Kanca and Baglan *Boundary Value Problems* 2013, **2013**:28
http://www.boundaryvalueproblems.com/content/2013/1/28

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Continuous dependence on data for a solution of the quasilinear parabolic equation with a periodic boundary condition

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Abstract
In this paper we consider a parabolic equation with a periodic boundary condition and we prove the stability of a solution on the data. We give a numerical example for the stability of the solution on the data.

1 Introduction

Consider the following mixed problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t,u), \quad (x,t) \in D := \{0 < t < T, 0 < x < \pi\}, \tag{1}$$

$$u(0,t) = u(\pi,t), \quad t \in [0,T], \tag{2}$$

$$u_x(0,t) = u_x(\pi,t), \quad t \in [0,T], \tag{3}$$

$$u(x,0) = \varphi(x), \quad x \in [0,\pi] \tag{4}$$

for a quasilinear parabolic equation with the nonlinear source term $f = f(x,t,u)$.

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\bar{D} \times (-\infty,\infty)$ respectively. Denote the solution of problem (1)-(4) by $u = u(x,t)$. The existence, uniqueness and convergence of the weak generalized solution of problem (1)-(4) are considered in [1]. The numerical solution of problem (1)-(4) is considered [2].

In this study we prove the continuous dependence of the solution $u = u(x,t)$ upon the data $\varphi(x)$ and $f(x,t,u)$. In [3], a similar iteration method is used with this kind of a local boundary condition for a nonlinear inverse coefficient problem for a parabolic equation. Then we give a numerical example for the stability.

2 Continuous dependence upon the data

In this section, we will prove the continuous dependence of the solution $u = u(x,t)$ using an iteration method. The continuous dependence upon the data for linear problems by different methods is shown in [4, 5].

Theorem 1 *Under the following assumptions, the solution $u = u(x,t)$ depends continuously upon the data.*

(A_1) Let the function $f(x,t,u)$ be continuous with respect to all arguments in $\bar{D} \times (-\infty, \infty)$ and satisfy the following condition:

$$\left|f(t,x,u) - f(t,x,\tilde{u})\right| \leq b(x,t)|u - \tilde{u}|,$$

where $b(x,t) \in L_2(D)$, $b(x,t) \geq 0$,

(A_2) $f(x,t,0) \in C^2[0,\pi]$, $t \epsilon [0,\pi]$,

(A_3) $\varphi(x) \in C^2[0,\pi]$.

Proof Let $\phi = \{\varphi, f\}$ and $\overline{\phi} = \{\overline{\varphi}, \overline{f}\}$ be two sets of data which satisfy the conditions (A_1)-(A_3).

Let $u = u(x,t)$ and $v = v(x,t)$ be the solutions of problem (1)-(4) corresponding to the data ϕ and $\overline{\phi}$ respectively, and

$$\left|f(t,x,0) - \overline{f}(t,x,0)\right| \leq \varepsilon \quad \text{for } \varepsilon \geq 0.$$

The solutions of (1)-(4), $u = u(x,t)$ and $v = v(x,t)$, are presented in the following form, respectively:

$$\begin{aligned}
u_0(t) &= \varphi_0 + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au(\xi,\tau)\big)\,d\xi\,d\tau, \\
u_{ck}(t) &= \varphi_{ck} e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au(\xi,\tau)\big) e^{-(2\pi k)^2(t-\tau)} \cos 2k\xi\,d\tau, \\
u_{sk}(t) &= \varphi_{sk} e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au(\xi,\tau)\big) e^{-(2\pi k)^2(t-\tau)} \sin 2k\xi\,d\tau.
\end{aligned} \tag{5}$$

Let $Au(\xi,\tau) = \frac{u_{0(\tau)}}{2} + \sum_{k=1}^{\infty}[u_{ck}(\tau)\cos 2k\xi + u_{sk}(\tau)\sin 2k\xi]$.

$$\begin{aligned}
v_0(t) &= \overline{\varphi}_0 + \frac{2}{\pi}\int_0^t \int_0^\pi \overline{f}\big(\xi,\tau,Av(\xi,\tau)\big)\,d\xi\,d\tau, \\
v_{ck}(t) &= \overline{\varphi}_{ck}\, e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t \int_0^\pi \overline{f}\big(\xi,\tau,Av(\xi,\tau)\big) e^{-(2\pi k)^2(t-\tau)} \cos 2k\xi\,d\tau, \\
v_{sk}(t) &= \overline{\varphi}_{sk} e^{-(2k)^2 t} + \frac{2}{\pi}\int_0^t \int_0^\pi \overline{f}\big(\xi,\tau,Av(\xi,\tau)\big) e^{-(2\pi k)^2(t-\tau)} \sin 2k\xi\,d\tau.
\end{aligned} \tag{6}$$

Let $Av(\xi,\tau) = \frac{v_0(\tau)}{2} + \sum_{k=1}^{\infty}[v_{ck}(\tau)\cos 2k\xi + v_{sk}(\tau)\sin 2k\xi]$.

From the condition of the theorem, we have $u^{(0)}(t)$ and $v^{(0)}(t) \in B$. We will prove that the other sequential approximations satisfy this condition.

$$\begin{aligned}
u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au^{(N)}(\xi,\tau)\big)\,d\xi\,d\tau, \\
u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au^{(N)}(\xi,\tau)\big) e^{-(2k)^2(t-\tau)} \cos 2k\xi\,d\tau, \\
u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t \int_0^\pi f\big(\xi,\tau,Au^{(N)}(\xi,\tau)\big) e^{-(2k)^2(t-\tau)} \sin 2k\xi\,d\tau, \\
v_0^{(N+1)}(t) &= v_0^{(0)}(t) + \frac{2}{\pi}\int_0^t \int_0^\pi \overline{f}\big(\xi,\tau,Av^{(N)}(\xi,\tau)\big)\,d\xi\,d\tau,
\end{aligned} \tag{7}$$

$$v_{ck}^{(N+1)}(t) = v_{ck}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi \overline{f}\big(\xi,\tau,Av^{(N)}(\xi,\tau)\big)e^{-(2k)^2(t-\tau)}\cos 2k\xi\,d\tau, \tag{8}$$
$$v_{sk}^{(N+1)}(t) = v_{sk}^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi \overline{f}\big(\xi,\tau,Av^{(N)}(\xi,\tau)\big)e^{-(2k)^2(t-\tau)}\sin 2k\xi\,d\tau,$$

where $u_0^{(0)}(t) = \varphi_0$, $u_{ck}^{(0)}(t) = \varphi_{ck}e^{-(2k)^2t}$, $u_{sk}^{(0)}(t) = \varphi_{sk}e^{-(2k)^2t}$ and $v_0^{(0)}(t) = \overline{\varphi}_0$, $v_{ck}^{(0)}(t) = \overline{\varphi}_{ck}e^{-(2k)^2t}$, $v_{sk}^{(0)}(t) = \overline{\varphi}_{sk}e^{-(2k)^2t}$.

First of all, we write $N = 0$ in (6)-(7). We consider $u^{(1)}(t) - v^{(1)}(t)$

$$\begin{aligned}
u^{(1)}(t) - v^{(1)}(t) &= \frac{u_0^{(1)}(t) - v_0^{(1)}(t)}{2} \\
&\quad + \sum_{k=1}^{\infty}\big[\big(u_{ck}^{(1)}(t) - v_{ck}^{(1)}(t)\big) + \big(u_{sk}^{(1)}(t) - v_{sk}^{(1)}(t)\big)\big] \\
&= (\varphi_0 - \overline{\varphi_0}) \\
&\quad + \frac{2}{\pi}\int_0^t\int_0^\pi \big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau)\big) - \overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau)\big)\big]\,d\xi\,d\tau \\
&\quad + (\varphi_{ck} - \overline{\varphi_{ck}})e^{-(2k)^2t} \\
&\quad + \frac{2}{\pi}\int_0^t\int_0^\pi \big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau)\big) - \overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau)\big)\big] \\
&\quad \times e^{-(2\pi k)^2(t-\tau)}\cos 2\pi k\xi\,d\xi\,d\tau + (\varphi_{sk} - \overline{\varphi_{sk}})e^{-(2k)^2t} \\
&\quad + \frac{2}{\pi}\int_0^t\int_0^\pi \big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau)\big) - \overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau)\big)\big] \\
&\quad \times e^{-(2\pi k)^2(t-\tau)}\sin 2\pi k\xi\,d\xi\,d\tau.
\end{aligned} \tag{9}$$

Adding and subtracting

$$\int_0^t\int_0^\pi f(\xi,\tau,0)\,d\xi\,d\tau, \qquad \int_0^t\int_0^\pi e^{-(2k)^2(t-\tau)}f(\xi,\tau,0)\cos 2\pi k\xi\,d\xi\,d\tau,$$
$$\int_0^t\int_0^\pi e^{-(2k)^2(t-\tau)}f(\xi,\tau,0)\sin 2\pi k\xi\,d\xi\,d\tau$$

to both sides and applying the Cauchy inequality, Hölder inequality, Lipschitz condition and Bessel inequality to the right-hand side of (8) respectively, we obtain

$$\begin{aligned}
\big|u^{(1)}(t) - v^{(1)}(t)\big| &\le 2\big|u_0^{(1)}(t) - v_0^{(1)}(t)\big| + 4\sum_{k=1}^{\infty}\big(\big|u_{ck}^{(1)}(t) - v_{ck}^{(1)}(t)\big| + \big|u_{sk}^{(1)}(t) - v_{sk}^{(1)}(t)\big|\big) \\
&\le \|\varphi - \overline{\varphi}\| \\
&\quad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\big|\overline{u}^{(0)}(t)\big| \\
&\quad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\big|\overline{v}^{(0)}(t)\big| \\
&\quad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0) - \overline{f}^2(\xi,\tau,0)\,d\xi\,d\tau\right)^{\frac{1}{2}},
\end{aligned}$$

$$A_T = \|\varphi - \overline{\varphi}\| + \left[\left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\|b(x,t)\|\left|\bar{u}^{(0)}(t)\right| + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\|b(x,t)\|\left|\overline{v}^{(0)}(t)\right|\right] + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\|f-\overline{f}\|,$$

$$\|\varphi - \overline{\varphi}\| = \max\frac{|\varphi_0 - \overline{\varphi_0}|}{2} + \sum_{k=1}^{\infty}\max|\varphi_{ck} - \overline{\varphi_{ck}}| + \max|\varphi_{sk} - \overline{\varphi_{sk}}|.$$

For $N = 1$,

$$\begin{aligned}\left|u^{(2)}(t) - v^{(2)}(t)\right| &\le \frac{|u_0^{(2)}(t) - v_0^{(2)}(t)|}{2} + \sum_{k=1}^{\infty}\left(\left|u_{ck}^{(2)}(t) - v_{ck}^{(2)}\right| + \left|u_{sk}^{(2)}(t) - v_{sk}^{(2)}(t)\right|\right)\\ &\le \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}A_T\\ &\quad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}A_T.\end{aligned}$$

For $N = 2$,

$$\begin{aligned}&\left|u^{(3)}(t) - v^{(3)}(t)\right|\\ &\quad\le \frac{|u_0^{(3)}(t) - v_0^{(3)}(t)|}{2} + \sum_{k=1}^{\infty}\left(\left|u_{ck}^{(3)}(t) - v_{ck}^{(3)}(t)\right| + \left|u_{sk}^{(3)}(t) - v_{sk}^{(3)}(t)\right|\right)\\ &\quad\le \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|\overline{u}^{(2)}(t) - \overline{v}^{(2)}(t)\right|^2 d\xi\,d\tau\right)^{\frac{1}{2}}\\ &\qquad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)\left(\int_0^t\int_0^\pi \overline{b}^2(\xi,\tau)\left|\overline{u}^{(2)}(t) - \overline{v}^{(2)}(t)\right|^2 d\xi\,d\tau\right)^{\frac{1}{2}}\\ &\quad\le \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^2 A_T\left[\int_0^t\int_0^1 b^2(\xi,\tau)\left(\int_0^\tau\int_0^\pi b^2(\xi_1,\tau_1)\,d\xi_1\,d\tau_1\right)d\xi\,d\tau\right]^{\frac{1}{2}}\\ &\qquad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^2 A_T\left[\int_0^t\int_0^1 b^2(\xi,\tau)\left(\int_0^\tau\int_0^1 \overline{b}^2(\xi_1,\tau_1)\,d\xi_1\,d\tau_1\right)d\xi\,d\tau\right]^{\frac{1}{2}}\\ &\quad\le \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{1}{2}}\\ &\qquad + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

In the same way, for a general value of N, we have

$$\begin{aligned}\left|u^{(N+1)}(t) - v^{(N+1)}(t)\right| &\le \frac{|u_0^{(N+1)}(t) - v_0^{(N+1)}(t)|}{2}\\ &\quad + \sum_{k=1}^{\infty}\left(\left|u_{ck}^{(N+1)}(t) - v_{ck}^{(N+1)}(t)\right| + \left|u_{sk}^{(N+1)}(t) - v_{sk}^{(N+1)}(t)\right|\right)\\ &\le A_T\cdot a_N = a_N\left(\|\varphi - \overline{\varphi}\| + C(t) + M_1\|f - \overline{f}\|\right),\end{aligned} \tag{10}$$

where

$$a_N = \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}} + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^\pi \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}}$$

and

$$M_1 = \left(\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\right)^N.$$

(The sequence a_N is convergent, then we can write $a_N \le M$, $\forall N$.)

It follows from the estimation ([1, pp.76-77]) that $\lim_{N\to\infty} u^{(N+1)}(t) = u(t)$.

Then let $N \to \infty$ for the last equation

$$\left|u(t)-v(t)\right| \le M\|\varphi-\overline{\varphi}\| + M_2\|f-\overline{f}\|,$$

where $M_2 = M\cdot M_1$.

If $\|f-\overline{f}\| \le \varepsilon$ and $\|\varphi-\overline{\varphi}\| \le \varepsilon$, then $|u(t)-v(t)| \le \varepsilon$. □

3 Numerical example

In this section we consider an example of numerical solution of (1)-(4) to test the stability of this problem. The numerical procedure of (1)-(4) is considered in [2].

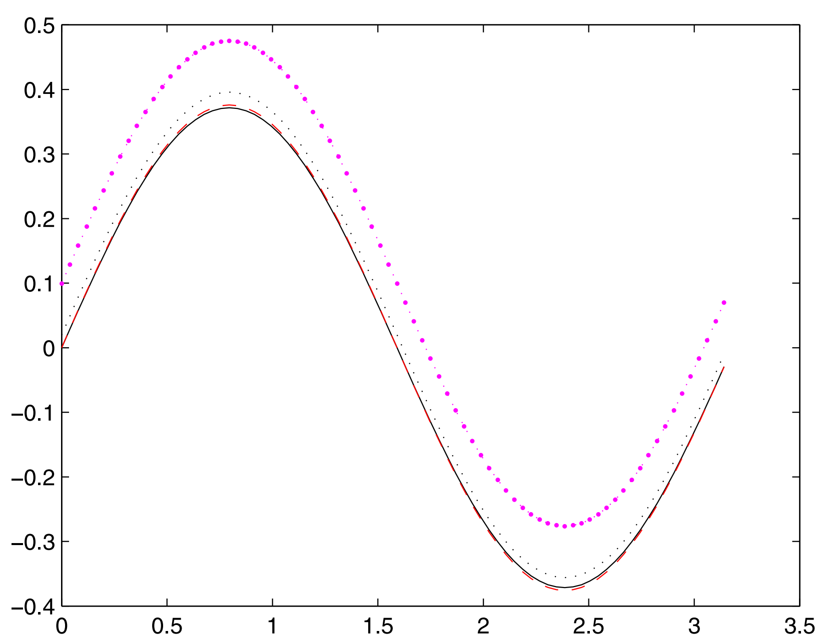

Figure 1 The exact and numerical solutions of $u(x,1)$. The exact and numerical solutions of $u(x,1)$, (–) for $\varepsilon = 0$, (–·) for $\varepsilon = 0.05$, (..) for $\varepsilon = 0.01$, the exact solution is shown with a dashed line.

Example 1 Consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 3u, \tag{11}$$

$$u(x,0) = \sin 2x, \quad x \in [0,\pi], \tag{12}$$

$$u(0,t) = u(\pi,t), \quad t \in [0,T], \qquad u_x(0,t) = u_x(\pi,t), \quad t \in [0,T]. \tag{13}$$

It is easy to see that the analytical solution of this problem is

$$u(x,t) = \sin 2x \exp(-t).$$

In this example, we take $f(x,t,u) = f(x,t,u) + \varepsilon$ and $\varphi(x) = \varphi(x) + \varepsilon$ for different ε values.

The comparisons between the analytical solution and the numerical finite difference solution for $\varepsilon = 0,01$, $\varepsilon = 0,05$ values when $T = 1$ are shown in Figure 1.

The computational results presented are consistent with the theoretical results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FK conceived the study, participated in its design and coordination and prepared computing section. ISB participated in the sequence alignment and achieved the estimation.

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Acknowledgements
Dedicated to Professor Hari M Srivastava.

Received: 7 January 2013 Accepted: 29 January 2013 Published: 14 February 2013

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doi:10.1186/1687-2770-2013-28
Cite this article as: Kanca and Baglan: **Continuous dependence on data for a solution of the quasilinear parabolic equation with a periodic boundary condition.** *Boundary Value Problems* 2013 **2013**:28.