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Generalized Einstein Tensor for a Weyl Manifold and Its Applications

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Abstract It is well known that the Einstein tensor G for a Riemannian manifold defined by $G_\alpha^\beta = R_\alpha^\beta - \frac{1}{2}R\delta_\alpha^\beta$, $R_\alpha^\beta = g^{\beta\gamma}R_{\gamma\alpha}$ where R_{γ_α} and R are respectively the Ricci tensor and the scalar curvature of the manifold, plays an important part in Einstein's theory of gravitation as well as in proving some theorems in Riemannian geometry. In this work, we first obtain the generalized Einstein tensor for a Weyl manifold. Then, after studying some properties of generalized Einstein tensor, we prove that the conformal invariance of the generalized Einstein tensor implies the conformal invariance of the curvature tensor of the Weyl manifold and conversely. Moreover, we show that such Weyl manifolds admit a one-parameter family of hypersurfaces the orthogonal trajectories of which are geodesics. Finally, a necessary and sufficient condition in order that the generalized circles of a Weyl manifold be preserved by a conformal mapping is stated in terms of generalized Einstein tensors at corresponding points.

Keywords Weyl manifold, Einstein–Weyl manifold, Einstein tensor, generalized Einstein tensor, generalized circle

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1 Preliminaries

A differentiable manifold of dimension n having a torsion-free connection ∇ and a conformal class $C[g]$ of metrics preserved by ∇ is called a Weyl manifold which will be denoted by $W_n(g,\omega)$ where $g \in C[g]$ and ω is a 1-form satisfying the compatibility condition (see [1–3])

$$\nabla g = 2(\omega \otimes g). \tag{1.1}$$

Under the conformal re-scaling (normalization)

$$\bar{g} = \lambda^2 g, \quad \lambda > 0 \tag{1.2}$$

of the representative metric tensor g, ω is transformed by the law

$$\bar{\omega} = \omega + d\ln\lambda. \tag{1.3}$$

A tensor field A defined on $W_n(g,\omega)$ is called a satellite of g of weight $\{p\}$ if it admits a transformation of the form

$$\bar{A} = \lambda^p A \tag{1.4}$$

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under the conformal re-scaling (1.2) of g (see [1–3]).

It can be easily seen that the pair $(\bar{g}, \bar{\omega})$ generates the same Weyl manifold. The process of passing from (g, ω) to $(\bar{g}, \bar{\omega})$ is called a gauge transformation.

The curvature tensor, covariant curvature tensor, the Ricci tensor and the scalar curvature of $W_n(g, \omega)$ are respectively defined by

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) v^p = v^j W^p_{jkl}, \tag{1.5}$$

$$W_{hjkl} = g_{hp} W^p_{jkl}, \tag{1.6}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \tag{1.7}$$

$$W = g^{ij} W_{ij}. \tag{1.8}$$

It is clear that W^p_{jkl} and W_{jk} are gauge invariants [4].

It follows from (1.5) that

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk} \Gamma^h_{jl} - \Gamma^p_{hl} \Gamma^h_{jk}, \quad \partial_k = \frac{\partial}{\partial x^k}, \tag{1.9}$$

where Γ^i_{kl} are the coefficients of the Weyl connection ∇ given by

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - (\delta^i_k \omega_l + \delta^i_l \omega_k - g_{kl} g^{im} \omega_m), \tag{1.10}$$

in which $\left\{ {i \atop kl} \right\}$ are the coefficients of the Levi–Civita connection formed with respect to g.

By straightforward calculations it is not difficult to see that

$$W_{ijkl} + W_{ijlk} = 0, \tag{1.11}$$

$$W_{ijkl} + W_{jikl} = 4 g_{ij} \nabla_{[l} \omega_{k]}, \tag{1.12}$$

$$W_{[ij]} = n \nabla_{[i} w_{j]}, \tag{1.13}$$

where brackets indicate the antisymmetric parts of the corresponding tensors (see [4, 5]).

The prolonged (extended) covariant derivative of the satellite A of weight $\{p\}$ in the direction of the vector field X is defined by [1, 3]

$$\dot{\nabla}_X A = \nabla_X A - p\, \omega(X) A. \tag{1.14}$$

From (1.1) and (1.14) it follows that

$$\dot{\nabla}_X g = 0, \quad g \in C[g]. \tag{1.15}$$

We note that the prolonged covariant differentiation preserves the weights of the satellites of g.

A Riemannian manifold is called an *Einstein manifold* if its Ricci tensor is proportional to its metric tensor.

A Weyl manifold is said to be an *Einstein–Weyl* manifold [6], if the symmetric part of its Ricci tensor is proportional to the representative metric tensor $g \in C[g]$, and hence we have

$$W_{(ij)} = \frac{W}{n} g_{ij}. \tag{1.16}$$

In [7], as a generalization of geodesic circles in a Riemannian manifold, we defined the so-called *generalized circles* by means of prolonged covariant differentiation as follows: Let C be a smooth curve belonging to the Weyl manifold $W_n(g, \omega)$ and let ξ_1 be the tangent vector to

C at the point p, normalized by the condition $g(\xi_1, \xi_1) = 1$. C is called a generalized circle in $W_n(g, w)$ if there exist a vector field ξ_2, normalized by the condition $g(\xi_2, \xi_2) = 1$, and a positive prolonged covariant constant function κ_1 of weight $\{-1\}$ along C such that

$$\dot{\nabla}_{\xi_1}\xi_1 = \kappa_1\xi_2, \quad \dot{\nabla}_{\xi_1}\xi_2 = -\kappa_1\xi_1. \tag{1.17}$$

According to Frenet' s formulas

$$\dot{\nabla}_{\xi_1}\xi_m = -\kappa_{m-1}\xi_{m-1} + \kappa_m\xi_{m+1}, \quad m = 1, 2, \ldots, n; \quad \kappa_0 = \kappa_n = 0$$

given in [1], the equations (1.17) imply that C will be a generalized circle if and only if the first curvature κ_1 of C is prolonged covariant constant and the second curvature κ_2 is zero along C. Namely,

$$\dot{\nabla}_{\xi_1}\kappa_1 = \nabla_{\xi_1}\kappa_1 + \kappa_1\omega(\xi_1) = 0, \quad \kappa_2 = 0. \tag{1.18}$$

A conformal mapping of a Weyl manifold upon another Weyl manifold will be called *generalized concircular* if it preserves the generalized circles [7].

Concerning generalized concircular mappings we have

Theorem 1.1 ([7]) *The conformal mapping $\tau : W_n(g, \omega) \to \tilde{W}_n(\tilde{g}, \tilde{\omega})$ will be generalized concircular if and only if*

$$P_{kl} = \phi g_{kl}, \quad P_{kl} = \nabla_l P_k - P_k P_l + \frac{1}{2} g_{kl} g^{rs} P_r P_s, \tag{1.19}$$

where

$$P = w - \tilde{w} \tag{1.20}$$

is the covector field of the conformal mapping of weight $\{0\}$ and ϕ is a smooth scalar function of weight $\{-2\}$ defined on $W_n(g, w)$.

2 Generalized Einstein Tensor for a Weyl Manifold

The Einstein tensor G_α^β for the Riemannian manifold M of dimension n is defined by $G_\alpha^\beta = R_\alpha^\beta - \frac{1}{2}R\delta_\alpha^\beta, R_\alpha^\beta = g^{\beta\gamma}R_{\alpha\gamma}$, where $R_{\alpha\gamma}$ and R are respectively, the Ricci tensor and the scalar curvature of M (see [8–10]). It is well known that Einstein tensor for a Riemannian manifold is identically zero for $n = 2$ and that its divergence is zero for $n > 2$ (see [9]).

In this section, as a generalization of Einstein tensor for a Riemannian manifold ,we define the Einstein tensor for the Weyl manifold $W_n(g, \omega)$ and call it the generalized Einstein tensor since it reduces to G_α^β when ω becomes zero or locally a gradient.

To derive the generalized Einstein tensor for $W_n(g, \omega)$, we will use the second Bianchi identity for $W_n(g, \omega)$ which is obtained in [5, 11] as

$$\dot{\nabla}_l W_{mijk} + \dot{\nabla}_k W_{milj} + \dot{\nabla}_j W_{mikl} = 0. \tag{2.1}$$

Transvecting (2.1) by g^{mk} and remembering that the prolonged covariant derivatives of g and its reciprocal tensor are zero, we obtain

$$\dot{\nabla}_l W_{ij} + \dot{\nabla}_k g^{mk} W_{milj} - \dot{\nabla}_j W_{il} = 0, \tag{2.2}$$

in which (1.8) and (1.11) have been used.

On the other hand, using (1.12) we find that

$$g^{mk}W_{milj} = 4\delta_i^k \nabla_{[j}\omega_{l]} - g^{mk}W_{imlj}. \tag{2.3}$$

Transvecting (2.2) by g^{ij} and using (1.8), (1.11) and (2.3), we get

$$\dot{\nabla}_l W + \dot{\nabla}_k[4g^{kj}\nabla_{[j}\omega_{l]} - g^{mk}W_{ml}] - \dot{\nabla}_j g^{ij} W_{il} = 0. \tag{2.4}$$

Putting

$$g^{ij}W_{il} = W_l^j \tag{2.5}$$

in (2.4), using the relation $\dot{\nabla}_l W = \dot{\nabla}_j(\delta_l^j W)$ and dividing (2.4) through by 2, we find that

$$\dot{\nabla}_j \left(W_l^j - \frac{1}{2} W \delta_l^j - 2g^{jk}\nabla_{[k}\omega_{l]} \right) = 0. \tag{2.6}$$

The tensor with components

$$G_l^j = W_l^j - \frac{1}{2} W \delta_l^j - 2g^{jk}\nabla_{[k}\omega_{l]} \tag{2.7}$$

will be named as the *generalized Einstein tensor* since it reduces to Einstein tensor for the Riemannian space M in the special case when w is zero or a gradient. It is clear that G_l^j is a satellite of g of weight $\{-2\}$.

We may define the *generalized divergence* of G_l^j as $\dot{\nabla}_j G_l^j$. Then from (2.6) it follows that

$$\dot{\nabla}_j G_l^j = \dot{\nabla}_j \left(W_l^j - \frac{1}{2} W \delta_l^j - 2g^{jk}\nabla_{[k}\omega_{l]} \right) = 0. \tag{2.8}$$

This is the generalization of the fact that the divergence of Einstein tensor for a Riemannian manifold is zero, to the case of a Weyl manifold. From (2.8), we obtain

$$\dot{\nabla}_j W_l^j = \frac{1}{2}\dot{\nabla}_l W + 2g^{jk}\dot{\nabla}_j(\nabla_{[k}\omega_{l]}). \tag{2.9}$$

We note that, if ω is zero or a gradient, (2.9) reduces to the well-known equation

$$\nabla_j R_l^j = \frac{1}{2}\partial_l R, \quad \partial_l = \frac{\partial}{\partial x^l},$$

which is important in the general theory of relativity [10, 12].

Transvecting (2.7) by g_{ij} and using (2.5), we obtain the gauge invariant tensor

$$g_{ij}G_l^j = G_{il} = W_{il} - \frac{1}{2} W g_{il} - 2\nabla_{[i}\omega_{l]}.$$

Suppose now that $W_n(g, \omega)$ is an Einstein–Weyl manifold. Then by (1.13), (1.16) and (2.5), we find

$$W_l^j = g^{kj}W_{kl} = g^{kj}\left(W_{(kl)} + W_{[kl]}\right) = \frac{W}{n}\delta_l^j + ng^{kj}\nabla_{[k}\omega_{l]}. \tag{2.10}$$

Substitution of (2.10) into (2.7) gives the generalized Einstein tensor for an Einstein–Weyl manifold in the form

$$G_l^j = \frac{2-n}{2}\left(\frac{W}{n}\delta_l^j - 2g^{jk}\nabla_{[k}\omega_{l]}\right). \tag{2.11}$$

It follows from (1.11) that the generalized Einstein tensor for an Einstein–Weyl manifold vanishes identically for $n = 2$. According to (2.8) and (2.11), for $n > 2$, we have

$$\frac{1}{n}(\dot{\nabla}_l W) - 2g^{jk}\dot{\nabla}_j(\nabla_{[k}\omega_{l]}) = 0. \tag{2.12}$$

It is clear from (2.12) that, unlike the Riemannian case, W need not, in general, be a constant.

In particular, if ω is locally a gradient, i.e., if the Einstein–Weyl manifold $W_n(g,\omega)$ is conformal to an Einstein manifold, the second term in (2.12) vanishes and (2.12) reduces to

$$\dot{\nabla}_l W = \nabla_l W + 2W\omega_l = 0, \tag{2.13}$$

which means that W is prolonged covariant constant.

3 Conformal Change of Generalized Einstein Tensor

In this section we will study the conformal change of the generalized Einstein tensor since it is closely related to the invariance of the curvature tensor of $W_n(g,\omega)$. In this section we will prove Theorems 1.1, 3.2 and 3.3. In particular, when the conformal mapping under consideration is a generalized concircular one, we have Theorem 3.4.

Let $\tau : W_n(g,w) \to \tilde{W}_n(\tilde{g},\tilde{w})$ be a conformal mapping of $W_n(g,w)$ upon $\tilde{W}_n(\tilde{g},\tilde{w})$. By suitable conformal re-scalings on $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$, at corresponding points of these manifolds we can make [2, 3]

$$g = \tilde{g}. \tag{3.1}$$

It is clear that the covector field $P = w - \tilde{w}$ of τ has zero weight.

Let ∇ and $\tilde{\nabla}$ be the connections of $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$ and let the connection coefficients be denoted by Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ respectively. Then, by (1.10) and (3.1), we have

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j P_k + \delta^i_k P_j - g^{im} P_m g_{jk}. \tag{3.2}$$

Replacing Γ^i_{jk} in (1.9) by $\tilde{\Gamma}^i_{jk}$ in (3.2), we obtain the curvature tensor of $\tilde{W}_n(\tilde{g},\tilde{w})$ as [7],

$$\tilde{W}^p_{jkl} = W^p_{jkl} + \delta^p_l P_{jk} - \delta^p_k P_{jl} + g_{jk} g^{pm} P_{ml} - g_{jl} g^{pm} P_{mk} + 2\delta^p_j \nabla_{[k} P_{l]}, \tag{3.3}$$

where $\nabla_{[k} P_{l]}$ is the antisymmetric part of $\nabla_k P_l$ and

$$P_{kl} = \nabla_l P_k - P_k P_l + \frac{1}{2} g_{kl} g^{rs} P_r P_s. \tag{3.4}$$

Contraction on the indices p and l in (3.3) gives

$$\tilde{W}_{jk} = W_{jk} + (n-2)P_{jk} + g_{jk} g^{lm} P_{ml} + 2\nabla_{[k} P_{j]}, \tag{3.5}$$

in which we have used the relation $g_{jk} g^{km} = \delta^m_j$.

Transvecting (3.5) by $\tilde{g}^{jk} = g^{jk}$ and using (1.8), we obtain

$$\tilde{W} = W + 2(n-1) g^{jk} P_{jk},$$

from which it follows that

$$g^{jk} P_{jk} = \frac{\tilde{W} - W}{2(n-1)}. \tag{3.6}$$

By virtue of (3.6), (3.5) becomes

$$\tilde{W}_{jk} = W_{jk} + (n-2)P_{jk} + \frac{\tilde{W} - W}{2(n-1)} g_{jk} + 2P_{[jk]}, \quad \nabla_{[k} P_{j]} = P_{[jk]}. \tag{3.7}$$

Transvecting (3.7) by $\tilde{g}^{jl} = g^{jl}$ and putting $g^{jl} W_{jk} = W^l_k$, $\tilde{g}^{jl} \tilde{W}_{jk} = \tilde{W}^l_k$, we find that

$$\tilde{W}^l_k = W^l_k + (n-2) g^{jl} P_{jk} + \frac{\tilde{W} - W}{2(n-1)} \delta^l_k + 2g^{jl} P_{[jk]}. \tag{3.8}$$

According to (2.7), the generalized Einstein tensor for $\tilde{W}_n(\tilde{g},\tilde{w})$ is

$$\tilde{G}^l_k = \tilde{W}^l_k - \frac{1}{2}\tilde{W}\delta^l_k - 2\tilde{g}^{lm}\tilde{\nabla}_{[m}\tilde{\omega}_{k]}. \tag{3.9}$$

On the other hand, by using the definition of the covariant derivative and the relation (3.2) and remembering that $P = \omega - \tilde{\omega}$, we obtain

$$\tilde{\nabla}_{[m}\tilde{\omega}_{k]} = \nabla_{[m}\omega_{k]} - \nabla_{[m}P_{k]}, \quad P_{[km]} = \nabla_{[m}P_{k]}. \tag{3.10}$$

By using (3.8) and (3.10), we can write (3.9) in the form

$$\tilde{G}^l_k = G^l_k + (n-2)\left[g^{jl}P_{jk} - \frac{\tilde{W}-W}{2(n-1)}\delta^l_k\right], \tag{3.11}$$

connecting the generalized Einstein tensors of $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$.

In the special case where $W_n(g,w)$ and $\tilde{W}_n(\tilde{g},\tilde{w})$ are Einstein–Weyl manifolds, (3.11) takes the form

$$\tilde{G}^l_k = G^l_k + (n-2)\left[g^{jl}P_{[jk]} - \frac{\tilde{W}-W}{2n}\delta^l_k\right]. \tag{3.12}$$

3.1 Conformal Invariance of Generalized Einstein Tensor

We first mention that any 2-dimensional Weyl manifold is an Einstein–Weyl manifold, as can be seen by direct calculation, and that the generalized Einstein tensor for such a manifold is identically zero. So, in what follows we will assume that $n > 2$.

Concerning the conformal invariance of the generalized Einstein tensor for $W_n(g,w)$, we prove

Theorem 3.1 *The generalized Einstein tensor for the Weyl manifold $W_n(g,w)$ $(n > 2)$ will be preserved by the conformal mapping $\tau : W_n(g,w) \to \tilde{W}_n(\tilde{g},\tilde{w})$ if and only if the curvature tensor of $W_n(g,w)$ is preserved.*

Proof According to (3.11), the necessary and sufficient condition for the generalized Einstein tensor G^l_k of $W_n(g,w)$ to be preserved by τ is

$$(n-2)\left(g^{jl}P_{jk} - \frac{\tilde{W}-W}{2(n-1)}\delta^l_k\right) = 0. \tag{3.13}$$

For $n > 2$, we have

$$g^{jl}P_{jk} - \frac{\tilde{W}-W}{2(n-1)}\delta^l_k = 0 \tag{3.14}$$

or, multiplying (3.14) by g_{lm} and summing for l, we get

$$P_{mk} - \frac{\tilde{W}-W}{2(n-1)}g_{km} = 0.$$

Separating P_{mk} into its symmetric and antisymmetric parts, we obtain

$$\left(P_{(mk)} - \frac{\tilde{W}-W}{2(n-1)}g_{km}\right) + P_{[km]} = 0,$$

from which it follows that

$$P_{[mk]} = \nabla_{[k}P_{m]} = 0 \quad (P = \text{grad}), \quad P_{(mk)} = P_{mk} = \frac{\tilde{W}-W}{2(n-1)}g_{km}. \tag{3.15}$$

Transvecting the second equation in (3.15) by g^{km} and using (3.6), we conclude that $W = \tilde{W}$ and consequently $P_{mk} = 0$. In this case, (3.3) yields $\tilde{W}^p_{jkl} = W^p_{jkl}$.

Conversely, suppose that the curvature tensor is preserved by τ. Then, clearly the Ricci tensors at corresponding points are equal. On the other hand, by (1.8) and (3.1), the scalar curvature is also preserved. Under these conditions, (3.7) reduces to

$$(n-2)P_{jk} + 2P_{[jk]} = (n-2)P_{(jk)} + nP_{[jk]} = 0,$$

from which it follows that

$$P_{(jk)} = P_{[jk]} \Rightarrow P_{jk} = 0.$$

We then have $\tilde{G^l_k} = G^l_k$. This completes the proof of the theorem. □

3.2 Conformal Mapping of Weyl Manifolds Preserving the Ricci Tensor

Let φ be a conformal mapping of $W_n(g,w)$ upon $\tilde{W}(\tilde{g},\tilde{w})$ and suppose that φ preserves the Ricci tensor of $W_n(g,w)$, i.e.,

$$Ric_g = Ric_{\tilde{g}} \quad (W_{jk} = \tilde{W}_{jk}). \tag{3.16}$$

We first prove the following theorem which will be used later on.

Theorem 3.2 *The only conformal mapping of a Weyl manifold* (*of* $\dim > 2$) *upon another Weyl manifold which preserves the Ricci tensor of the manifold, is the one that preserves the curvature tensor.*

Proof We first suppose that the conformal mapping $\varphi : W_n(g,w) \to \tilde{W}_n(\tilde{g},\tilde{w})$ preserves the Ricci curvature tensor. Then, by (3.16), (3.7) reduces to

$$(n-2)P_{jk} + \frac{\tilde{W} - W}{2(n-1)} g_{jk} + 2P_{[jk]} = 0. \tag{3.17}$$

Separating P_{jk} into its symmetric and antisymmetric parts and remembering that

$$W = g^{jk}W_{jk} = \tilde{g}^{jk}\tilde{W}_{jk} = \tilde{W},$$

we obtain

$$(n-2)P_{(jk)} + nP_{[jk]} = 0,$$

from which it follows (for $n > 2$) that $P_{(jk)} = P_{[jk]} = P_{jk} = 0$. Consequently, the equation (3.3) reduces to $\tilde{W}^p_{jkl} = W^p_{jkl}$.

Conversely, if the conformal mapping φ preserves the curvature tensor, it is clear from (3.1) and the definition of the Ricci tensor that φ preserves the Ricci tensor. □

Combining Theorems 3.1 and 3.2, we deduce the following corollary:

Corollary 3.3 *The only conformal mapping of* $W_n(g,w)$ $(n>2)$ *upon* $\tilde{W}(\tilde{g},\tilde{w})$ *preserving the generalized Einstein tensor is the one which preserves the Ricci tensor.*

It is clear from Theorems 3.1 and 3.2 that the vanishing of the gauge-invariant tensor P_{jk} is necessary and sufficient for generalized Einstein tensor to be a conformal invariant.

We now proceed to obtain the differential equation satisfied by $f \in C^2(W_n)$, where $P = \mathrm{grad} f$. Multiplying the equations

$$0 = \nabla_k P_j - P_j P_k + \frac{1}{2} g_{jk} g^{rs} P_r P_s \tag{3.18}$$

by g^{jk}, summing for j and k and using the relation $g_{ij}g^{ij} = n$, we obtain

$$g^{jk}\nabla_k P_j + \frac{n-2}{2}\,|P|^2 = 0, \quad |P|^2 = g^{rs}P_rP_s. \tag{3.19}$$

Remembering that the weight of P is zero, we have

$$\dot{\nabla}_k P_j = \nabla_k P_j \tag{3.20}$$

so that (3.19) can be written in the form

$$\dot{\nabla}_k(g^{jk}P_j) + \frac{n-2}{2}\,|P|^2 = 0, \tag{3.21}$$

in which we have used (1.15). If we put $g^{jk}P_j = P^k$, (3.21) becomes

$$\dot{\nabla}_k P^k + \frac{n-2}{2}\,|P|^2 = 0. \tag{3.22}$$

Clearly, the weight of P^k is $\{-2\}$. Since $\dot{\nabla}$ preserves the weights, the weight of $\dot{\nabla}_k P^k$ is also $\{-2\}$.

We define $\dot{\nabla}_k P^k$ to be *the generalized divergence* of P^k since it reduces to the divergence of P^k in the Riemannian case. Putting $P = \operatorname{grad} f$, the first term in (3.22) becomes the generalized Laplacian of f which will be denoted by $\dot{\Delta} f$. Therefore, f is the solution of the equation

$$\dot{\Delta} f + \frac{n-2}{2}\,|\nabla f|^2 = 0. \tag{3.23}$$

We note that the left-hand member of this equation differs by the factor λ^{-2} under a conformal change of g.

We can obtain an alternative form of the equation (3.23) by using the Levi–Civita connection D formed with respect to the representative metric g. Since the weight of P^k is $\{-2\}$, by using (1.14), we find that

$$\dot{\nabla}_k P^k = \nabla_k P^k + 2\omega_k P^k. \tag{3.24}$$

According to (1.10), ∇ and D are related by

$$\nabla_i P^j = D_i P^j + \gamma^j_{ik} P^k, \tag{3.25}$$

where

$$\gamma^j_{ik} = -(\delta^j_i \omega_k + \delta^j_k \omega_i - g_{ik} g^{jm} \omega_m).$$

Then, by (3.22), (3.24) and (3.25), we obtain

$$D_k P^k + \frac{n-2}{2}\left(|P|^2 - 2\omega_l P^l\right) = 0, \quad |P|^2 = |\nabla f|^2$$

or,

$$\Delta f + \frac{n-2}{2}\left(|\nabla f|^2 - 2g(\omega, \nabla f)\right) = 0, \quad n > 2, \tag{3.26}$$

in which $\Delta f (= D_k P^k)$ is the Laplacian of f with respect to the Levi–Civita connection generated by the representative metric g.

3.3 A Geometrical Implication of the Condition $P_{jk} = 0$

Let $W_n(g, \omega)$ be a Weyl manifold whose generalized Einstein tensor (or, equivalently, whose curvature tensor) is a conformal invariant. In this case, according to Theorems 3.1 and 3.2, it is necessary and sufficient that $P_{jk} = 0$.

Multiplying (3.18) by g^{jl} and summing for j and remembering that

$$g^{jl}\nabla_k P_j = g^{jl}\dot{\nabla}_k P_j = \dot{\nabla}_k(g^{jl}P_j) = \dot{\nabla}_k P^l,$$

we obtain the condition $P_{jk} = 0$ in the form

$$\dot{\nabla}_k P^l - P_k P^l + \frac{1}{2}\delta^l_k |P|^2 = 0, \quad P^l = g^{jl}P_j. \tag{3.27}$$

Multiplication of (3.27) by P^k and summation with respect to k yields

$$P^k \dot{\nabla}_k P^l = \frac{1}{2}|P|^2 P^l. \tag{3.28}$$

Then, by (1.14), (3.28) gives

$$P^k \nabla_k P^l = \psi P^l, \quad \psi = \frac{1}{2}|P|^2 - 2\omega_k P^k, \tag{3.29}$$

which shows that a curve in $W_n(g, \omega)$ whose tangential direction coincides with that of the vector field P^l, is a geodesic.

On the other hand, let us consider a hypersurface of $W_n(g, \omega)$ defined by

$$f(x^1, x^2, \ldots, x^n) = \text{const.},$$

where $\text{grad} f = P$. Clearly, P is orthogonal to this hypersurface. Therefore we have

Theorem 3.4 *The Weyl manifold $W_n(g, \omega)$ whose generalized Einstein tensor (or, equivalently, whose curvature tensor) is a conformal invariant, admits a 1-parameter family of hypersurfaces the orthogonal trajectories of which are geodesics.*

3.4 Characterization of Generalized Concircular Mappings of Weyl Manifolds by Means of Generalized Einstein Tensor

Consider the conformal mapping $\tau : W_n(g, w) \to \tilde{W}_n(\tilde{g}, \tilde{w})$ of $W_n(g, w)$ upon $\tilde{W}_n(\tilde{g}, \tilde{w})$. τ will be named as a generalized concircular mapping if it preserves the generalized circles of $W_n(g, w)$. In this section we will prove

Theorem 3.5 *The conformal mapping τ will be generalized concircular if and only if the condition*

$$\tilde{G}^l_k - G^l_k = \rho\, \delta^l_k \tag{3.30}$$

or, equivalently, the (gauge invariant) condition

$$\tilde{G}_{mk} - G_{mk} = \rho\, g_{mk}, \quad G_{mk} = g_{lm}G^l_k \tag{3.31}$$

is satisfied for $n > 2$.

Proof We first suppose that τ is generalized concircular. Then, according to (1.19), $P_{kl} = \phi g_{kl}$ and consequently

$$P_{[kl]} = 0, \quad P_{(kl)} = P_{kl} = \frac{\tilde{W} - W}{2n(n-1)} g_{kl}, \tag{3.32}$$

in which we have used (3.6). Substitution of P_{kl} into (3.11) yields

$$\tilde{G}^l_k - G^l_k = \rho\,\delta^l_k, \quad \rho = -\frac{(n-2)}{2n}(\tilde{W} - W). \tag{3.33}$$

Conversely, assume that the equation (3.30) is valid. In this case, (3.11) gives

$$\tilde{G}^l_k - G^l_k = \rho\,\delta^l_k = (n-2)\left(g^{jl}P_{jk} - \frac{\tilde{W} - W}{2(n-1)}\delta^l_k\right),$$

from which it follows that

$$g^{jl}P_{jk} = \mu\delta^l_k \tag{3.34}$$

for some function μ defined on $W_n(g, w)$.

Transvecting (3.34) by g_{lm}, we obtain $P_{mk} = \mu g_{mk}\big(\mu = \frac{\rho}{n-2} + \frac{\tilde{W}-W}{2(n-1)}\big)$ which states that τ is generalized concircular. This completes the proof of the theorem. □

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