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Orthogonal projection and liftings of Hamilton-decomposable Cayley graphs on abelian groups

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ABSTRACT

In this article we introduce the concept of (p, α)-switching trees and use it to provide sufficient conditions on the abelian groups G and H for when $\textsc{Cay}\,(G\times H; S\cup B)$ is Hamilton-decomposable, given that $\textsc{Cay}\,(G; S)$ is Hamilton-decomposable and B is a basis for H. Applications of this result to elementary abelian groups and Paley graphs are given.

1. Introduction

Let A be an abelian group and $S \subseteq A$ such that $0 \notin S$ and S is inverse-closed, that is, $s \in S$ if and only if $-s \in S$. The *Cayley graph* $\textsc{Cay}\,(A; S)$ is the graph whose vertices are the elements of A with x adjacent to y if and only if $x - y \in S$. The subset $S \subseteq A$ is called the *connection set* for the Cayley graph $\textsc{Cay}\,(A; S)$.

It frequently will be the case that it is more convenient to work with subsets S of abelian groups that are not inverse-closed, and yet we want a Cayley graph to be defined in terms of S. For this reason we introduce the *inverse closure* of S which is defined to be the smallest superset of S that is inverse-closed. We denote the inverse closure of S by $S^\star$.

Let X be a graph with m edges. Recall that the *edge space* $\mathcal{E}(X)$ of X is the vector space of dimension m over $\mathbb{F}_2$, where we associate the coordinates of $\mathcal{E}(X)$ with the edges of X. Thus, the elements of $\mathcal{E}(X)$ are in one-to-one correspondence with the subgraphs of X. Because we shall be working with more than one vector space in this paper, we use $\oplus$ to denote binary-addition for edge spaces. If X_1 and X_2 are subgraphs of X, note that the edge set of $X_1 \oplus X_2$ is the symmetric difference of $E(X_1)$ and $E(X_2)$.

A cycle that spans the vertices of a graph X is called a *Hamilton cycle* of X. A *Hamilton decomposition* of a regular graph X with valency $2d$ is a collection of d Hamilton cycles $H_1, H_2, \ldots, H_d$ such that $X = H_1 \oplus H_2 \oplus \cdots \oplus H_d$. A *Hamilton decomposition* of a regular graph with valency $2d + 1$ is a collection of d Hamilton cycles $H_1, H_2, \ldots, H_d$ and a single one-factor F such that $X = F \oplus H_1 \oplus \cdots \oplus H_d$. A graph admitting a Hamilton decomposition is said to be *Hamilton-decomposable*. Fig. 1 depicts a Hamilton decomposition of $\textsc{Cay}\,(\mathbb{Z}_5^2; \{(1, 1), (0, 1), (1, 0)\}^\star)$, where $\mathbb{Z}_5^2$ denotes the elementary abelian 5-group of rank 2. Alspach [1] conjectured in 1984 that Cayley graphs on abelian groups are Hamilton-decomposable. This conjecture remains unresolved. The main result of this paper, which we prove in Section 3, provides a framework for significant progress on the conjecture and we include several consequences with their proofs in subsequent sections.

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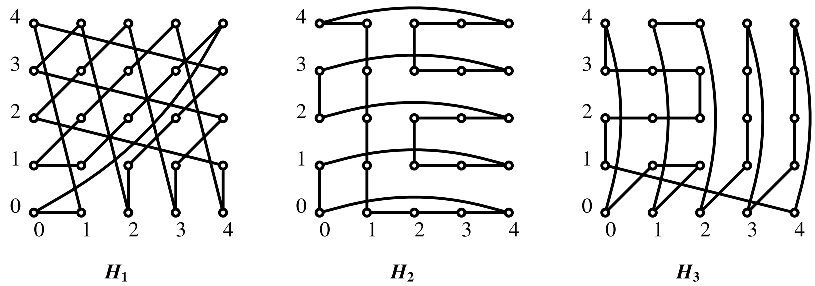


Fig. 1. Hamilton decomposition of CAY ($\mathbb{Z}_5^2$; {(1, 1), (0, 1), (1, 0)}*).

2. Basic tools

In this section we develop some basic tools that are used throughout the rest of the paper. The first tool is an outgrowth of a conjecture of Bermond [3] from 1978. He conjectured that the Cartesian product of Hamilton-decomposable graphs is Hamilton-decomposable. This conjecture also remains unresolved, but there is a very useful partial result due to Stong [6]. Stong's result includes the following theorem which we require.

Theorem 2.1. *If X_1 is a Hamilton-decomposable graph of valency $2r$ and X_2 is a Hamilton-decomposable graph of valency $2s$, with $r \le s$, then the Cartesian product $X_1 \Box X_2$ is Hamilton-decomposable if either of the following two conditions holds:*

1. $s \le 3r$, *or*
2. $r \ge 3$.

There are two partial results on the Cayley graph conjecture we use. The first was obtained by Bermond, Favaron and Maheo [4] in 1989. The second is a recent result by Westlund, Kreher and Liu [7].

Theorem 2.2. *Every connected Cayley graph of valency 4 on an abelian group is Hamilton-decomposable.*

Theorem 2.3. *Every connected Cayley graph of valency 6 on an odd order abelian group is Hamilton-decomposable.*

We now present two fundamental techniques used in the construction of Hamilton decompositions (see for example [5]). The proofs are straightforward and omitted.

Lemma 2.4. *If $C(0), C(1), C(2), \ldots, C(k-1)$ are pairwise vertex-disjoint cycles and $C = x_0y_0x_1y_1x_2y_2\cdots x_{k-1}y_{k-1}$ is a cycle of length $2k$ such that $x_iy_i \in E(C(0) \oplus C(1) \oplus \cdots \oplus C(k-1))$ for all i, and x_iy_i and x_jy_j do not intersect the same $C(\ell)$ when $i \ne j$, then the subgraph*

$$(C(0) \oplus C(1) \oplus \cdots \oplus C(k-1)) \oplus C$$

is a single cycle.

Lemma 2.5. *If C is a cycle of length ℓ with orientation $x_0x_1\cdots x_\ell$ and F is a 4-cycle $uvwy$ such that $uv, wy \in E(C)$, $vw, uy \notin E(C)$, and (u, v), (y, w) both agree with the orientation given to C, then the subgraph $C \oplus F$ is a cycle of length ℓ.*

The two preceding lemmas deal with what results after performing certain edge switchings. The first is used to tie together vertex-disjoint cycles into cycles of strictly greater length. The second is used to guarantee that certain edge switchings do not break a given cycle into two smaller cycles. Continuing in this vein, the next lemma provides another tool that guarantees a Hamilton cycle results from certain edge switchings.

Let T be a tree with maximum valency k and let $\mathcal{Z} : E(T) \to \{0, 1, \ldots, m\}$ denote a proper edge coloring of T with $m+1$ colors, where $m \ge k-1$. Consider the Cartesian product $T \Box C_r$ of T with an r-cycle, where $r \ge m+1$. Let the vertices of T be labeled $u_1, u_2, \ldots, u_n$ and let the vertices of the r-cycle replacing u_i be labeled $u_{i,0}, u_{i,1}, \ldots, u_{i,r-1}$, where $u_{i,j}$ is adjacent to $u_{i,j+1}$ for all j and subscript calculation is done modulo r. If the edge joining u_i and u_j in T is colored α, let $F_{i,j}$ be the 4-cycle $u_{i,\alpha}u_{i,\alpha+1}u_{j,\alpha+1}u_{j,\alpha}$. Let $\mathcal{F}$ denote the vertex-disjoint union

$$\bigoplus F_{i,j},$$

where the sum is taken over all edges $\{u_i, u_j\}$ of T. Let $\mathcal{D}$ denote the vertex-disjoint union of all the r-cycles in $T \Box C_r$. The graph $\mathcal{F} \oplus \mathcal{D}$ is called the *chromatic lift* of T in $T \Box C_r$.

Lemma 2.6. *Let T be a tree with maximum valency k and let $\mathcal{Z} : E(T) \to \{0, 1, \ldots, m\}$ denote a proper edge coloring of T with $m+1$ colors, where $m \ge k-1$, and all colors are used at least once. If $r \ge m+1$, then the chromatic lift of T in $T \Box C_r$ is a Hamilton cycle.*

Proof. Let the vertices of T be ordered $u_1, u_2, \ldots, u_n$ so that for each i satisfying $2 \le i \le n$, u_i has precisely one neighbor in $\{u_1, u_2, \ldots, u_{i-1}\}$. (Such an ordering exists for every tree and it need not be unique.) Let $C(u_i) = u_{i,0}u_{i,1}\cdots u_{i,r-1}u_{i,0}$ denote the r-cycle in $T\Box C_r$ with fixed coordinate u_i. Let $\mathcal{F}$ denote the 2-factor composed of the n vertex-disjoint r-cycles $C(u_1), C(u_2), \ldots, C(u_n)$. If the edge joining u_1 and u_2 is colored k, then in the chromatic lift of T, the edges $u_{1,k}u_{1,k+1}$ and $u_{2,k}u_{2,k+1}$ are replaced by the edges $u_{1,k}u_{2,k}$ and $u_{1,k+1}u_{2,k+1}$. The effect of this is to produce a single cycle spanning the vertices of $C(u_1) \cup C(u_2)$. Moving to u_3, there is an edge from u_3 to either u_1 or u_2. This edge is colored k' where $k' \ne k$. Thus, we remove the edge $u_{3,k'}u_{3,k'+1}$ from $C(u_3)$ and the corresponding edge from either $C(u_1)$ or $C(u_2)$, and replace them with the edges at levels k' and $k'+1$ joining the two cycles. This produces a single cycle spanning the vertices of $C(u_1)\cup C(u_2)\cup C(u_3)$.

It is easy to see that as we work along the tree in the specified order, the resulting graph is the chromatic lift of T in $T\Box C_r$ and is a single cycle by Lemma 2.6. Thus, the result follows. □

We now introduce several more concepts required for the forthcoming proofs.

Definition 2.7. If $H_0, H_1, H_2, \ldots, H_d$ is a Hamilton decomposition of the graph X, then a matching M of dk edges is a *chordal set of density k* for H_0 if $|M \cap E(H_j)| = k$ for all $j = 1, 2, \ldots, d$. The edges in a chordal set are called *chords*. They are chords to the cycle H_0. A vertex is a *chordal vertex* if it is incident to a chord in M. A subpath of $H_0 \oplus M$ is *internally chordal vertex-free* if no internal vertex of the subpath is a chordal vertex. A maximal internally chordal vertex-free subpath necessarily begins and ends with a chordal vertex.

Proposition 2.8. *If $H_0, H_1, H_2, \ldots, H_d$ is a Hamilton decomposition of the graph X and $|X| \ge 4dk$, then X has a chordal set of density k for H_0.*

Proof. Let k' be maximal such that X has chordal set M of density k'. We may assume $k' < k$, otherwise we are done. Further suppose ℓ is maximal such that there are edges $e_i \in H_i$, $i = 1, 2, \ldots, \ell$ extending M to a larger matching $M' = M \cup \{e_1, e_2, \ldots, e_\ell\}$. Consider the edges of $H_{\ell+1}$. Exactly k' of these edges are included in M' and at most $4(k'(d-1)+\ell)+2k'$ of them are adjacent to an edge in M. This leaves at least one edge of $H_{\ell+1}$ unaccounted for, contrary to the choice of ℓ and k'. □

Proposition 2.9. *Given integer $n \ge 2$, if $H_0, H_1, H_2, \ldots, H_d$ is a Hamilton decomposition of the graph X and $|X| \ge 2dkn$, then X has a chordal set M of density k for H_0 and H_0 has an internally chordal vertex-free path of length at least n.*

Proof. Because $n \ge 2$, then $|X| \ge 4dk$ and we can apply 2.8 to obtain a chordal set M of density k for H_0. The chordal vertices divide H_0 into $2|M| = 2dk$ paths. The average length of such a path is

$$\frac{|X|}{2|M|} = \frac{|X|}{2dk} \ge \frac{2dnk}{2dk} = n. \quad \square$$

Definition 2.10. A subset S of an abelian group A is *inverse-free* if whenever $s \in S$ either $s = -s$ or $-s \notin S$.

Definition 2.11. Let A be an abelian group and let $X = \textsc{Cay}\,(A; S^\star)$, where $S = \{s_0, s_1, \ldots, s_d\}$ is inverse-free. If Y is any subgraph of X, then for an odd integer $p \ge 3$ and a mapping $\alpha : S \to \mathbb{Z}_p$, we define $\textsc{Lift}_{p,\alpha}(Y)$ to be the subgraph of the Cayley graph $\textsc{Lift}_{p,\alpha}(X) = \textsc{Cay}\,(A \times \mathbb{Z}_p; \{(s, \alpha(s)) : s \in S\} \cup \{(0, 1)\}^\star)$ with edges

$$\left\{\{(u, i), (v, i + \alpha(s))\} : \{u, v\} \in E(Y),\ i \in \mathbb{Z}_p,\ \text{and } s = v - u\right\}.$$

The lift of $\overline{K_{|A|}}$, the graph with no edges, is $\textsc{Lift}_{p,\alpha}\left(\overline{K_{|A|}}\right) = \textsc{Cay}\,(A \times \mathbb{Z}_p; \{(0, 1)\}^\star)$ which consists of $|A|$ vertex-disjoint p-cycles.

Definition 2.12. The *switch* determined by an edge uv of X, with color $z = \mathcal{Z}(uv) \in \mathbb{Z}_p$, is the 4-cycle

$$\sigma(\mathcal{Z}; uv) = (u, z)(u, z + 1)(v, z + 1)(v, z)$$

in $\textsc{Lift}_{p,\alpha}(X)$. If uv is an uncolored edge, that is, $\mathcal{Z}(uv)$ is undefined, then $\sigma(\mathcal{Z}; \{u, v\})$ is the edgeless graph. If Y is a subgraph of X, then $\sigma(\mathcal{Z}; Y) = \bigoplus_{e\in E(Y)} \sigma(\mathcal{Z}; e)$.

Definition 2.13. A properly edge-colored spanning tree T of X with coloring $\mathcal{Z} : E(T) \to \mathbb{Z}_p$ is a (p, α)-*switching tree* T for the Hamilton decomposition $H_0, H_1, H_2, \ldots, H_d$ of X if

$$\textsc{Lift}_{p,\alpha}(H_0) \oplus \sigma(\mathcal{Z}; H_0),\ \textsc{Lift}_{p,\alpha}(H_1) \oplus \sigma(\mathcal{Z}; H_1),\ \ldots,\ \textsc{Lift}_{p,\alpha}(H_d) \oplus \sigma(\mathcal{Z}; H_d),\ \textsc{Lift}_{p,\alpha}\left(\overline{K_{|A|}}\right) \oplus \sigma(\mathcal{Z}; T)$$

is a Hamilton decomposition of $\textsc{Lift}_{p,\alpha}(X)$. Note that $\mathcal{Z}(e)$ remains undefined for edges e that are not in T. Thus $\sigma(\mathcal{Z}; T \cap H_i) = \sigma(\mathcal{Z}; H_i)$.

Proposition 2.14. *If θ is an automorphism of the abelian group A, then θ is an isomorphism from* $\textsc{Cay}\,(A; S^\star)$ *to* $\textsc{Cay}\,(A; \theta(S)^\star)$ *for any $S \subset A$.*

Proof. If xy is an edge of $\textsc{Cay}\,(A; S^\star)$, then $x - y = s$ for some $s \in S^\star$. Thus $\theta(x) - \theta(y) = \theta(x - y) = \theta(s) \in S^\star$. □

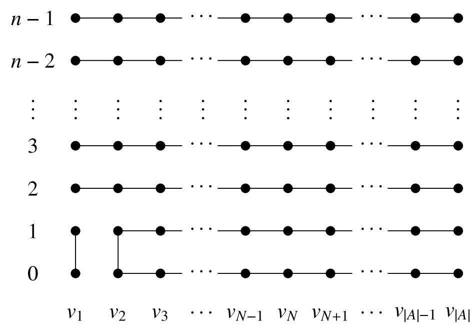


Fig. 2. The graph $G_1 = G_0 \oplus \sigma(\mathcal{Z}; v_1v_2)$ is the union of n vertex-disjoint paths. Here we have assumed $\alpha(s) = 0$, for all $s \in S$.

3. Proof of the main theorem

We now state and prove our main result.

Theorem 3.1. *Let $X = \text{Cay}(A; S^\star)$, where A is abelian and S is inverse-free. Given an odd integer $n \geq 3$ and a mapping $\alpha : S \to \mathbb{Z}_n$, if X has a Hamilton decomposition $H_0, H_1, \ldots, H_d$, with chordal set M of density $n - 1$ for H_0 such that H_0 has an internally chordal vertex-free path of length n, then $\text{Lift}_{n,\alpha}(X)$ is also Hamilton-decomposable.*

Proof. Let Q be a maximal internally chordal vertex-free path on H_0. Then Q has length at least n and begins and ends with a chordal vertex. We show that $H_0 \oplus M$ contains a cubic (n, α)-switching tree T and hence $X' = \text{Lift}_{n,\alpha}(X)$ is Hamilton-decomposable.

Write H_0 as the cycle $v_1v_2v_3 \cdots v_Nv_{N+1} \cdots v_{|A|}v_1$ such that $Q = v_Nv_{N+1}v_{N+2} \cdots v_{|A|}v_1$, and set $P = H_0 \oplus \{v_{|A|}v_1\}$ to be the path $P = v_1v_2v_3 \ldots v_{|A|}$. Then N is the index of the last chordal vertex on P. The subgraph $G_0 = \text{Lift}_{n,\alpha}(P)$ of $\text{Lift}_{n,\alpha}(X)$ consists of the n vertex-disjoint paths. We process the vertices of P in the order $v_1, v_2, v_3, \ldots$ to build the (n, α)-switching tree T, with coloring $\mathcal{Z} : E(T) \to \mathbb{Z}_n$.

Vertex v_1 is a chordal vertex and is incident to a chord $e \in M$. We include e in T and set $\mathcal{Z}(e) = 1$. We also include the edge v_1v_2 in T, set its color $\mathcal{Z}(v_1v_2) = 0$ and let $G_1 = G_0 \oplus \sigma(\mathcal{Z}; v_1v_2)$. Then G_1 consists of n vertex-disjoint paths. (See Fig. 2.)

Let $P_i = v_1v_2 \cdots v_i$. Suppose for $1 < i \leq N$, that every chord incident with a vertex of P_{i-1} has been colored and belongs to T, and that every edge $e \in P_i$ is either uncolored or included as an edge of T with $\mathcal{Z}(e)$ specified. Further suppose

$$G_{i-1} = G_{i-2} \oplus \sigma(\mathcal{Z}; v_{i-i}v_i) = G_0 \oplus \bigoplus_{j=2}^{i} \sigma(\mathcal{Z}; v_{j-1}v_j)$$

is the union of n vertex-disjoint paths. Consider the edges in $P_i \oplus M$ that are incident to v_i. There are three situations to resolve.

I: *v_i is a chordal vertex and the chord c_i incident to v_i has been colored.* In this situation the edge $e = v_iv_{i+1}$ is not included in T and consequently does not require coloring. Hence $\sigma(\mathcal{Z}; e)$ is the empty graph and $G_i = G_{i-1} \oplus \sigma(\mathcal{Z}; e) = G_{i-1}$ is the union of n vertex-disjoint paths.

II: *v_i is a chordal vertex and the chord c_i incident to v_i has not been colored.* In this situation we first include the edge $e = v_iv_{i+1}$ in T. The two edges $(v_i, x)(v_{i+1}, x)$ and $(v_i, x+1)(v_{i+1}, x+1)$ belong to the same path if and only if $(v_{|A|}, x)$ and $(v_{|A|}, x+1)$ are ends of the same path. Hence we let $L \subseteq \mathbb{Z}_n$ be the set of colors x such that $(v_{|A|}, x)$ and $(v_{|A|}, x+1)$ are ends of the same path in G_{i-1}. (If $v_{i-1}v_i$ was colored x, then $(v_{|A|}, x)$ and $(v_{|A|}, x+1)$ are path ends of G_{i-1}.) Then $|L| \leq \lfloor n/2 \rfloor$, and hence there are $n - \lfloor n/2 \rfloor = \lceil n/2 \rceil \geq 2$ colors not in L. Let $z \in \mathbb{Z}_n \setminus L$, set $\mathcal{Z}(e) = z$ and $G_i = G_{i-1} \oplus \sigma(\mathcal{Z}; e)$. It is easy to see that G_i is the union of n vertex-disjoint paths. The chord $c_i \in M \cap E(H_j)$, for some j, and possibly the other $n - 2$ edges in $M \cap E(H_j)$ have been colored. One of the remaining two colors, say z', is not z. We set $\mathcal{Z}(c_i) = z'$ and include c_i in T.

III: *v_i is not a chordal vertex.* In this situation we include $e = v_iv_{i+1}$ in T. To determine a color for e, let L be the set of colors x such that $(v_{|A|}, x)$ and $(v_{|A|}, x + 1)$ are ends of the same path in G_{i-1}. Then $|L| \leq \lfloor n/2 \rfloor$, and hence there are $n - \lfloor n/2 \rfloor = \lceil n/2 \rceil \geq 2$ colors not in L. Let $z \in \mathbb{Z}_n \setminus L$, set $\mathcal{Z}(e) = z$ and $G_i = G_{i-1} \oplus \sigma(\mathcal{Z}; e)$. It is easy to see that G_i is the union of n vertex-disjoint paths.

We conclude this process at the last chordal vertex, i.e. at $i = N$, obtaining a graph G_N consisting of n vertex-disjoint paths, a tree T and an edge-coloring $\mathcal{Z}$. We complete T by including the edges of the path $v_Nv_{N+1} \cdots v_{|A|}$. From P one edge adjacent to each chord has not been included in T and all the chords have been included in T. Thus T is a spanning tree of X. So far no two adjacent edges of T have been assigned identical colors and there are distinct colors on all the edges in $M_i = M \cap E(H_i)$, for each $i = 1, 2, \ldots, d$. It remains to color the edges of the path $v_Nv_{N+1}v_{N+2} \cdots v_{|A|}$. However, coloring these edges has no effect on $\text{Lift}_{n,\alpha}(H_i) \oplus \sigma(\mathcal{Z}; H_i)$, $i = 1, 2, \ldots, d$. Because the $n - 1$ matching edges of H_i receive $n - 1$ distinct colors, it is clear that $\textsc{Lift}_{n,\alpha}(H_i) \oplus \sigma(\mathcal{Z}; H_i)$ is a Hamilton cycle for $i = 1, 2, \ldots, d$. Moreover, because of Lemma 2.6, no matter how these edges are colored, we have that $\textsc{Lift}_{n,\alpha}\left(\overline{K_{|A|}}\right) \oplus \sigma(\mathcal{Z}; T)$ also is a Hamilton cycle. Thus, the scheme we describe for coloring the aforementioned edges is designed to guarantee that $\textsc{Lift}_{n,\alpha}(H_0) \oplus \sigma(\mathcal{Z}; T)$ is a Hamilton cycle.

Let W be the n-matching $\{0, 1, 2, \ldots, n-1\}\Box\{v_{|A|}v_1\}$. Then $\textsc{Lift}_{n,\alpha}(P) \oplus W = \textsc{Lift}_{n,\alpha}\left(P \oplus \{v_{|A|}v_1\}\right) = \textsc{Lift}_{n,\alpha}(H_0)$ and hence

$$G_N \oplus W = \textsc{Lift}_{n,\alpha}(H_0) \oplus \left(\bigoplus_{j=1}^{N-1} \sigma(\mathcal{Z}; v_j v_{j+1})\right)$$

consists of $k \le n$ vertex-disjoint cycles $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$.

If $k = 1$, then $\textsc{Lift}_{n,\alpha}\left(P \oplus \{v_{|A|}v_1\}\right)$ already is a Hamilton cycle and we omit the next step. If $k > 1$, then we choose $k-1$ distinct colors $x_1, x_2, \ldots, x_{k-1} \in \mathbb{Z}_n$ from the set

$$\{x : \{(|A|, x), (|A|, x+1)\} \not\subseteq V(\mathcal{C}_j),\ \text{for all } j = 1, 2, \ldots, k\},$$

where $x_1 \ne \mathcal{Z}(c)$ and c is the chord incident to v_N, and then setting $\mathcal{Z}(v_{N+j-1}v_{N+j}) = x_j,\ j = 1, 2, \ldots, k-1$, it follows that

$$\mathcal{C} = \textsc{Lift}_{n,\alpha}(H_0) \oplus \left(\bigoplus_{j=1}^{N+k-2} \sigma(\mathcal{Z}; v_j v_{j+1})\right)$$

is a Hamilton cycle. We now color the remaining $|A|-N-k-1$ edges one at a time such that each switch produces a Hamilton cycle. Suppose we wish to color the edge $v_j v_{j+1}$. If $j = N$ (that is, $k = 1$), then only the chord incident with v_N has been colored some color x. This implies that the current Hamilton cycle C uses all of the edges M of the form $\{0, 1, \ldots, n-1\}\Box\{v_N v_{N+1}\}$, the edge $v_{N,x}v_{N,x+1}$ and no other edges on the n-cycle replacing v_N. Hence, upon orienting the edges of C, the edges $v_{N,x}v_{N+1,x}$ and $v_{N,x+1}v_{N+1,x+1}$ have opposite orientation. Thus, there is some $y \ne x$ for which $v_{N,y}v_{N+1,y}$ and $v_{N,y+1}v_{N+1,y+1}$ have the same orientation, because n is odd. Hence, if we color the edge $v_N v_{N+1}$ with y, then the corresponding switch produces a Hamilton cycle by Lemma 2.5. The same argument applies to $v_j v_{j+1},\ j > N$, because only one edge incident with v_j is colored in this procedure. This completes the proof of the theorem. □

Putting Theorem 3.1, Propositions 2.9 and 2.14 together we arrive at Corollary 3.2.

Corollary 3.2. *Let $S = \{s_0, s_1, s_2, s_3, \ldots, s_d\}$ be an inverse-free subset of the odd order abelian group A and let n be an odd integer. Given $x_0, x_1, x_2, \ldots, x_d \in \mathbb{Z}_n$ and generator g of $\mathbb{Z}_n$, let $S' = \{(s_i, x_i) : i = 0, 1, 2, \ldots, d\} \cup \{(0, g)\}$. If $|A| \ge 2d(n^2-n)$ and $\textsc{Cay}(A; S^\star)$ is Hamilton-decomposable, then $\textsc{Cay}(A \times \mathbb{Z}_n; S'^\star)$ is Hamilton-decomposable.*

This corollary can be extended to Corollary 3.3.

Corollary 3.3. *Let $S = \{s_0, s_1, s_2, s_3, \ldots, s_d\}$ be an inverse-free subset of the odd order abelian group A and let $B = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ be a rank r odd order abelian group, where $n_r|n_{r-1}|n_{r-2}|\cdots|n_1$, with basis $G = \{g_1, g_2, \ldots, g_r\}$. Given $x_0, x_1, x_2, \ldots, x_d \in B$ let $S' = \{(s_i, x_i) : i = 0, 1, 2, \ldots, d\} \cup \{(0, g_i) : i = 1, 2, \ldots, r\}$. If $|A| \ge 2d(n_1^2 - n_1)^2$ and $\textsc{Cay}(A; S^\star)$ is Hamilton-decomposable, then $\textsc{Cay}(A \times B; S'^\star)$ is Hamilton-decomposable.*

Proof. Write $x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,r})$, where $x_{i,j} \in \mathbb{Z}_{n_j}$, for $i = 0, 1, 2, \ldots, d$. There is a group automorphism θ of B such that $\theta(g_i) = e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$. Thus by Proposition 2.14 we may assume without loss that $g_i = e_i$ for all $i = 1, 2, \ldots, r$. Because $|A| \ge 2d(n_1^2 - n_1)^2$, we apply Corollary 3.2 obtaining a Hamilton decomposition of $\textsc{Cay}(A \times \mathbb{Z}_{n_1}; S_1)$, where

$$S_1 = \{(s_i, x_{i,1}) : i = 0, 1, 2, \ldots, d\} \cup \{(0, 1)\}.$$

Now $|A \times \mathbb{Z}_{n_1}| > |A| \ge 2d(n_1^2 - n_1)^2 \ge 2d(n_2^2 - n_2)^2$. So we may again apply Corollary 3.2 to obtain a Hamilton decomposition of $\textsc{Cay}(A \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}; S_2)$, where

$$S_2 = \{(s_i, x_{i,1}, x_{i,2}) : i = 0, 1, 2, \ldots, d\} \cup \{(0, 1, 0), (0, 0, 1)\}.$$

Iterating this process k times we obtain a Hamilton decomposition of $\textsc{Cay}(A \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}; S_2)$, where

$$S_k = \{(s_i, x_{i,1}, x_{i,2}, \ldots, x_{i,k}) : i = 0, 1, 2, \ldots, d\} \cup \{(0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}.$$

Because $S' = S_r$, the desired result is obtained on the r-th iteration. □

We now explore some consequences of Theorem 3.1 and its corollaries.

4. Elementary abelian groups

We now focus on the elementary abelian group $A = \mathbb{Z}_p^n$ which we also consider as the vector space of dimension n over the field $\mathbb{F}_p = \mathbb{Z}_p$. Alspach, Bryant and Dyer [2] established the following lemma in 2010.

Lemma 4.1. *If $S = \{s_1, s_2, \ldots, s_t\}$ is a set of linearly independent vectors in $\mathbb{Z}_p^n$, then the components of the Cayley graph $\textsc{Cay}(\mathbb{Z}_p^n; S^\star)$ are all isomorphic to the Cartesian product of t p-cycles.*

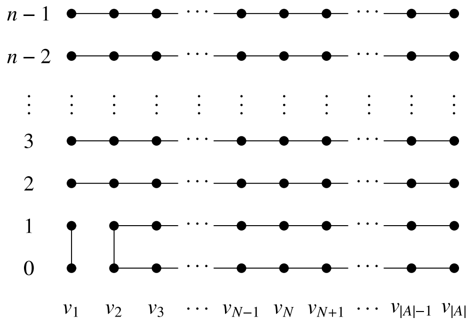


Fig. 3. Hamilton decomposition of Cay $(\mathbb{Z}_3^2; \{(1,1),(1,0),(0,1)\}^\star)$.

It has an interesting corollary which also appears in [2].

Corollary 4.2. *If S is a basis of $\mathbb{Z}_p^n$, then the Cayley graph* Cay $(\mathbb{Z}_p^n; S^\star)$ *has a Hamilton decomposition.*

The remainder of this section establishes Theorem 4.5 which is a generalization of Corollary 4.2. Namely we will show that if the set $S \subseteq \mathbb{Z}_p^n$ has $|S| = n+1$ and rank n, then Cay $(\mathbb{Z}_p^n; S^\star)$ is Hamilton decomposable. First in Section 4.1 we reduce to where S has a row reduced echelon form. In Sections 4.2–4.4, and 4.4, we settle the problem for dimension $n = 2$, and also for $n = 3$ when $p = 3$. These are the initial ingredients needed for an inductive proof using Corollary 3.2.

4.1. Reduction

The automorphism group of $\mathbb{Z}_p^n$ is $\mathrm{GL}_n(p)$ the group of n by n invertible matrices over $\mathbb{Z}_p$. If $M \in \mathrm{GL}_n(p)$, then it is easy to see that the mapping $x \mapsto Mx$ on $\mathbb{Z}_p^n$ is a graph isomorphism from Cay $(\mathbb{Z}_p^n; S^\star)$ to Cay $(\mathbb{Z}_p^n; MS^\star)$. In particular if S of cardinality n is a linearly independent subset of $\mathbb{Z}_p^n$, then the matrix M whose columns are the elements of S is invertible and hence $M \in \mathrm{GL}_n(p)$. It follows that Cay $(\mathbb{Z}_p^n; S^\star)$ is isomorphic Cay $(\mathbb{Z}_p^n; \{e_1, e_2, \ldots, e_n\}^\star)$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for $\mathbb{Z}_p^n$. That is

$$e_j = [0, 0, \ldots, 0, \underbrace{1}_{j\text{–th}}, 0, \ldots, 0].$$

Thus if p is a prime and S is a rank n cardinality $n+1$ inverse-free subset of $\mathbb{Z}_p^n$, we may assume that $X =$ Cay $(\mathbb{Z}_p^n; S^\star)$ has

$$S = \{r, e_1, e_2, \ldots, e_n\},$$

with $r \neq \pm e_j$, for all $j = 1, 2, \ldots, n$. Also because we may multiply any coordinate by -1 and preserve $S^\star$, we may assume the entries of r are each less than or equal to $(p-1)/2$. Moreover we may put the entries in r in descending order, because permuting the coordinates fixes the set $\{e_1, e_2, \ldots, e_n\}$. We record these observations with the following lemma.

Lemma 4.3. *Let p be an odd prime. If $S \subseteq \mathbb{Z}_p^n$ has cardinality $n+1$ and rank n, then* Cay $(\mathbb{Z}_p^n; S^\star)$ *is isomorphic to* Cay $(\mathbb{Z}_p^n; \{r, e_1, e_2, \ldots, e_n\}^\star)$*, where $r \neq \pm e_j$, for all $j = 1, 2, \ldots, n$, each entry of r is at most $(p-1)/2$ and the entries of r are in descending order.*

4.2. $p = 3, n \in \{2, 3\}$

Applying Lemma 4.3 we see that all 6-valent Cayley graphs on $\mathbb{Z}_3^2$ whose connection sets have full rank are isomorphic to

$$X_{3,2} = \text{Cay}\,(\mathbb{Z}_3^2; \{(1,1),(1,0),(0,1)\}^\star).$$

A Hamilton decomposition of this graph is depicted in Fig. 3.

Also using Lemma 4.3 we find that there are exactly two non-isomorphic 8-valent Cayley graphs on $\mathbb{Z}_3^3$ whose connection sets have full rank. Namely:

1. $X_{3,3_1} = \text{Cay}\,(\mathbb{Z}_3^2; \{(1,1,0),(1,0,0),(0,1,0),(0,0,1)\}^\star)$
2. $X_{3,3_2} = \text{Cay}\,(\mathbb{Z}_3^2; \{(1,1,1),(1,0,0),(0,1,0),(0,0,1)\}^\star)$.

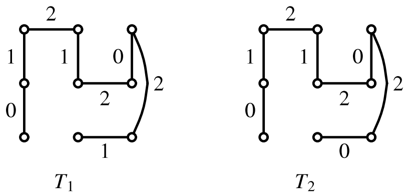
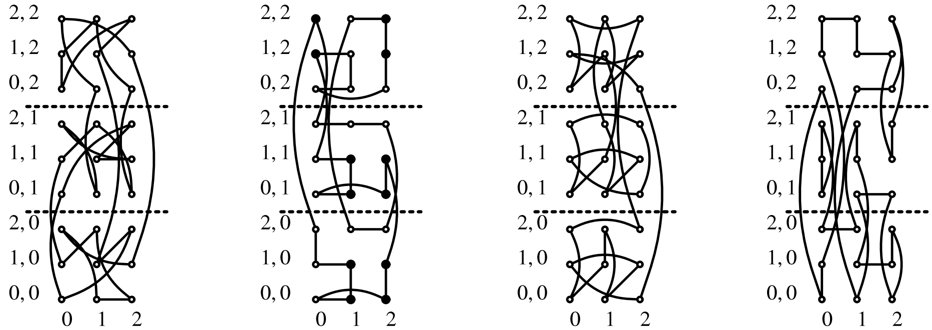
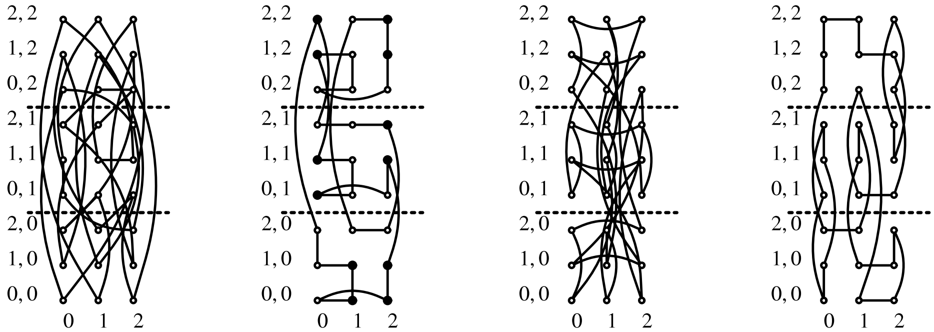


Fig. 4. Switching trees for Fig. 3.



$\text{LIFT}_{3,\alpha_1}(H_0) \oplus \sigma(\mathcal{Z}_1; H_0)$ $\quad$ $\text{LIFT}_{3,\alpha_1}(H_1) \oplus \sigma(\mathcal{Z}_1; H_1)$ $\quad$ $\text{LIFT}_{3,\alpha_1}(H_2) \oplus \sigma(\mathcal{Z}_1; H_2)$ $\quad$ $\text{LIFT}_{3,\alpha_1}\left(\overline{K_9}\right) \oplus \sigma(\mathcal{Z}_1; T_1)$

Fig. 5. Hamilton decomposition of CAY $(\mathbb{Z}_3^2; \{(1, 1, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}^{\star})$.



$\text{LIFT}_{3,\alpha_2}(H_0) \oplus \sigma(\mathcal{Z}_2; H_0)$ $\quad$ $\text{LIFT}_{3,\alpha_2}(H_1) \oplus \sigma(\mathcal{Z}_2; H_1)$ $\quad$ $\text{LIFT}_{3,\alpha_2}(H_2) \oplus \sigma(\mathcal{Z}_2; H_2)$ $\quad$ $\text{LIFT}_{3,\alpha_2}\left(\overline{K_9}\right) \oplus \sigma(\mathcal{Z}_2; T_2)$

Fig. 6. Hamilton decomposition of CAY $(\mathbb{Z}_3^2; \{(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}^{\star})$.

Defining functions

$$\alpha_1 = \begin{pmatrix} (1,1) & (1,0) & (0,1) \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\alpha_2 = \begin{pmatrix} (1,1) & (1,0) & (0,1) \\ 1 & 0 & 0 \end{pmatrix},$$

it is easily verified for $i = 1$ and 2 that the $\mathbb{Z}_3$-labeled tree T_i with coloring $\mathcal{Z}_i$ depicted in Fig. 4 is a $(3, \alpha_i)$-switching tree for the decomposition given in Fig. 3. The resulting decompositions of $X_{3,3_i}$, $i = 1, 2$, are provided in Figs. 5 and 6, respectively. (The vertex in row y, z and column x has coordinates (x, y, z).)

It is also easy to verify that M_1 and M_2 given below are chordal sets of density 2 for $\text{LIFT}_{3,\alpha_i}(H_1) \oplus \sigma(\mathcal{Z}_i; H_1)$, $i = 1$ and 2.

$$M_1 = \big\{\{(1,0,0),(2,0,0)\}, \{(1,1,1),(2,1,1)\}, \{(0,2,2),(2,2,2)\}, \{(0,1,2),(2,1,2)\},$$
$$\{(1,1,0),(2,1,0)\}, \{(1,0,1),(2,0,1)\}\big\}$$

$$M_2 = \big\{\{(0,0,1),(0,1,1)\},\{(2,1,1),(2,2,1)\},\{(0,1,2),(2,1,2)\},\{(0,2,2),(2,2,2)\},\\ \{(1,1,0),(2,1,0)\},\{(1,0,0),(2,0,0)\}\big\}.$$

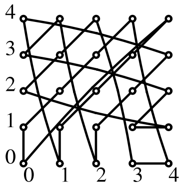
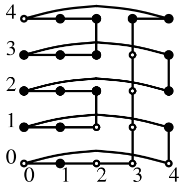
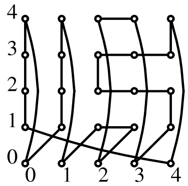
Chordal vertices are blackened in Figs. 5 and 6. An internally chordal vertex-free path of length 3 in $\mathrm{LIFT}_{3,\alpha_i}(H_1) \oplus \sigma(\mathcal{Z}; H_1),\ i = 1$ or 2, is

$$P = (1,1,2)(1,0,2)(0,0,2)(2,0,2).$$

4.3. $p = 5, n = 2$

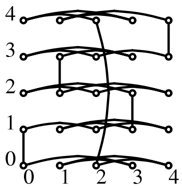
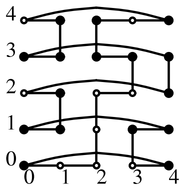
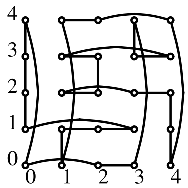
Applying Lemma 4.3 we see that there are exactly 4 non-isomorphic 6-valent Cayley graphs on $\mathbb{Z}_5^2$ whose connection sets have full rank. For each we provide a Hamilton decomposition (H_1, H_2, H_3), a chordal set $M = M_1 \cup M_3$ of density 4 for H_2 and an internally chordal vertex-free path P of length 5 in $H_2 + M$. Chordal vertices have been blackened.

4.3.1. CAY $(Z_5^2; \{(1,1),(1,0),(0,1)\}^\star)$

$H_1 =$  $H_2 =$  $H_3 =$ 

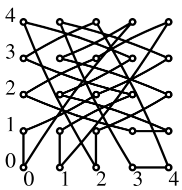
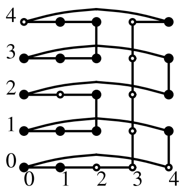
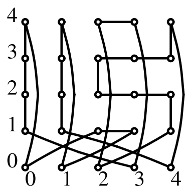
$$M_1 = \left\{\begin{matrix}\{(0,3),(1,4)\},\\ \{(0,2),(1,3)\},\\ \{(0,1),(1,2)\},\\ \{(1,0),(1,1)\}\end{matrix}\right\} \qquad P = (1,0)(2,0)(3,0)(3,1)(3,2)(3,3) \qquad M_3 = \left\{\begin{matrix}\{(4,1),(4,2)\},\\ \{(4,3),(4,4)\},\\ \{(2,4),(3,4)\},\\ \{(2,2),(2,3)\}\end{matrix}\right\}.$$

4.3.2. CAY $(Z_5^2; \{(2,0),(1,0),(0,1)\}^\star)$

$H_1 =$  $H_2 =$  $H_3 =$ 

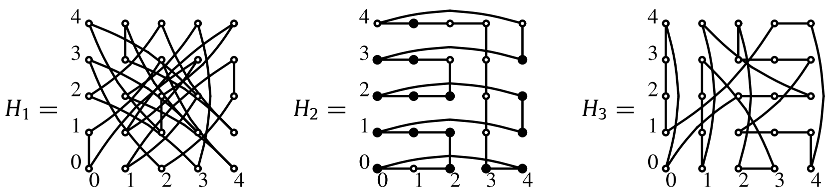
$$M_1 = \left\{\begin{matrix}\{(0,0),(0,1)\},\\ \{(1,2),(1,3)\},\\ \{(1,1),(3,1)\},\\ \{(0,3),(2,3)\}\end{matrix}\right\} \qquad P = (0,0)(1,0)(2,0)(2,1)(2,2)(3,2) \qquad M_3 = \left\{\begin{matrix}\{(1,4),(2,4)\},\\ \{(4,0),(4,4)\},\\ \{(4,1),(4,2)\},\\ \{(3,3),(4,3)\}\end{matrix}\right\}.$$

4.3.3. CAY $(Z_5^2; \{(2,1),(1,0),(0,1)\}^\star)$

$H_1 =$  $H_2 =$  $H_3 =$ 

$$M_1 = \left\{\begin{matrix}\{(0,3),(2,4)\},\\ \{(0,2),(2,3)\},\\ \{(0,1),(2,2)\},\\ \{(1,0),(1,1)\}\end{matrix}\right\} \qquad P = (1,0)(2,0)(3,0)(3,1)(3,2)(3,3) \qquad M_3 = \left\{\begin{matrix}\{(0,0),(2,1)\},\\ \{(4,1),(4,2)\},\\ \{(1,3),(1,4)\},\\ \{(4,3),(4,4)\}\end{matrix}\right\}.$$

4.3.4. Cay $(Z_5^2; \{(2,2), (1,0), (0,1)\}^{\star})$



$$M_1 = \left\{\begin{matrix} \{(0,0),(0,1)\}, \\ \{(2,1),(2,2)\}, \\ \{(1,3),(1,4)\}, \\ \{(4,2),(4,3)\} \end{matrix}\right\} \qquad P = (3,0)(3,1)(3,2)(3,3)(3,4)(2,4) \qquad M_3 = \left\{\begin{matrix} \{(4,0),(4,1)\}, \\ \{(1,1),(1,2)\}, \\ \{(0,2),(0,3)\}, \\ \{(2,0),(3,0)\} \end{matrix}\right\}.$$

4.4. $p > 5, n = 2$

Let $p > 5$ be a prime and let $e_1 = (1,0), e_2 = (0,1)$. Choose any $r = (a,b) \in \mathbb{Z}_p^2 \setminus \{e_1, e_2\}^{\star}$. In this section we consider the Cayley graph

$$X = \text{Cay}\,(\mathbb{Z}_p^2; \{r, e_1, e_2\}^{\star})$$

and construct a Hamilton decomposition H_1, H_2, H_3 of X and a chordal set M of density $p - 1$ for H_2, such that $H_2 \oplus M$ has an internally chordal vertex-free path P of length p. The existence of the Hamilton decomposition of X guaranteed by Theorem 2.3 need not yield a decomposition with the desired chordal set.

To begin we start with the edge partition

$$H_1' = \text{Cay}\,(\mathbb{Z}_p^2; \{r\}^{\star}), \qquad H_2' = \text{Cay}\,(\mathbb{Z}_p^2; \{e_1\}^{\star}), \qquad H_3' = \text{Cay}\,(\mathbb{Z}_p^2; \{e_2\}^{\star}).$$

An example when $p = 7$ is given in Fig. 7.

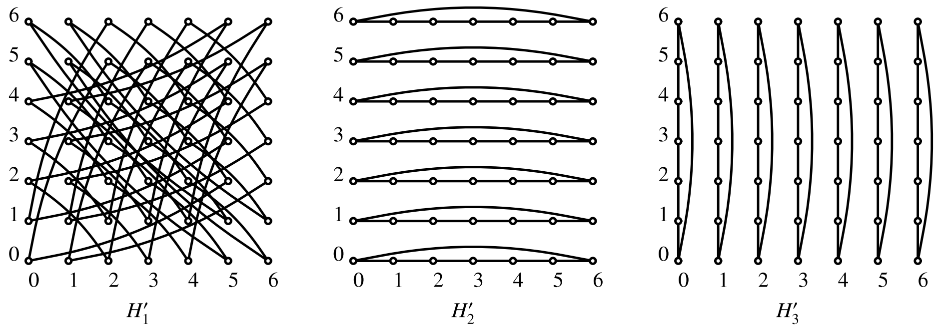


Fig. 7. Cay $(\mathbb{Z}_7^2; \{(2,5), (0,1), (1,0)\}^{\star})$.

Let C be the cycle defined by the length $2p$ alternating $r, -e_2$ sequence

$$(w_1, w_2, \ldots, w_{2p}) = (r, -e_2, r, -e_2, \ldots, r, -e_2)$$

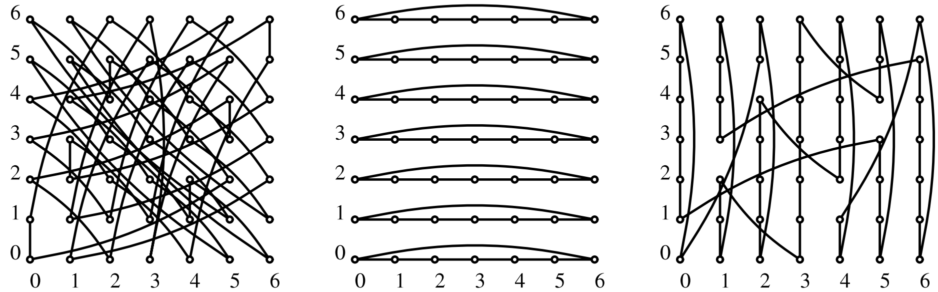
and the vertex $(0,0)$. That is

$$C = \left((0,0) + \sum_{i=1}^{j} w_i : j = 0, 1, 2, \ldots, 2p - 1 \right).$$

This is a cycle of length $2p$, because r and e_2 are linearly independent. The edges of C alternate between edges of H_1' and H_3'. The r-edges of C join the cycles of H_3' and the e_2-edges of C join the cycles of H_1'. Thus by Lemma 2.4 the symmetric differences $H_1' \oplus C$ and $H_3' \oplus C$ are Hamilton cycles. (See Fig. 8.) It is not difficult to see that the e_2-edges used in the cycle C are

$$S = \{(ka, -k(1-b)), (ka, 1 - k(1-b))\},$$

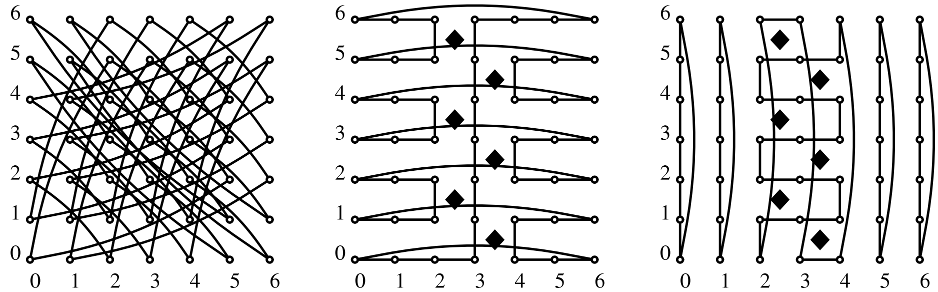
where $k = 0, 1, 2, \ldots, p - 1$. We may assume $a \neq 0$. There are three cases to consider.



$H'_1 \oplus C$ — A Hamilton cycle; H'_2 — p p-cycles; $H'_3 \oplus C$ — A Hamilton cycle

$C = (0,0)(2,5)(2,4)(4,2)(4,1)(6,6)(6,5)(1,3)(1,2)(3,0)(3,6)(5,4)(5,3)(0,1)$

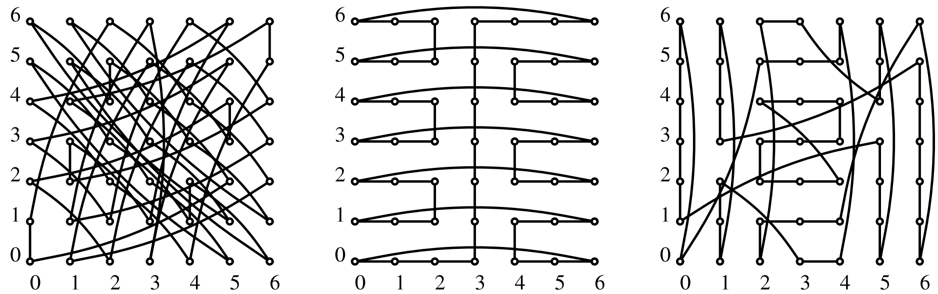
Fig. 8. Symmetric difference with the cycle C.



H'_1 — p p-cycles; $H'_2 \oplus Z$ — A Hamilton cycle; $H'_3 \oplus Z$ — $p-3$ p-cycles and one $3p$-cycle

$Z = F(3,0) + F(2,1) + F(3,2) + F(2,3) + F(3,4) + F(2,5)$

Fig. 9. Symmetric difference with zig–zag Z marked with ◆.



$H'_1 \oplus C$ — A Hamilton cycle; $H'_2 \oplus Z$ — A Hamilton cycle; $H'_3 \oplus (C \oplus Z)$ — Two cycles

Fig. 10. Symmetric difference with C and Z.

Case 1, $b \notin \{0, 1\}$: Setting $x = ka$ and $z = -(b-1)^{-1}a$ we find the e_2-edges used in the cycle C are:

$$S = \left\{\{(x, -z^{-1}x), (x, 1 - z^{-1}x)\} : x \in \mathbb{Z}_p\right\}. \quad (1)$$

If the edge $s_x = \{(x, y_1), (x, y_2)\} \in S$ and $y_2 = y_1 + 1$, then we call y_2 the *top* of s_x and y_1 the *bottom* of s_x; otherwise y_1 is the top and y_2 is the bottom. Let F_x, where $x \in \mathbb{Z}_p^2$, be the 4-cycle defined by the sequence $(e_1, e_2, -e_1, -e_2)$ and the vertex x, that is, F_x is the subgraph with edge set

$$E(F_x) = \{\{x, x + e_1\}, \{x + e_1, x + e_1 + e_2\}, \{x + e_1 + e_2, x + e_2\}, \{x + e_2, x\}\}.$$

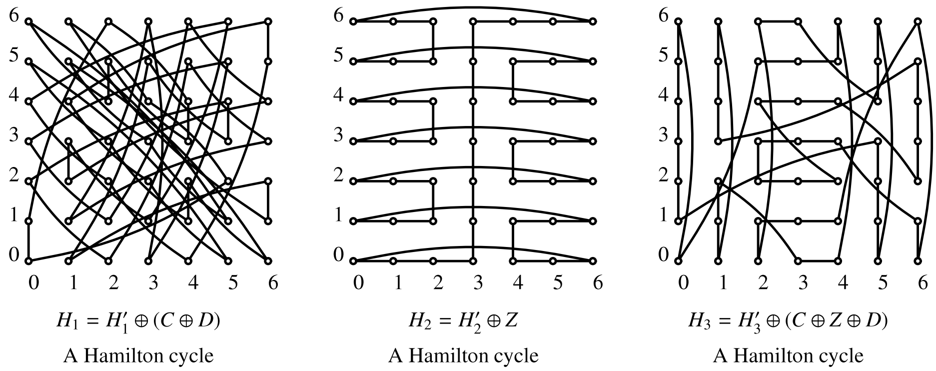


Fig. 11. Symmetric difference with C, Z, and $D = (4, 3)(4, 4)(6, 2)(6, 1)$.

Then focusing on $s_z = \{(z, -1), (z, 0)\}$ we define the *zig–zag* to be

$$Z = \begin{cases} F_{(z-1,0)} + F_{(z,1)} + F_{(z-1,2)} + F_{(z,3)} + \cdots + F_{(z-1,p-2)} & \text{if } [z^{-1}] \text{ is even;} \\ F_{(z,0)} + F_{(z-1,1)} + F_{(z,2)} + F_{(z-1,3)} + \cdots + F_{(z,p-2)} & \text{if } [z^{-1}] \text{ is odd,} \end{cases}$$

where $[z^{-1}]$ is the unique integer such that $0 \le [z^{-1}] < p$ and $[z^{-1}] \equiv z^{-1} \pmod p$. It should be observed that $S \cap E(Z) = \emptyset$. The zig–zag Z is a length $4(p-1)$ closed trail with edges alternating between H_2' and H_3'. Thus applying Lemma 2.4 we find that the e_2-edges of Z join the cycles of H_2' and consequently the symmetric difference $H_2' \oplus Z$ is a Hamilton cycle. The e_1-edges of Z span only the cycles of H_3' that have first coordinate among $z-1$, z and $z+1$, thus these cycles are joined by Lemma 2.4 into a cycle of length $3p$ in the symmetric difference $H_3' \oplus Z$. The remaining vertices are in cycles of length p. An example when $p = 7$ is given in Fig. 9. Consequently the symmetric differences $H_1' \oplus C$ and $H_2' \oplus Z$ are Hamilton cycles whereas $H_3' \oplus (C \oplus Z)$ may not be. (See Fig. 10.) We now show that $H_3' \oplus (C \oplus Z)$ is either a Hamilton cycle or consists of exactly two vertex-disjoint cycles. The $3p$-cycle of e_1- and e_2-edges formed by the symmetric difference $H_3' \oplus Z$ is broken into three paths when the edges s_{z-1}, s_z and s_{z+1} are removed by the symmetric difference $H_3' \oplus (C \oplus Z)$. These three paths of e_1- and e_2-edges are

the top of s_z to the top of s_{z-1} path P_1,
the bottom of s_{z-1} to the top of s_{z+1} path P_2,
the bottom of s_{z+1} to the bottom of s_z path P_3,
} when $[z^{-1}] > [-z^{-1}]$

or

the top of s_z to the top of s_{z+1} path P_1,
the bottom of s_{z+1} to the top of s_{z-1} path P_2,
the bottom of s_{z-1} to the bottom of s_z path P_3,
} when $[z^{-1}] < [-z^{-1}]$.

Each r-edge in $H_3' \oplus (C \oplus Z)$ is adjacent to exactly two edges in S; it is adjacent to one at the bottom end and another at the top end. When traversing the cycle containing an r-edge $\{(x-a, y_2-b), (x, y_2)\}$, where $x \notin \{z-1, z, z+1\}$, then it follows the path

$$(x, y_2 + 1)(x, y_2 + 2) \cdots (x, y_2 + k) \cdots (x, y_2 - 1)$$

and then exits on the r-edge $\{(x, y_2-1), (x+a, y_2-1+a)\}$. Hence it enters at the top of s_x and leaves at the bottom of s_x. It follows that the cycles containing P_1, P_2 or P_3 must join their top ends to bottom ends. Hence because P_1 has two top ends, P_2 has a top and bottom end and P_3 has two bottom ends, then we can only complete the traversal of cycles by either

1. Joining P_1 and P_3 with intermediate edges into a cycle and simultaneously joining P_3 with intermediate edges into a cycle, thus obtaining two cycles.
2. Joining P_1, P_2, P_3 with intermediate edges into a single cycle.

In the second case as mentioned earlier the graph X has been successfully decomposed into the Hamilton cycles: $H_1 = H_1' \oplus C$, $H_2 = H_2' \oplus Z$, and $H_3 = H_3' \oplus (C \oplus Z)$. In the first case let K_1 and K_2 be the two cycles. Then because vertices with first coordinate x are joined by an r-edge to vertices with first a coordinate $x + a$, there must exist an $x \in \mathbb{Z}_p \setminus \{z\}$ where all of the edges $\{(x+a, i), (x+a, i+1)\}$ are edges of K_2 except the edge s_{x+a} and an edge $\{(x, y), (x, y+1)\}$ in K_1 where $\{(x+a, y), (x+a, y+1)\} \ne s_{x+a}$. Let D be the 4-cycle

$$(x, y)(x, y + 1)(x + a, y + 1 + b)(x + a, y + b).$$

The edges of D alternate between $H_1' \oplus C$ and $K_1 + K_2 = H_3' \oplus (C \oplus Z)$. Also when the edges of the Hamilton cycle $H_1' \oplus C$ are traversed, parallel edges are traversed in the same direction. Consequently, applying Lemma 2.5, we see that $H_1' \oplus (C \oplus D)$ and $H_3' \oplus (C \oplus Z \oplus D)$ are Hamilton cycles (see Fig. 11). Now X has been successfully decomposed into the Hamilton cycles: $H_1 = H_1' \oplus (C \oplus D)$, $H_2 = H_2' \oplus Z$, and $H_3 = H_3' \oplus (C \oplus Z \oplus D)$.

To construct a chordal set of density $p-1$ for H_2, we use the set S given in Eq. (1). Set

$$M_1 = S \setminus \{s_z\} = \left\{\{(x, -z^{-1}x), (x, 1-z^{-1}x)\} : x \in \mathbb{Z}_p \setminus \{z\}\right\}.$$

Then M_1 is a matching in H_1 that has a unique e_2-edge with first coordinate x for each $x \in \mathbb{Z}_p \setminus \{z\}$. Let $x \in \mathbb{Z}_p$. If $x \notin \{z-1, z, z+1\}$, the only e_2-edge with first coordinate x that is not in H_3 is $s_x = \{(x, -z^{-1}x), (x, 1-z^{-1}x)\}$. Hence there are $p-3$ e_2-edges in H_3 with first coordinate x that are not adjacent to s_x. At most one of these was used by D. Thus there remains at least $(p-3)-1 \geq 1$ edges in H_3 with first coordinate x that are non-adjacent to an edge in M_1. If $x = z-1$ or $x = z+1$, there are $(p-1)/2$ e_2-edges with first coordinate x used by Z and at most one was used by D. There remains at least $p-(p-1)/2-1 = (p-1)/2 \geq 3$ e_2-edges in H_3 with first coordinate x. Of these at most two are adjacent to s_x and hence there is at least one that is non-adjacent to s_x. Therefore we may choose a coordinate y_x for each $x \in \mathbb{Z}_p \setminus \{z\}$ such that $M_3 = \{\{(x, y_x), (x, y_x+1)\} : x \in \mathbb{Z}_p \setminus \{z\}\}$ is a matching in H_3 vertex-disjoint from M_1. Consequently, $M = M_1 \cup M_3$ is a chordal set of density $p-1$ for H_2. An internally chordal vertex-free path of length p in $H_2 + M$ is

$$P = (z-1, 0)(z, 0)(z, 1)(z, 2)\cdots(z, p-1).$$

Case 2, $b = 1$: In this case the e_2-edges used in the cycle C are:

$$S = \left\{\{(x, 0), (x, 1)\} : x \in \mathbb{Z}_p\right\}. \tag{2}$$

Similar to Case 1 we employ the zig–zag

$$Z = F_{(0,0)} + F_{(1,1)} + F_{(0,2)} + F_{(1,3)} + \cdots + F_{(0,p-2)}.$$

Only the 4-cycle $F(0,0)$ has non-empty intersection with S. Thus, $F(0,0)$ alternates edges between $H_1' \oplus C$ and H_2', whereas the edges of the other 4-cycles in Z alternate between H_2' and $H_3' \oplus C$. The e_2-edges of Z join the cycles of H_2' and thus by Lemma 2.4 $H_2 = H_2' \oplus Z$ is a Hamilton cycle. Furthermore, because parallel e_2-edges of $H_3' \oplus Z$ have the same orientation it follows by Lemma 2.5 that $H_3 = H_3' \oplus (Z - F(0,0))$ is a Hamilton cycle. Also the edges $\{(0,0), (0,1)\}$ and $\{(1,0), (1,1)\}$ have the same orientation in $H_1' \oplus C$ so it follows that $H_1 = H_1' \oplus (C \oplus F(0,0))$ is a Hamilton cycle. Thus X has been successfully decomposed into the Hamilton cycles: H_1, H_2 and H_3. An example is provided in Fig. 12.

To construct a chordal set of density $p-1$ for H_2 we use the set S given in Eq. (2). Set

$$\begin{aligned} M_1 &= S \setminus \{\{(0,0), (0,1)\}, \{(1,0), (1,1)\}\} \cup \{\{(0,0), (1,0)\}\} \\ &= \{\{(x,0), (x,1)\} : x = 2, 3, 4, \ldots, p-1\} \cup \{\{(0,0), (1,0)\}\} \\ M_3 &= (S + (0,2)) \setminus \{\{(0,2), (0,3)\}, \{(1,2), (1,3)\}\} \cup \{\{(0,1), (0,2)\}\} \\ &= \{\{(x,2), (x,3)\} : x = 2, 3, 4, \ldots, p-1\} \cup \{\{(0,1), (0,2)\}\}. \end{aligned}$$

Then M_i is a partial matching in H_i, $i = 1, 3$ and M_1 and M_3 are vertex disjoint. Consequently $M = M_1 \cup M_3$ is a chordal set of density $p-1$ for H_2. An internally chordal vertex-free path of length p in $H_2 + M$ is

$$P = (1,0)(1,1)(1,2)\cdots(1,p-1)(0,p-1).$$

In the Fig. 12 example chordal vertices have been blackened.

Case 3, $b = 0$: Here we must find a Hamilton decomposition, chordal set and an internally chordal vertex-free path for

$$\text{Cay}\,(\mathbb{Z}_p^2; \{(a,0), (1,0), (0,1)\}^\star),$$

for all $p > 3$ and $1 < a \leq (p-1)/2$.

- Let F_x be as defined in Case 1. That is F_x is the 4-cycle with edge set

$$E(F_x) = \{\{x, x+e_1\}, \{x+e_1, x+e_1+e_2\}, \{x+e_1+e_2, x+e_2\}, \{x+e_2, x\}\}.$$

- For $r = (a, 0)$ let G_x, where $x \in \mathbb{Z}_p^2$, be the 4-cycle defined by the sequence $(r, e_2, -r, -e_2)$ and the vertex x that is G_x is the subgraph with edge set

$$E(G_x) = \{\{x, x+r\}, \{x+r, x+r+e_2\}, \{x+r+e_2, x+e_2\}, \{x+e_2, x\}\}.$$

- Let $\mathcal{F} = F_{(0,0)} + F_{(1,1)} + \cdots + F_{(p-4,p-4)} + F_{(p-3,p-3)} + F_{(p-2,p-2)}$.
- Let $\mathcal{G} = G_{(2,0)} + G_{(3,1)} + \cdots + G_{(p-2,p-4)} + G_{(p-1,p-3)} + G_{(0,p-2)}$.

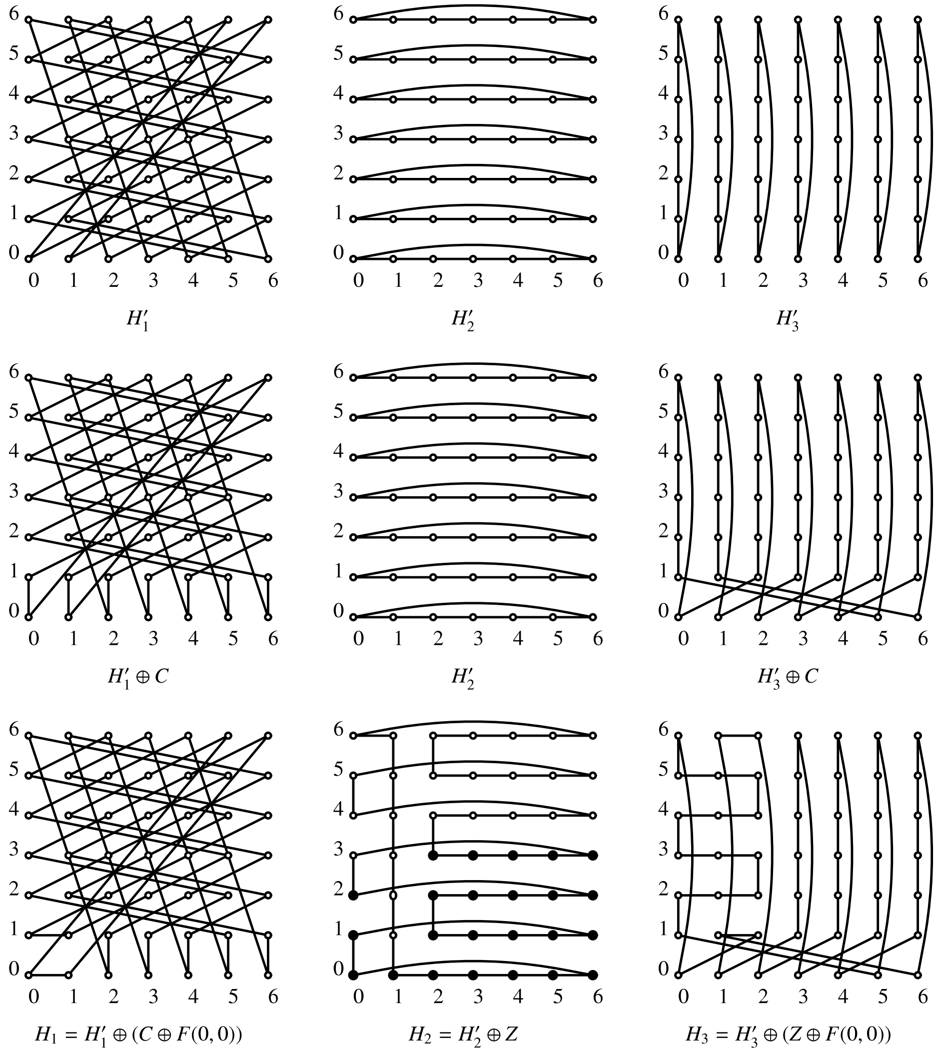


Fig. 12. CAY $(Z_7^2; \{(2,1),(1,0),(0,1)\}^\star)$.

Then it is routine to see that $H_1 = H_1' \oplus \mathcal{G}, H_2 = H_2' \oplus \mathcal{F}, H_3 = H_3' \oplus (\mathcal{F} \oplus \mathcal{G})$ are Hamilton cycles and thus H_1, H_2, H_3 is a Hamilton decomposition of X. An example is provided in Fig. 13.

To construct a chordal set of density $p-1$ for H_2, we set

$$M_1 = \{\{(2+x,x),(2+x,x+1)\},\{(2+a+x,x),(2+a+x,x+1)\} : x = 1,\ldots,(p-1)/2\}$$
$$M_3 = \{\{(x,0),(x,p-1)\} : x = 0,1,2,3,4,\ldots,p-2\}.$$

Then M_i is a matching in $H_i,\ i = 1,3$ and M_1 and M_3 are vertex-disjoint. Consequently $M = M_1 \cup M_3$ is a chordal set of density $p-1$ for H_2. Chordal vertices of H_2 are blackened in Fig. 13. An internally chordal vertex-free path of length p in $H_2 \oplus M$ is for example:

$$P = (p-1,p-2)(0,p-2)(1,p-2)(2,p-2)(3,p-2)\cdots(p-3,p-2)(p-3,p-3)(p-4,p-3).$$

In the Fig. 13 example chordal vertices have been blackened.

We summarize with the following theorem.

Theorem 4.4. *For every odd prime p and $(a,b) \in \mathbb{Z}_p$, the Cayley graph*

$$\text{CAY}\ (\mathbb{Z}_p^2; \{(a,b),(1,0),(0,1)\}^\star)$$

has a decomposition into Hamilton cycles H_1, H_2, H_3 and a chordal set M of density $p-1$ for H_2 such that $H_2 + M$ has an internally chordal vertex-free path P of length p.

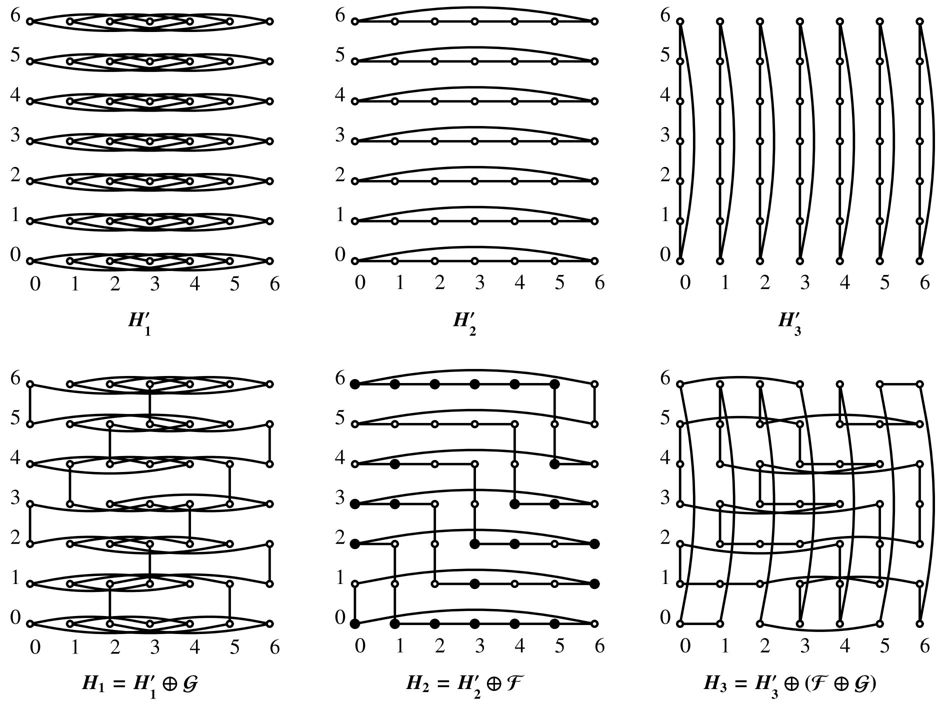


Fig. 13. Cay $(Z_7^2; \{(3,0), (1,0), (0,1)\}^\star)$.

4.5. Key result

We close this section with a key result.

Theorem 4.5. *Let $\mathcal{B}$ be a basis of $\mathbb{Z}_p^n$, p an odd prime, and let r be any non-zero vector of $\mathbb{Z}_p^n \setminus \mathcal{B}^\star$. Then the Cayley graph $X = \text{Cay}\,(V; (\mathcal{B} \cup \{r\})^\star)$ has a Hamilton decomposition.*

Proof. As discussed in the introduction to Section 4, we may assume $S = \{r, e_1, e_2, \ldots, e_n\}$, with $r \neq \pm e_j$, for all $j = 1, 2, \ldots, n$. Set $r_j = (r_1, r_2, \ldots, r_j)$, where $r_n = r$, and let $X_j = \text{Cay}\,(\mathbb{Z}_p^j; S_j)$, where $S_j = \{r_j, e_1, e_2, \ldots, e_j\}$.

If $n = 2$ we may use Theorem 4.4 to obtain a Hamilton decomposition H_0, H_1, H_2 of X_2 and a chordal set M of density $p - 1$ for H_0 such that H_0 has an internally chordal vertex-free path of length p.

If $p = 3$ and $n = 3$, we can use the construction given in Section 4.2 to decompose X_3. If $n \geq 3$, then

$$|X_n| = p^n \geq 2pn(p-1) = 2n(p^2 - p)$$

and we may apply Proposition 2.9 to any Hamilton decomposition $H_0, H_1, \ldots, H_n$ of X_n and obtain a chordal set M of density $p - 1$ for H_0 such that H_0 has an internally chordal vertex-free path of length p. Then taking $g = 1$ and defining $\alpha : (S \cup \{r_{n-1}\}) \to \mathbb{Z}_p$ by $\alpha(e_i) = 0,\ i = 1, 2, \ldots, n-1$ and $\alpha(r_{n-1}) = r_n - 1$, we can apply Corollary 3.2. (We assign n to d and p to n.) Using Corollary 3.2 we have by induction that $X_n = \text{Cay}\,(V; \{S_n{}^\star\})$ is Hamilton-decomposable for all n and odd primes p. □

Theorem 4.5 is our extension of Corollary 4.2 and is key to the Sub-Paley graph Hamilton decomposition problem, which we settle in the next section.

5. Sub-Paley graphs

We are interested in a particular family of Cayley graphs on abelian groups we call the Sub-Paley graphs. Let $\mathbb{F}_q$ denote the finite field of order q. For even m dividing $q - 1$, let $\text{R}(q, m)$ be the unique multiplicative subgroup of $\mathbb{F}_q \setminus \{0\}$ of order m. We define the Sub-Paley graph $\text{P}(q, m)$ of order q as the Cayley graph on $\mathbb{F}_q$ with connection set $\text{R}(q, m)$. Hence, the vertices of $\text{P}(q, m)$ are labeled with the elements of the field and there is an edge joining g and h if and only if $g - h \in \text{R}(q, m)$. The reason we insist that m be even is because then $\{1, -1\}$ is a subgroup of $\text{R}(q, m)$ and thus we have $g - h \in \text{R}(q, m)$ if and only if $h - g \in \text{R}(q, m)$. Because multiplicative subgroups of $\mathbb{F}_q \setminus \{0\}$ are cyclic, $\text{R}(q, m) = \{1, \beta^1, \beta^2, \ldots, \beta^{m-1}\}$ for some $\beta \in \mathbb{F}_q$. Let $\text{R}_h(q, m) = \{1, \beta^1, \beta^2, \ldots, \beta^{m/2-1}\}$. Then either $g \in \text{R}_h(q, m)$ or $-g \in \text{R}_h(q, m)$, but not both. Hence, $|\text{R}_h(q, m)| = m/2$ and $\text{R}_h(q, m)^\star = \text{R}(q, m)$.

Note that if $q \equiv 1 (\text{mod } 4)$, then $\text{R}(q, (q-1)/2)$ is the set of quadratic residues and $\text{P}(q, (q-1)/2)$ is the *Paley graph* of order q. In [2] all Paley graphs were shown to be Hamilton-decomposable.

Theorem 5.1. *Let $q = p^n$, where p is an odd prime, and let $m \geq 2n^2$ be an even divisor of $q-1$. If the sub-Paley graph $X = \text{Cay}(\mathbb{F}_q; \mathrm{R}(q, m))$ is connected, then X is Hamilton-decomposable.*

Proof. Let $g(X)$ be the minimum polynomial for β over $\mathbb{F}_p$ and let $d = \deg(g(X))$. Then

$$A_0 = \{1, \beta, \beta^2, \ldots, \beta^{d-1}\}$$

considered as vectors over $\mathbb{F}_p$ is a maximal linear independent set in $\mathrm{R}_h(q, m)$. If the graph X is connected, then $\mathrm{R}_h(q, m)$ must span $\mathbb{F}_q$ and therefore in this case $d = n$. Thus writing $m/2 = tn + r$, where $0 \leq r < n$, we partition $\mathrm{R}_h(q, m)$ into the linearly independent sets

$$A_0, A_1, \ldots, A_t$$

where

$$A_i = (\beta^d)^i A_0 = \{\beta^{di}, \beta^{di+1}, \ldots, \beta^{di+d-1}\},$$

$i = 0, 1, 2, \ldots, t-1$ and $A_t = \{\beta^{tn}, \beta^{tn+1}, \beta^{tn+2}, \ldots, \beta^{m/2-1}\}$. Now $t = \left\lfloor \frac{m}{2n} \right\rfloor \geq n > r$. Thus we may apply Theorem 4.5 to $A_j \cup \{\beta^{tn+j}\}$, for $j = 0, 1, 2, \ldots, m/2 - tn - 1$, decomposing $\text{Cay}(\mathbb{F}_q; A_j \cup \{\beta^{tn+j}\})$ into Hamilton cycles, for $j = 0, 1, 2, \ldots, m/2 - tn - 1$. We apply Corollary 4.2 to decompose $\text{Cay}(\mathbb{F}_q; A_\ell)$ into Hamilton cycles for $\ell = m/2 - tn, m/2 - tn + 1, \ldots, t-1$. □

The result of Alspach, Bryant and Dyer on Paley graphs in [2] can be obtained as a simple consequence of Theorem 5.1.

Corollary 5.2 (*Alspach, Bryant, Dyer, 2010*)**.** *All Paley graphs are Hamilton-decomposable.*

Proof. If $q = p^n \equiv 1 \pmod 4$, where p is a prime and n a positive integer, then $(q-1)/2 \geq 2n^2$, except when $q = 9$. Applying Theorem 5.1 we obtain the result. For $q = 9$, the Paley graph is 4-regular and is Hamilton decomposable by Theorem 2.2. □

Theorem 5.1 leaves open the sub-Paley graphs $X = \text{Cay}(\mathbb{F}_q; \mathrm{R}(q, m))$, where q is odd and $2n \leq m < 2n^2$ or where q is even.

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