, will live for at least t time durations can be expressed:

$$\begin{aligned} P\left(T(u)>t,T(v)>t\right) &= P\left(X_{r:n}>u+t,Y_{[r:n]}>v+t\mid X_{r:n}>u,Y_{[r:n]}>v\right) \\ &= \frac{P\left(X_{r:n}>u+t,Y_{[r:n]}>v+t\right)}{P\left(X_{r:n}>u,Y_{[r:n]}>v\right)}=\frac{S_{X_{r:n},Y_{[r:n]}}(u+t,v+t)}{S_{X_{r:n},Y_{[r:n]}}(u,v)}={}_tp_{uv}. \end{aligned} \tag{B.14}$$

The probability that the concomitant live lives for a time duration of t, at least, is

$$\begin{aligned} P\left(T(u)>t\right) &= P\left(X_{r:n}>u+t\mid X_{r:n}>u\right) \\ &= \frac{P\left(X_{r:n}>u+t\right)}{P\left(X_{r:n}>u\right)}=\frac{S_{X_{r:n}}(u+t)}{S_{X_{r:n}}(u)}={}_tp_u. \end{aligned} \tag{B.15}$$

The joint hazard function, then, is found as

$$h(x,y)=\frac{\lambda\left(e^{\mu_1 x}-1\right)e^{\mu_1 x}e^{\mu_2 y}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}\right)\right]}{\frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}\exp\left[\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}-e^{\mu_1 x}\left(\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}+\frac{\lambda}{\mu_1}\right)+\frac{\lambda}{\mu_1}\right]}. \tag{B.16}$$

Further, the conditional hazard function, that uses (B.16) and $h(x)=\lambda e^{\mu_1 x}$, is derived as

$$\begin{aligned} h(y\mid x) &= \frac{h(x,y)}{h(x)} \\ &= \frac{\dfrac{\lambda\left(e^{\mu_1 x}-1\right)e^{\mu_1 x}e^{\mu_2 y}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}\right)\right]}{\frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}\exp\left[\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}-e^{\mu_1 x}\left(\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}+\frac{\lambda}{\mu_1}\right)+\frac{\lambda}{\mu_1}\right]}}{\lambda e^{\mu_1 x}} \\ &= \frac{\left(e^{\mu_1 x}-1\right)e^{\mu_2 y}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}\right)\right]}{\frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}\exp\left[\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}-e^{\mu_1 x}\left(\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}+\frac{\lambda}{\mu_1}\right)+\frac{\lambda}{\mu_1}\right]}. \end{aligned} \tag{B.17}$$

As the result; using (B.6) and (B.17) above, the sought joint hazard function is obtained; $h_{X_{r:n},Y_{[r:n]}}(x,y)=h(y\mid x)h_{X_{r:n}}(x)$

$$\begin{aligned} &= \left(\frac{\left(e^{\mu_1 x}-1\right)e^{\mu_2 y}\exp\left[-\left(e^{\mu_1 x}-1\right)\left(\frac{\lambda}{\mu_1}+\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}\right)\right]}{\frac{\lambda\mu_2}{\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda}\exp\left[\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}-e^{\mu_1 x}\left(\frac{\left(e^{\mu_2 y}-1\right)}{\mu_2}+\frac{\lambda}{\mu_1}\right)+\frac{\lambda}{\mu_1}\right]}\right) \\ &\quad\times\left(\frac{\frac{\lambda e^{\mu_1 x}n!}{(r-1)!(n-r)!}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+h+1)}}{\frac{n!}{(r-1)!(n-r+1)!}e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+1)}{}_2F_1\left(n-r+1,1-r,n-r+2,e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)}\right)}\right). \end{aligned} \tag{B.18}$$

After the necessary simplifications, a simpler form of this function is obtained: $h_{X_{r:n},Y_{[r:n]}}(x,y)=\frac{\left(\mu_1\left(e^{\mu_2 y}-1\right)+\mu_2\lambda\right)}{\mu_2}$ $\frac{(n-r+1)\left(e^{\mu_1 x}-1\right)e^{\mu_2 y}e^{\mu_1 x}\sum_{h=0}^{r-1}(-1)^h\binom{r-1}{h}e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+h+1)}}{e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)(n-r+1)}{}_2F_1\left(n-r+1,1-r,n-r+2,e^{-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x}-1\right)}\right)}$.

References

[1] A. Marshall, I. Olkin, Life Distributions, Springer, 2007.
[2] P. Hougaard, Life table methods for heterogenous populations: distributions describing heterogeneity, Biometrica 71 (1984) 75–85.
[3] J.F. Carriere, Parametric models for life tables, Transactions of the Society of Actuaries 44 (1992) 77–100.
[4] J.H. Pollard, E. Valkovics, The Gompertz distribution and its applications, Genus 48 (1992) 15–27.
[5] W.J. Ewens, The diffusion equation and a pseudo-distribution in genetics, Journal of the Royal Statistical Society: Series B 25 (1963) 405–412.

[6] J.A. Diaz-Garcia, R.G. Jaimez, K.V. Mardia, Wishart and Pseudo-Wishart distributions and some applications to shape theory, Journal of Multivariate Analysis 63 (1997) 73–87.
[7] S.A. Adham, S.G. Walker, A multivariate Gompertz-Type distibution, Journal of Applied Statistics 28 (2001) 1051–1065.
[8] W.J. Willemse, R. Kaas, Rational reconstruction of frailty-based mortality models by a generalisation of Gompertz' law of mortality, Insurance: Mathematics and Economics 40 (2007) 468–484.
[9] S. Shahbaz, M.Q. Shahbaz, M. Ahsanullah, M. Mohsin, On a new class of probability distributions, Applied Mathematics Letters 24 (2011) 545–557.
[10] H.A. David, H.N. Nagaraja, Concomitants of order statistics, in: N. Balakrishnan, C.R. Rao (Eds.), Order Statistics: Theory & Methods, Elsevier, 1998, pp. 487–513.
[11] H.A. David, H. Nagaraja, Order Statistics, third ed., John Wiley and Sons, 2003.
[12] O.L. Gebizlioglu, B. Yagcı, Tolerance intervals for quantiles of bivariate risk and risk measurement, Insurance: Mathematics and Economics 42 (2008) 1022–1027.
[13] M.I. Beg, M. Ahsanullah, Concomitants of generalized order statistics from Farlie–Gumbel–Morgenstern distributions, Statistical Methodology 5 (2008) 1–20.
[14] P.K. Bhattacharya, Induced order statistics: theory and applications, in: P.R. Krishnaiah, P.K. Sen (Eds.), Handbook of Statistics, Vol. 4, Elsevier BV, 1984, pp. 383–403.
[15] J.K. Filus, L.Z. Filus, On some new classes of multivariate probability distributions, Pakistan Journal Statistics 22 (1) (2006) 21–42.
[16] S. Shahbaz, M. Ahmad, Concomitants of order statistics for bivariate Pseudo-Weibull distribution, World Applied Sciences Journal 6 (10) (2009) 1409–1412.
[17] S. Shahbaz, M.Q. Shahbaz, M. Mohsin, On concomitant of order statistics for bivariate pseudo exponential distribution, World Applied Sciences Journal 6 (8) (2009) 1151–1156.
[18] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series and Producst, seventh ed., Academic Press, 2007.
[19] D.F. Connon, Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers, Volume II(a), Cornell University Library, 2007. arxiv: 0710.4047v2 [math.HO].
[20] T. Bedford, R. Cooke, Probabilistic Risk Analysis, Foundations and Method, Cambridge University Press, 2001.
[21] A. Melnikov, Risk Analysis in Finance and Insurance, Chapman and Hall/CRC, 2004.
[22] V.I. Rotar, Actuarial Models: The Mathematics of Insurance, Chapman and Hall/CRC, 2007.
[23] H.U. Gerber, Life Insurance Mathematics, Springer, 1997.
[24] S.S. Yang, General distribution theory of the concomitants of order statistics, The Annals of Statistics 5 (1977) 996–1002.