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Computing trade-offs in robust design: Perspectives of the mean squared error ☆

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ABSTRACT

Researchers often identify robust design as one of the most effective engineering design methods for continuous quality improvement. When more than one quality characteristic is considered, an important question is how to trade off robust design solutions. In this paper, we consider a bi-objective robust design problem for which Pareto solutions of two quality characteristics need to be obtained. In practical robust design applications, a second-order polynomial model is adequate to accommodate the curvature of process mean and variance functions, thus mean-squared robust design models, frequently used by many researchers, would contain fourth-order terms. Consequently, the associated Pareto frontier might be non-convex and supported and non-supported efficient solutions needs to be generated. So, the objective of this paper is to develop a lexicographic weighted-Tchebycheff based bi-objective robust design model to generate the associated Pareto frontier. Our numerical example clearly shows the advantages of this model over frequently used weighted-sums model.

1. Introduction

In responses to the increasing pressure from global competitiveness, there is a growing investment of efforts to enhance productivity quality. It is also recognized that quality improvement activities are most efficient and cost-effective when implemented during the design stage. Based on this awareness, Taguchi (1986) introduced a systematic method for applying experimental design, which has become known as robust design. The primary goal of this method is to determine the best design factor settings by minimizing performance variability and product bias, i.e., the deviation from the target value of a product. Because of their practicability in reducing the inherent uncertainty associated with design factors and system performance, the widespread application of robust design techniques has resulted in significant improvements in product quality, manufacturability and reliability at low cost.

Even though the *ad hoc* robust design methods suggested by Taguchi remain controversial due to various mathematical flaws, there is little disagreement among researchers and practitioners about his basic philosophy. The controversy surrounding Taguchi's assumptions, experimental design, and statistical analysis has been well addressed by Leon, Shoemaker, and Kackar (1987), Box (1988), Box, Bisgaard, and Fung (1988), Nair (1992), and Tsui (1992). Consequently, researchers have closely examined alternatives using well-established statistical tools from traditional theories of experimental designs. In an early attempt of such research, Vining and Myers (1990) introduced the dual response approach based on response surface methodology (RSM) as a superior alternative for modeling process relationships by separately estimating the response functions of the process mean and variance; thus, it achieved the primary goal of robust design by minimizing the process variance while adjusting the process mean at the target. Del Castillo and Montgomery (1993) and Copeland and Nelson (1996) showed that the solution technique used by Vining and Myers (1990) does not always guarantee optimal robust design solutions, and proposed that the standard nonlinear programming techniques such as the generalized reduced gradient method and the Nelder–Mead simplex method may provide more effective alternatives. The response surface modeling based on estimating robust parameters and the dual response approach using fuzzy optimization methodology were further developed by Khattree (1996) and Kim and Lin (1998), respectively. However, Cho (1994) and Lin and Tu (1995) pointed out that the robust design solutions obtained from the dual-response model may not necessarily be optimal since this model forces the process mean to be located at the target value, so they proposed the mean-squared-error model, relaxing the zero-bias assumption. While allowing

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some process bias, the resulting process variance was less than or at most equal to the variance obtained from the Vining and Myers model (1990); hence, the mean-squared-error model may provide better (or at least equal) robust design solutions unless the zero-bias assumption must be met. Further modifications to the mean-squared-error model have been discussed by Jayaram and Ibrahim (1999), Cho, Kim, Kimber, and Phillips (2000), Kim and Cho (2000, 2002), Yue (2002), Miro-Quesada and Del Castillo (2004), and Shin and Cho (2005).

Most robust design models discussed above find the robust design solutions for a single quality characteristic. Recently, Tang and Xu (2002) and Koksoy and Doganaksoy (2003) provided the Pareto solutions for the two process attributes (i.e., process bias and variability) of a quality characteristic using an additive mean-squared-error model based on weights. This weighted-sums approach minimizes $w(\hat{\mu}(\boldsymbol{x})-\tau)^2+(1-w)\hat{\sigma}^2(\boldsymbol{x})$ where w, $\hat{\mu}(\boldsymbol{x})$, $\hat{\sigma}^2(\boldsymbol{x})$, and τ denote a weight, a response function of the process mean, a response function of the process variance, and a desired target value, respectively. In fact, this simple weighted-sums approach is often used by many researchers, and is considered one of the standard optimization techniques (Steuer, 1986). However, care must be exercised when this approach is applied to robust design problems. In most such problems, second-order models are often adequate for representing $\hat{\mu}(\boldsymbol{x})$ and $\hat{\sigma}^2(\boldsymbol{x})$, as evidenced by Vining and Myers (1990), Del Castillo and Montgomery (1993), Cho (1994), Lin and Tu (1995), Kim and Cho (2000, 2002), Tang and Xu (2002), Koksoy and Doganaksoy (2003), and Shin and Cho (2005, 2006). Further, Myers, Brenneman, and Myers (2005), Park and Cho (2005), and Robinson, Wulff, Montgomery, and Khuri (2006) developed a dual-response model using generalized linear model, a robust design model using the weighted-least-square method for unbalanced data, and a robust design model using a generalized linear mixed model for nonnormal quality characteristics, respectively. Govindaluri and Cho (2007) investigated the effect of correlations of quality characteristics on robust design solutions. Furthermore, Egorov, Kretinin, Leshchenko, and Kuptzov (2007) and Kovach, Cho, and Antony (2008) studied on optimal robust design solutions by using the indirect optimization algorithm and physical programming, respectively. More recently, Shin and Cho (2009) studied a bi-objective dual-response based robust design problem which minimizes $\hat{\sigma}^2(\boldsymbol{x})$, subject to $\hat{\mu}(\boldsymbol{x})=$ target.

In real-world industrial settings, though, there are many situations in which a decision maker often needs a balance between two quality characteristics. Reasonable control of one can apparently be achieved only at the expense of sacrificing the other, and vice versa. When two quality characteristics are considered simultaneously, a bi-objective robust design problem needs to be solved. A closer look at the mean-squared-error models for these two quality characteristics reveals that $(\hat{\mu}_1(\boldsymbol{x})-\tau_1)^2+\hat{\sigma}_1^2(x)$ and $(\hat{\mu}_2(\boldsymbol{x})-\tau_2)^2+\hat{\sigma}_2^2(x)$ then become fourth-order functions, which are often neither convex nor concave. When at least one of the objective functions for this bi-objective case has a higher order than the second, it is known that obtaining all efficient solutions with the weighted-sums approach is unlikely (Mattson & Messac, 2003; Messac, Sundararaj, Taapetta, & Renaud, 2000; Tind & Wiecek, 1999). In a minimization problem, with weighted-sums approach, supported efficient solutions which lie in the convex hull of the Pareto front can be found. However, it is impossible to obtain non-supported efficient solutions which are located on the non-convex portions of the Pareto optimal set in the criterion space. In that case, other methods such as lexicographic weighted Tchebycheff (LWT) method or augmented weighted Tchebycheff method needs to be used to find all efficient solutions.

In this paper, we formulate a bi-objective robust design problem in order to simultaneously consider two quality characteristics, and develop a methodology to obtain the Pareto frontier when objective functions have higher-order terms and objective space is neither convex nor concave. Here, LWT approach is preferred over the frequently used weighted-sums approach because regardless of the shape of the feasible region, all nondominated criterion vectors returned by the LWT method are nondominated and all nondominated criterion vectors are uniquely computable. Thus, this method can be used in linear, nonlinear, finite-discrete, infinite-discrete and polyhedral cases (Steuer, 1986). To our knowledge, the LWT approach to a robust design problem has not been addressed in the research community. The main purpose of this paper is twofold. First, we show how experimental results can be integrated into a robust design paradigm by proposing a LWT-based robust design model. Then, we show how this proposed model can effectively find the Pareto frontier with a numerical example where we compare the results with the ones obtained with the frequently used weighted-sums method.

Following this introduction, the response surface design is presented in Section 2, while the mean-squared-error model is discussed in Section 3. The proposed bi-objective robust design model and the methodology to obtain the Pareto frontier are presented in Section 4. Finally, comparison studies are conducted in Section 5, followed by the conclusion in Section 6.

2. Response surface design

Researchers have sought to combine Taguchi's robust design principles with conventional RSM to model the response directly as a function of design variables. RSM is a statistical tool that is useful for modeling and analysis in situations where the response of interest is affected by several input factors. In addition, it is typically used to optimize this response by estimating a functional form for an input response when the exact functional relationship is not known or is very complicated. Therefore, RSM is often used in model fitting and optimization. Using this method, the response functions of the process mean and variance for each quality characteristic are given by

$$\hat{\mu}_i(\boldsymbol{x})=\hat{\alpha}_0^i+\boldsymbol{x}^T\boldsymbol{a}_i+\boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} \tag{1}$$

where

$$\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_k\end{bmatrix},\quad \boldsymbol{a}_i=\begin{bmatrix}\hat{\alpha}_1^i\\\hat{\alpha}_2^i\\\vdots\\\hat{\alpha}_k^i\end{bmatrix},\quad\text{and}\quad \boldsymbol{A}_i=\begin{bmatrix}\hat{\alpha}_{11}^i & \hat{\alpha}_{12}^i/2 & \cdots & \hat{\alpha}_{1k}^i/2\\ \hat{\alpha}_{12}^i/2 & \hat{\alpha}_{22}^i & \cdots & \hat{\alpha}_{2k}^i/2\\ \vdots & \vdots & \ddots & \vdots\\ \hat{\alpha}_{1k}^i/2 & \hat{\alpha}_{2k}^i/2 & \cdots & \hat{\alpha}_{kk}^i\end{bmatrix},$$

and

$$\hat{\sigma}_i^2(\boldsymbol{x})=\hat{\beta}_0^i+\boldsymbol{x}^T\boldsymbol{b}_i+\boldsymbol{x}^T\boldsymbol{B}_i\boldsymbol{x} \tag{2}$$

where

$$\boldsymbol{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_k\end{bmatrix},\quad \boldsymbol{b}_i=\begin{bmatrix}\hat{\beta}_1^i\\\hat{\beta}_2^i\\\vdots\\\hat{\beta}_k^i\end{bmatrix},\quad\text{and}\quad \boldsymbol{B}_i=\begin{bmatrix}\hat{\beta}_{11}^i & \hat{\beta}_{12}^i/2 & \cdots & \hat{\beta}_{1k}^i/2\\ \hat{\beta}_{12}^i/2 & \hat{\beta}_{22}^i & \cdots & \hat{\beta}_{2k}^i/2\\ \vdots & \vdots & \ddots & \vdots\\ \hat{\beta}_{1k}^i/2 & \hat{\beta}_{2k}^i/2 & \cdots & \hat{\beta}_{kk}^i\end{bmatrix}.$$

For both equations, the term $\boldsymbol{x}$ is the vector of the design factors, $\boldsymbol{a}_i$ and $\boldsymbol{A}_i$ represent the vector and matrix forms of the estimated regression coefficients for the process mean for ith quality characteristic, and $\boldsymbol{b}_i$ and $\boldsymbol{B}_i$ represent the vector and matrix forms of the estimated regression coefficients for the process variance for ith quality characteristic. The ordinary method of least squares can be used to determine all the coefficient vectors ($\boldsymbol{a}_i$, $\boldsymbol{b}_i$) and matrices ($\boldsymbol{A}_i$, $\boldsymbol{B}_i$). The experimental format is shown in Table 1, where y_{ijk}

denotes the ith quality characteristic, the jth experimental run, and the kth replication. $\bar{y}_{1j}$, $\bar{y}_{2j}$, s_{1j}^2, and s_{2j}^2 represent the sample means of Y_1 and Y_2, and the sample variances of Y_1 and Y_2, respectively.

3. The mean-squared-error model

The dual-response model proposed by Vining and Myers (1990) minimizes $\hat{\sigma}(\boldsymbol{x})$ subject to $\hat{\mu}(\boldsymbol{x}) = \tau$ and $\boldsymbol{x} \in \Omega$, where Ω denotes a sample space. This dual-response model implies that the process mean is adjusted to the target first and then the variability is minimized. As Cho (1994), Lin and Tu (1995), and Shin and Cho (2009) demonstrated, this optimization scheme based on zero-bias logic can be misleading due to the unrealistic constraint of forcing the estimated mean to a specific value. Thus, they proposed the mean-squared-error model to minimize $(\hat{\mu}(\boldsymbol{x}) - \tau)^2 + \hat{\sigma}^2(\boldsymbol{x})$ subject to $\boldsymbol{x} \in \Omega$. To illustrate this method graphically, the two process distributions – process A and process B as shown in Fig. 1 – are considered. Denoting τ as the desired target value for both processes, Fig. 1 clearly shows the advantage of the mean-squared-error model since further variability reduction would be achieved by allowing a small magnitude of process bias.

4. The proposed bi-objective robust design paradigm

A bi-objective problem for a mean-squared-error robust design model is as follows:

$$\text{Minimize } [MSE_1(\boldsymbol{x}),\ MSE_2(\boldsymbol{x})]^T \quad (3)$$
$$\text{Subject to } \boldsymbol{x} \in X$$

where $MSE_1(\boldsymbol{x}) = (\hat{\mu}_1(\boldsymbol{x}) - \tau_1)^2 + \hat{\sigma}_1^2(\boldsymbol{x}), MSE_2(\boldsymbol{x}) = (\hat{\mu}_2(\boldsymbol{x}) - \tau_2)^2 + \hat{\sigma}_2^2(\boldsymbol{x})$. The set X of feasible solutions is closed and bounded, and the set of all criterion vectors for all feasible solutions is denoted by Ψ. A point $\boldsymbol{x}^* \in X$ is an efficient solution of the bi-objective problem if there does not exist another $\boldsymbol{x} \in X$ for which $MSE_1(\boldsymbol{x}) \leqslant MSE_1(\boldsymbol{x}^*)$ and $MSE_2(\boldsymbol{x}) \leqslant MSE_2(\boldsymbol{x}^*)$ with at least one strict inequality. The image of an efficient solution $\boldsymbol{x}^*$ in the objective space, is then a Pareto (non-dominated, noninferior) solution.

Table 1
Experimental format.

Run	$\boldsymbol{X}$	Replications (Y_1)	$\bar{y}_{1j}$	s_{1j}^2	Replications (Y_2)	$\bar{y}_{2j}$	s_{2j}^2
1		$y_{111} \ldots y_{11l}$	$\bar{y}_{11}$	s_{11}^2	$y_{211} \ldots y_{21l}$	$\bar{y}_{21}$	s_{21}^2
2		$y_{121} \ldots y_{12l}$	$\bar{y}_{12}$	s_{12}^2	$y_{221} \ldots y_{22l}$	$\bar{y}_{22}$	s_{22}^2
...		...	...	...	...	...	...
j		$y_{1j1} \ldots y_{1jl}$	$\bar{y}_{1j}$	s_{1j}^2	$y_{2j1} \ldots y_{2jl}$	$\bar{y}_{2j}$	s_{2j}^2
...		...	...	...	...	...	...
η		$y_{1\eta 1} \ldots y_{1\eta l}$	$\bar{y}_{1\eta}$	$s_{1\eta}^2$	$y_{2\eta 1} \ldots y_{2\eta l}$	$\bar{y}_{2\eta}$	$s_{2\eta}^2$

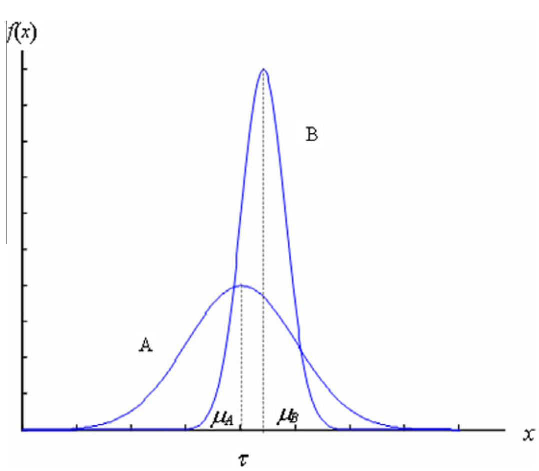

Fig. 1. Process distribution functions.

The bi-objective problem in Eq. (3) is convex if the feasible set X is convex, and the objective functions $MSE_1(\boldsymbol{x})$ and $MSE_2(\boldsymbol{x})$ are also convex. It is a well-known fact that the set Ψ on R^2 of the convex bi-objective problem is convex and that the Pareto set can be viewed as a convex curve in R^2. When the feasible set X is not convex and/or at least one objective function is not convex, the bi-objective problem becomes a non-convex problem. It is possible that for some non-convex bi-objective problems, the set Ψ on R^2 remains convex in R^2, a situation which is difficult to check analytically (see Shin & Cho, 2009). In general, for non-convex bi-objective problems, the Pareto curve may be non-convex and even not connected. Since determining the general shape of the Pareto curve is crucial for the approximation of this set, the knowledge of the convexity is critical.

4.1. The criticism of the robust design model using the weighted-sums method

In robust design literature, the weighted-sums approach is often utilized to generate efficient solutions of the bi-objective problem. Its associated applications are conducted by Lin and Tu (1995), Cho et al. (2000), Tang and Xu (2002), Memtsas (2003), Koksoy and Doganaksoy (2003), and Liu, Tang, and Song (2006). A weighted-sums approach based on the mean-squared-error concept can be formulated as follows:

$$\text{Minimize } w\ MSE_1(\boldsymbol{x}) + (1 - w)\ MSE_2(\boldsymbol{x}) \quad (4)$$
$$\text{Subject to } \boldsymbol{x} \in X$$

where $0 \leqslant w \leqslant 1$. The solutions of the weighted-sums approach as shown in problem (4) is Pareto optimal if the weighting coefficient is positive. While this weighted-sums method can generate all the efficient solutions of convex bi-objective problems, it cannot, in general, find all efficient points of non-convex problems (Tind & Wiecek, 1999). With this method, only supported efficient solutions which lie in the convex hull of the Pareto front can be found, however non-supported efficient solutions which lie in the non-convex portions of the Pareto front cannot be found. Further, drawbacks of the weighted-sums method are reported by Das and Dennis (1997), Mattson and Messac (2003), and Shin and Cho (2009). Solving the robust design optimization defined in Eq. (4), $MSE_1(\boldsymbol{x})$ and $MSE_2(\boldsymbol{x})$ would be of a fourth-order function when $\hat{\mu}_1(\boldsymbol{x})$ and $\hat{\mu}_2(\boldsymbol{x})$ are quadratic; hence, $(\hat{\mu}_1(\boldsymbol{x}) - \tau_1)^2 + \hat{\sigma}_1^2(\boldsymbol{x})$ and $(\hat{\mu}_2(\boldsymbol{x}) - \tau_2)^2 + \hat{\sigma}_2^2(\boldsymbol{x})$ would be neither convex nor concave. Even in the simplest practical cases, we suggest a lexicographic weighted Tchebycheff formulation for this type of bi-objective robust design optimization problem when non-convex Pareto frontiers are potentially present.

4.2. The proposed LWT-based robust design optimization

The lexicographic weighted Tchebycheff (LWT) formulation of this problem is given as:

$$\begin{aligned} lex \quad & \min\ \{\alpha, e^T(u - u^*)\} \\ s.t \quad & \alpha \geqslant \lambda(MSE_1(x) - u_1^*) \\ & \alpha \geqslant (1 - \lambda)(MSE_2(x) - u_2^*) \\ & x \in X \end{aligned} \quad (5)$$

where $\lambda > 0$ is the weight, u_i^* ($i = 1, 2$) is the utopia point defined as $u_i^* = \min_{\boldsymbol{x} \in X} MSE_i(\boldsymbol{x}) - \delta_i$ for $i = 1, 2$ ($\delta_i > 0$) and e^T is the sum vector of ones. Regardless of the shape of the feasible region, all criterion vectors obtained by the LWT program are nondominated and all are uniquely computable. Here, a two-stage minimization process is used – the first stage is a weighted Tchebycheff program and the second stage is a L_1 metric. If the first stage does not yield a unique criterion vector (in case of alternative optima), then second stage is used to break ties (Steuer, 1986).

Note that, the weight (w) in the weighted-sums method and the weight (λ) in the LWT method have different meanings. In general, if the same weights are used for these two methods, the results are generally two distinct Pareto points in the objective space. There exists, however, a particular selection of weights so that under some conditions both methods yield the same solutions (Tind & Wiecek, 1999).

4.3. The methodology to obtain the Pareto frontier

In robust design applications, factorial and central composite designs are often used. In this case, the feasible set X is always convex. For example, the constraints $g_m(\boldsymbol{x}) \leqslant 0$ may assume the form $-1 \leqslant x_i \leqslant 1$, $\forall i$ for factorial design or $\sum_{i=1}^{k} x_i \leqslant k$, where k is the number of factors, for central composite design. So, in fact, convexity of the bi-objective robust design problem is closely related to the convexity of the objective functions. The proposed methodology obtains the Pareto frontier for the robust design problem regardless of the shape of the objective space. The optimization procedure is given by the following steps.

Step 1: Estimate the response functions of the process parameters.
Step 2: Convert objective functions to minimization formulations. Check the objective functions for convexity within the feasible set with contour and surface plots. If both objective functions are convex go to step 3. Otherwise, go to step 4.
Step 3: Use weighted-sums method to obtain the Pareto frontier.
Step 4: Use LWT method to obtain the Pareto frontier.

5. Numerical example

An experiment has been done to examine the effects of three design variables – cutting speed (*sfpm*), cutting depth (*in*), and cutting feed (*ipr*) – on the metal thickness (*mm*), denoted by Y_1, and the metal removal rate (mm^3/min), denoted by Y_2, of a metal cutting machine. The design variables have been coded as x_1 = (cutting speed-25.5/30), x_2 = (cutting feed-55/9), and x_3 = (depth of cut-1.1/0.6). The experimental design shown in Table 2 is a central composite design consisting of eight factorial points, six axial points, and six center points, with three replicates.

From the viewpoint of the customer, the target values τ_1 and τ_2 of Y_1 and Y_2 are 96.5 and 57.5, respectively. The estimated response functions of $\hat{\mu}_1(\boldsymbol{x})$, $\hat{\mu}_2(\boldsymbol{x})$, $\hat{\sigma}_1^2(\boldsymbol{x})$, and $\hat{\sigma}_2^2(\boldsymbol{x})$ are $\hat{\mu}_1(\boldsymbol{x}) = 81.09 + \boldsymbol{X}^T\boldsymbol{a}_1 + \boldsymbol{X}^T\boldsymbol{A}_1\boldsymbol{X}$ and $\hat{\mu}_2(\boldsymbol{x}) = 59.85 + \boldsymbol{X}^T\boldsymbol{a}_2 + \boldsymbol{X}^T\boldsymbol{A}_2\boldsymbol{X}$ where

$$\boldsymbol{a}_1 = \begin{bmatrix} 1.03 \\ 4.04 \\ 6.20 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_1 = \begin{bmatrix} -1.83 & 2.13 & 11.38 \\ 2.13 & 2.94 & -3.88 \\ 11.38 & -3.88 & 5.19 \end{bmatrix}$$

$$\boldsymbol{a}_2 = \begin{bmatrix} 3.58 \\ 0.25 \\ 2.23 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{bmatrix} 0.83 & 0.39 & -0.04 \\ 0.39 & 0.07 & 0.31 \\ -0.04 & 0.31 & 0.06 \end{bmatrix}$$

and $\hat{\sigma}_1^2(\boldsymbol{x}) = 7.03 + \boldsymbol{X}^T\boldsymbol{b}_1 + \boldsymbol{X}^T\boldsymbol{B}_1\boldsymbol{X}$, $\hat{\sigma}_2^2(\boldsymbol{x}) = 15.11 + \boldsymbol{X}^T\boldsymbol{b}_2 + \boldsymbol{X}^T\boldsymbol{B}_2\boldsymbol{X}$ where

Table 2
Experimental results for the metal cutting experiment.

Run	x_1	x_2	x_3	$\bar{y}_{1j}$	s_{1j}^2	$\bar{y}_{2j}$	s_{2j}^2
1	−1	−1	−1	74	6.8	53.2	14.6
2	1	−1	−1	51	7.4	62.9	12.3
3	−1	1	−1	88	8.4	53.4	13.5
4	1	1	−1	70	12.5	62.6	10.5
5	−1	−1	1	71	6.2	57.3	9.6
6	1	−1	1	90	7.0	67.9	18.6
7	−1	1	1	66	8.0	59.8	20.0
8	1	1	1	97	3.2	67.8	10.3
9	−1.682	0	0	76	2.4	59.1	22.4
10	1.682	0	0	79	16.0	65.9	19.9
11	0	−1.682	0	85	9.0	60.0	12.6
12	0	1.682	0	97	4.4	60.7	9.6
13	0	0	−1.682	55	20.4	57.4	18.4
14	0	0	1.682	81	9.9	63.2	25.4
15	0	0	0	81	6.5	59.2	15.0
16	0	0	0	75	5.9	60.4	14.0
17	0	0	0	76	7.4	59.1	15.6
18	0	0	0	83	6.8	60.6	13.8
19	0	0	0	80	7.8	60.8	16.0
20	0	0	0	91	7.2	58.9	15.4

$$\boldsymbol{b}_1 = \begin{bmatrix} 1.73 \\ -0.22 \\ -2.08 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_1 = \begin{bmatrix} 0.14 & -0.26 & -1.09 \\ -0.26 & -0.74 & -1.09 \\ -1.09 & -1.09 & 2.25 \end{bmatrix}.$$

$$\boldsymbol{b}_2 = \begin{bmatrix} -0.75 \\ -0.43 \\ 1.42 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_2 = \begin{bmatrix} 1.22 & -2.43 & 0.58 \\ -2.43 & -2.33 & 0.63 \\ 0.58 & 0.63 & 1.49 \end{bmatrix}$$

The optimization model for this bi-objective robust design problem is now formulated as minimizing $[MSE_1(\boldsymbol{x}), MSE_2(\boldsymbol{x})]^T$ subject to $\boldsymbol{x} \in X$ where $MSE_1(\boldsymbol{x}) = (\hat{\mu}_1(\boldsymbol{x}) - \tau_1)^2 + \hat{\sigma}_1^2(\boldsymbol{x})$, $MSE_2(\boldsymbol{x}) = (\hat{\mu}_2(\boldsymbol{x}) - \tau_2)^2 + \hat{\sigma}_2^2(\boldsymbol{x})$. The set $X = \{\boldsymbol{x} \in R^3 : g(\boldsymbol{x}) \leqslant 0\}$ where $g(\boldsymbol{x}) = \sum_{\delta=1}^{k} x_\delta^2 - k \leqslant 0$ with k denoting number of design factors.

In order to scale (normalize) objective functions, we multiplied each objective function with $\pi_i = \frac{1}{R_i}$ where R_i is the range width of the ith criterion value over the efficient set and estimated by the difference between the approximated nadir objective vector and the ideal objective vector. For this problem, R_i values are calculated as $R_1 = 24.404 - 3.1684 = 21.2356$, $R_2 = 23.0785 - 8.5858 = 14.4927$, so corresponding π_i values are $(\pi_1, \pi_2) = (0.047091, 0.069)$. Note that, these normalized objective functions are only used in calculations (Eqs. (4) and (5)), but restored objective function values in the original scales are presented in the following tables and figures in order to prevent confusion.

Based on the methodology given in Section 4.3, the convexity of the objective functions within the feasible set is checked by using contour and surface plots. As shown in Figs. 2 and 3, which are the contour and surface plots for the two objective functions holding a control factor one by one, both of the objective functions $MSE_1(\boldsymbol{x})$ and $MSE_2(\boldsymbol{x})$ are non-convex. This result is consistent with the previous discussions in Section 1 about the non-convexity of functions that are including more than two factors.

As a result, based on the proposed methodology, to obtain the Pareto frontier, LWT method is preferred over the weighted-sums method. To illustrate the Pareto frontiers obtained by both methods and for comparison; we have generated around one thousand points by increasing weights, λ and w (in (4) and (5)) gradually from zero to one by 0.001. These Pareto optimal solutions are plotted in the objective space with the x-axis being $MSE_1(\boldsymbol{x})$ and the y-axis being $MSE_2(\boldsymbol{x})$ as shown in Figs. 4 and 5. MATLAB software package (MATLAB, 2008) is used to generate all the solutions.

As shown in the figures, the robust design solutions obtained from the weighted-sums model and the LWT model are distinctively

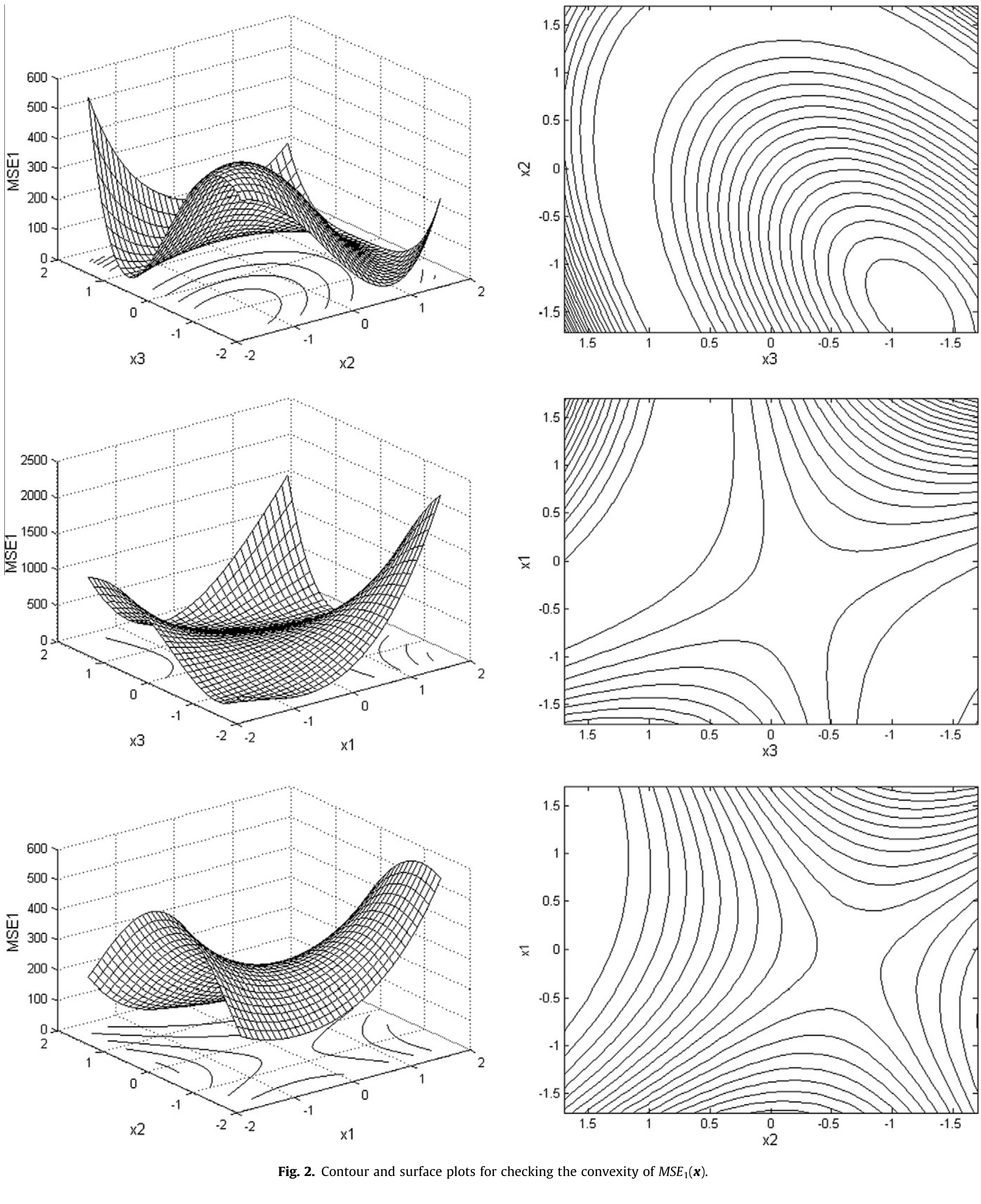

Fig. 2. Contour and surface plots for checking the convexity of $MSE_1(\boldsymbol{x})$.

different for various weights. Reflected in Table 3 and Fig. 4, the efficient solutions from the weighted-sums model are grouped into two distinct clusters. Again, this method fails to identify non-supported efficient solutions between the two clusters because the non-convexity occurs approximately between $3.9525 \leqslant MSE_1(\boldsymbol{x}) \leqslant 11.0917$ and $8.7695 \leqslant MSE_2(\boldsymbol{x}) \leqslant 15.825$ due to the nature of the fourth-order model generated by $MSE_1(\boldsymbol{x})$ and $MSE_2(\boldsymbol{x})$. Table 4 and Fig. 5 show that the proposed LWT based model generates the Pareto frontier consisting of supported and non-supported efficient solutions. The results obtained for this particular numerical example clearly demonstrates the advantage of the proposed model over the frequently used weighted-sums

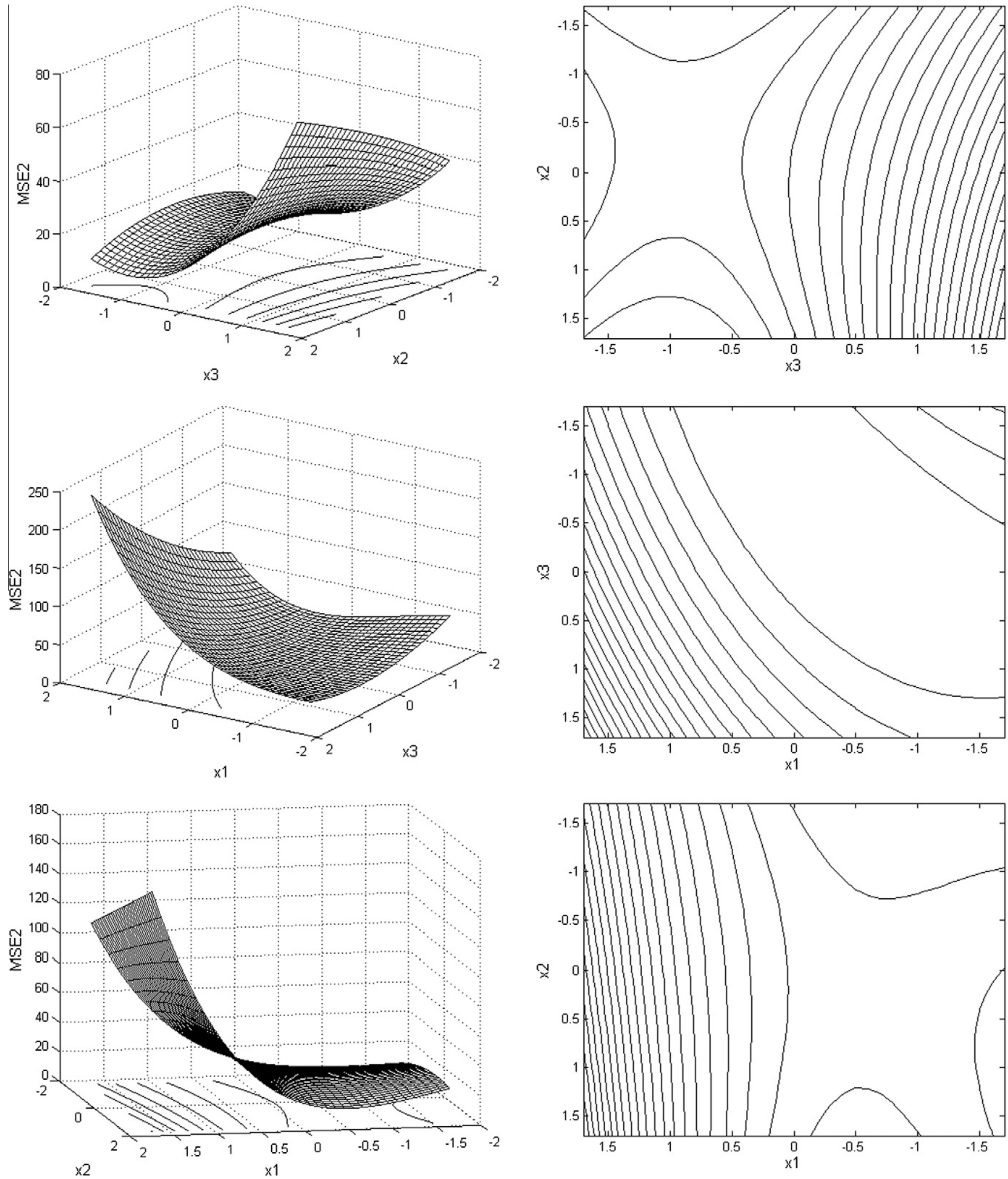

Fig. 3. Contour and surface plots for checking the convexity of $MSE_2(\boldsymbol{x})$.

approach in finding the Pareto frontier for robust design when higher-order functions are considered.

6. Conclusion and further study

In this paper, we have developed a lexicographic weighted Tchebycheff based bi-objective robust design model and a methodology to generate the Pareto frontier. Compared to the existing models for robust design, such as the dual response and additive weighted-sum models, the proposed approach has a significant advantage when determining the efficient solutions of a non-convex Pareto frontier. Models based on weighted-sums method cannot find non-supported efficient solutions which lie in the non-convex portions of the Pareto front. The proposed model, on

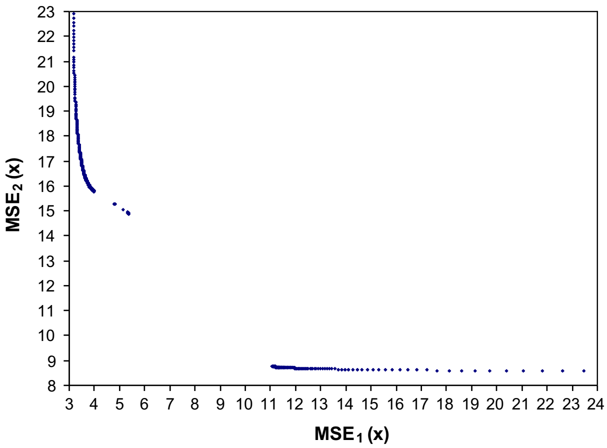

Fig. 4. Pareto frontier with the weighted-sums method.

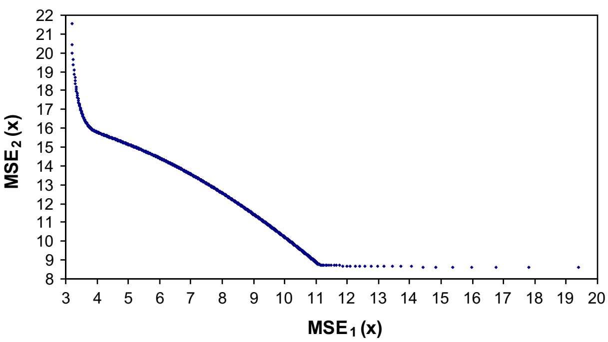

Fig. 5. Pareto frontier with the lexicographic weighted Tchebycheff method.

Table 3
Pareto optimal solutions with the weighted-sums model.

w_1	w_2	x_1	x_2	x_3	$MSE_1(\boldsymbol{x})$	$MSE_2(\boldsymbol{x})$
0.00	1.00	0.0237	1.5285	−0.8144	24.4053	8.5858
0.05	0.95	0.1008	1.4612	−0.9244	12.087	8.6912
0.10	0.90	0.1133	1.4496	−0.9411	11.3765	8.7266
0.15	0.85	0.1187	1.4452	−0.9473	11.2166	8.7415
0.20	0.80	0.122	1.4429	−0.9504	11.1583	8.7498
0.25	0.75	0.1246	1.4416	−0.952	11.1312	8.7551
0.30	0.70	0.1268	1.4407	−0.953	11.1163	8.7589
0.35	0.65	0.129	1.4402	−0.9535	11.1072	8.7619
0.40	0.60	0.1312	1.4399	−0.9536	11.1009	8.7644
0.45	0.55	0.1337	1.4398	−0.9534	11.096	8.7669
0.50	0.50	0.1366	1.4399	−0.953	11.0917	8.7695
0.55	0.45	−0.9669	1.4362	0.0497	3.9525	15.825
0.60	0.40	−0.9607	1.4398	0.0627	3.8952	15.8779
0.65	0.35	−0.9522	1.4448	0.0779	3.8417	15.9389
0.70	0.30	−0.9406	1.4512	0.0961	3.7872	16.0164
0.75	0.25	−0.9249	1.4596	0.1189	3.7272	16.1252
0.80	0.20	−0.903	1.4706	0.1488	3.6564	16.2933
0.85	0.15	−0.871	1.485	0.19	3.568	16.5819
0.90	0.10	−0.8208	1.5043	0.2514	3.4526	17.1483
0.95	0.05	−0.7323	1.5292	0.354	3.3009	18.4995
1.00	0.00	−0.5376	1.5465	0.5651	3.1684	23.0787

Table 4
Pareto optimal solutions with the lexicographic weighted Tchebycheff method.

λ_1	λ_2	x_1	x_2	x_3	$MSE_1(\boldsymbol{x})$	$MSE_2(\boldsymbol{x})$
0.00	1.00	0.0237	1.5285	−0.8144	24.404	8.5858
0.05	0.95	0.1453	1.4336	−0.9419	11.0166	8.8669
0.10	0.90	0.1205	1.4049	−0.9171	10.799	9.1637
0.15	0.85	0.0933	1.3744	−0.8904	10.5664	9.476
0.20	0.80	0.0644	1.3421	−0.8609	10.3173	9.8048
0.25	0.75	0.0322	1.3078	−0.8288	10.0504	10.1507
0.30	0.70	−0.0027	1.2715	−0.7935	9.7637	10.5142
0.35	0.65	−0.0423	1.2333	−0.7547	9.4556	10.8956
0.40	0.60	−0.0844	1.1936	−0.7114	9.124	11.2949
0.45	0.55	−0.1321	1.1528	−0.6634	8.7671	11.7115
0.50	0.50	−0.1855	1.112	−0.6098	8.3827	12.1439
0.55	0.45	−0.2456	1.0733	−0.5497	7.9692	12.5899
0.60	0.40	−0.3137	1.0401	−0.4825	7.5253	13.0456
0.65	0.35	−0.3913	1.0179	−0.4074	7.0505	13.5059
0.70	0.30	−0.4798	1.0158	−0.3248	6.5452	13.9631
0.75	0.25	−0.5803	1.0455	−0.2365	6.0118	14.4076
0.80	0.20	−0.6913	1.1172	−0.1475	5.455	14.8283
0.85	0.15	−0.8076	1.2295	−0.0647	4.8834	15.2191
0.90	0.10	−0.9213	1.3659	0.0062	4.3082	15.5887
0.95	0.05	−0.9303	1.4568	0.1113	3.7465	16.0871
1.00	0.00	−0.5376	1.5465	0.5651	3.1684	23.079

the other hand, is able to find supported and non-supported efficient solutions since it is based on LWT program. When lexicographic weighted Tchebycheff program is used, regardless of the feasible region, all criterion vectors computed are nondominated and all of these vectors are uniquely computable. Lexicographic program can be used for linear, nonlinear, infinite-discrete, finite-discrete, and polyhedral cases. The only disadvantage of lexicographic approach is that, two stages of optimization are required when the first stage, a weighted Tchebycheff program, results in alternative optima. In that case, a L_1 metric is used to break ties, however, this does not happen very often to be important. A numerical example was presented to illustrate the application of the model and to compare with the commonly used weighted-sums method. This example clearly demonstrates the proposed methodology's advantage over the traditional weighted-sums approach in finding the Pareto frontier when the model has higher-order terms or is neither convex nor concave. For further research, interactive lexicographic weighted Tchebycheff method can be applied to this problem to focus on a preferred part of the Pareto frontier. Furthermore, this approach can be expanded to multi-objective design optimization problems in order to consider more than two quality characteristics.

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