SCALING LAWS FOR DISTRIBUTED ESTIMATION OVER ORTHOGONAL FADING CHANNELS

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ABSTRACT

We analyze the outage for distributed estimation over orthogonal fading channels as a function of the number of sensors, K. We consider a scenario of fixed power per-sensor with an asymptotically large number of sensors. We characterize the scaling law of the outage and show that the outage decays faster than exponentially in the number of sensors and slower than $\exp(-K \log K)$.

1. INTRODUCTION

In recent years, research on distributed estimation has been evolving very rapidly [1]. Universal decentralized estimators of a source observed in additive noise have been considered in [2, 3]. The observations of the sensors can be delivered to the fusion center (FC) by analog or digital transmission methods. Amplify-and-forward is one analog option, whereas in digital transmission, observations are quantized, encoded and transmitted via digital modulation. The optimality of amplify and forward is described in [4, 5, 6, 7]. In [7], an amplify-and-forward approach is employed over an orthogonal multiple access fading channel, where the concept of estimation diversity is introduced, and shown to be given by the number of sensors. This seminal result is obtained under the assumption of asymptotically large number of sensors, and large total transmission powers. In [8], a definition of diversity order which assumes a finite number of sensors, but asymptotically large power is used. It is found that the diversity order need not be equal to the number of sensors, and depends on both the sensing signal-to-noise-ratios (SNRs), and the threshold used to define the outage. This means that it is possible to add new sensors into the system without any diversity benefit, and is unlike the result in [7] which considers asymptotically large sensors.

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In this work, we consider a similar orthogonal fading channel model but focus on a scenario where both the number of sensors, and the total transmit power are increased with their ratio (the power per sensor) remaining fixed. In this regime, we show that the outage decays faster than exponentially in the number of sensors and slower than $\exp(-K \log K)$.

2. SYSTEM MODEL

Consider a distributed estimation problem in a WSN with orthogonal channels as shown in Fig. 1. We assume that there are K sensors and focus on a single time snapshot. The sensor measurements $\{x_k\}_{k=1}^K$ are related to the source parameter θ with zero-mean and variance σ_θ^2 by

$$x_k = h_k \theta + n_k \ , \ \ k = 1, \ldots, K \ , \tag{1}$$

where $n_k \sim \mathcal{CN}(0, \sigma_{n_k}^2)$ is the sensing noise, and h_k is a parameter that controls the k^{th} sensing SNR given by $\gamma_k := |h_k|^2/\sigma_{n_k}^2$. The sensing SNRs $\{\gamma_k\}_{k=1}^K$ can be modelled as deterministic or random variables depending on the application. We will focus on deterministic $\gamma_k > 0$ in this work. The

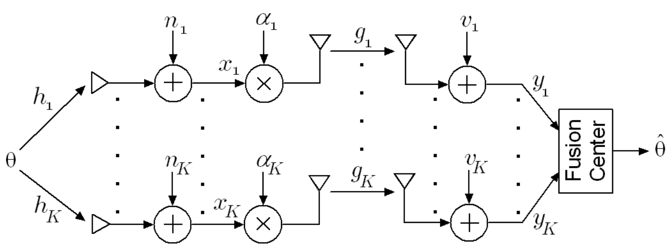

Fig. 1. Wireless Sensor Network with Orthogonal Channels

sensors amplify and forward their measurements which are separately received by the FC over orthogonal channels:

$$y_k = \alpha_k g_k (h_k \theta + n_k) + v_k \ , \quad k = 1, \ldots, K \tag{2}$$

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where $g_k \sim \mathcal{CN}(0, \sigma_{g_k}^2)$ is the k^{th} channel coefficient, $v_k \sim \mathcal{CN}(0, \sigma_{v_k}^2)$ is the receiver noise, and α_k is the amplification coefficient which controls the power of the k^{th} sensor. We assume that n_k, v_k, and g_k are statistically independent of each other and across sensors. We consider equal power transmission in the sequel. Since each sensor's average power is given by

$$P_0 := E_{\theta, n_k}[|\alpha_k x_k|^2] = |\alpha_k|^2 \left(|h_k|^2 \sigma_\theta^2 + \sigma_{n_k}^2\right) \ , \qquad (3)$$

in order to ensure that the total power P_{tot} is equally distributed among all sensors, the per-sensor power is $P_0 := P_{tot}/K$, which implies

$$|\alpha_k|^2 = \frac{P_{tot}}{K(|h_k|^2 \sigma_\theta^2 + \sigma_{n_k}^2)} \ . \qquad (4)$$

We assume that the FC knows $\alpha_k, g_k, h_k, \forall k$, and the noise variances $\sigma_{n_k}^2, \sigma_{v_k}^2$, $\forall k$, and thus can employ maximal ratio combining before doing estimation of the source parameter θ. Combining the separately received signals y_k in (2) to get the maximum possible SNR at the output of the FC amounts to multiplying with the conjugate of the coefficient of θ when the noise variances are equal [9]. Since the k^{th} noise term in (2) is given by $w_k := \alpha_k g_k n_k + v_k$ with variance $\sigma_{w_k}^2 := \text{var}(w_k|g_k) = |\alpha_k|^2 |g_k|^2 \sigma_{n_k}^2 + \sigma_{v_k}^2$, we can normalize (2) with σ_{w_k} so that the k^{th} noise term has unit variance:

$$\frac{y_k}{\sigma_{w_k}} = \frac{\alpha_k g_k h_k}{\sigma_{w_k}} \theta + \frac{w_k}{\sigma_{w_k}} \ . \qquad (5)$$

The maximal ratio combining coefficients are given by $\frac{\alpha_k^* g_k^* h_k^*}{\sigma_{w_k}}$ [9]. We denote the resulting SNR at the output of the FC with $\texttt{snr}$ given by

$$\texttt{snr} = \sum_{k=1}^{K} \frac{|\alpha_k|^2 |g_k|^2 |h_k|^2}{|\alpha_k|^2 |g_k|^2 \sigma_{n_k}^2 + \sigma_{v_k}^2} \ . \qquad (6)$$

Recalling that $\gamma_k := |h_k|^2/\sigma_{n_k}^2$, defining $\eta_k := |g_k|^2/\sigma_{v_k}^2$, and substituting for α_k in (4) into (6), we obtain

$$\texttt{snr} = \sum_{k=1}^{K} \frac{\eta_k \gamma_k}{\eta_k + \frac{K(\gamma_k \sigma_\theta^2 + 1)}{P_{tot}}} \ . \qquad (7)$$

The $\texttt{snr}$ in (7) is random because the instantaneous SNR on the k^{th} channel, η_k, is random. In what follows, in order to simplify the analysis, the random variable η_k will be assumed independent, and identically exponential distributed with mean ζ, which amounts to assuming that the channel coefficients and noise on each channel are i.i.d.

3. OUTAGE AND DIVERSITY

The outage probability is defined as the probability that $\texttt{snr}$ is below a threshold z

$$P_{out} := Pr\left[\sum_{k=1}^{K} \frac{\eta_k \gamma_k}{\eta_k + \frac{K(\gamma_k \sigma_\theta^2 + 1)}{P_{tot}}} \leq z\right] \qquad (8)$$

where the randomness of $\texttt{snr}$ stems from the instantaneous channel SNRs η_k. In (8), noting that $0 \leq \texttt{snr} < \sum_{k=1}^K \gamma_k$, we are interested in a threshold range of $0 < z < \sum_{k=1}^K \gamma_k$, because when $z \leq 0$, $P_{out} = 0$, and when $z \geq \sum_{k=1}^K \gamma_k$, $P_{out} = 1$. Examining (8), it is observed also that if $z \in (0, \sum_{k=1}^K \gamma_k)$, then $P_{out} \to 0$ as $P_{tot} \to \infty$.

The following table

k^{th} sensing SNR	$\gamma_k := \|h_k\|^2/\sigma_{n_k}^2$
Instantaneous k^{th} channel SNR	$\eta_k := \|g_k\|^2/\sigma_{v_k}^2$
Average k^{th} channel SNR	$\zeta_k := \sigma_{g_k}^2/\sigma_{v_k}^2$
Instantaneous SNR at the output of FC	$\texttt{snr}$
Outage probability	P_{out}

summarizes some parameters that recur throughout the paper for convenience.

4. OUTAGE FOR FIXED POWER PER-SENSOR

An scenario of practical importance is when the transmission power of each sensor is fixed to a certain value. In this context, an important question is how the outage performance scales with the number of sensors. In our analysis, we allow both $P_{tot} \to \infty$ and $K \to \infty$ with a fixed per-sensor power of $P_{tot}/K = P_0$. This is a natural scenario where each sensor that is deployed has a fixed individual power, and an increasingly larger number of sensors is introduced. For simplicity, we assume $\gamma_k = \gamma, \forall k$ and $\zeta_k = E[\eta_k] = \zeta, \forall k$, so that $f_{\eta_k}(x) = \zeta^{-1}\exp(-x/\zeta)$. Since P_{tot}/K is fixed, the outage in (8) is given by

$$P_{out} = Pr\left[\sum_{k=1}^{K} \frac{\eta_k \gamma}{\eta_k + c} \leq z\right] \ , \qquad (9)$$

where $c := (\gamma\sigma_\theta^2 + 1)/P_0$. Since the number of sensors is allowed to increase with the total power, the sum in the outage expression in (9) grows without bound, and therefore the outage probability converges to zero for any finite z. This hints that the behavior of the outage is markedly different from in [8]. In fact, we soon show that the outage behaves approximately like $\exp(-K\log K)$ in this regime. We have the following theorem.

Theorem 1 *For any $\gamma > 0$ and $z > 0$, if $P_{tot} = KP_0$, then*

$$\lim_{K\to\infty} -\frac{\log P_{out}}{K} = \infty \qquad (10)$$

Proof : We start with deriving a general expression that is useful for both Theorem 1 and part (2) of Theorem 2. Taking the logarithm of the Chernoff bound of P_{out} in (9) we obtain:

$$\log P_{out} \leq \nu z + K\log\Big[\int_0^\infty f_{\eta_k}(x)\exp\Big(-\frac{\nu\gamma x}{x+c}\Big)dx\Big], \quad (11)$$

where $c = (\gamma\sigma_\theta^2 + 1)/P_0$, and ν is any positive function of K which satisfies $\nu \to \infty$ as $K \to \infty$. Dividing through with

ν, and splitting the integral into two pieces for an arbitrary $g := g(K) > 0$ we obtain

$$\frac{\log P_{out}}{\nu} \le z + \frac{K}{\nu}\log\Big[\int_0^g f_{\eta_k}(x)\exp\Big(-\frac{\nu\gamma x}{x+c}\Big)dx + \int_g^\infty f_{\eta_k}(x)\exp\Big(-\frac{\nu\gamma x}{x+c}\Big)dx\Big]. \quad (12)$$

Equation (12) can be further upper bounded if the lower limits of the integrals are substituted for x in the argument of the exponentials:

$$\frac{\log P_{out}}{\nu} \le z+\frac{K}{\nu}\log\Big[\int_0^g f_{\eta_k}(x)dx+\exp\Big(-\frac{\nu g\gamma}{g+c}\Big)\Big], \quad (13)$$

where we also used $\int_g^\infty f_{\eta_k}(x)dx \le 1$. Recalling that both ν and g can be chosen as arbitrary positive functions of K, we focus on a choice that ensures that $g \to 0$, $\nu g \to \infty$, and $\nu g^2 \to 0$, as $K \to \infty$. Rewriting the exponential in (13) we get $\exp(-\frac{\nu g\gamma}{g+c}) = \exp\big(-\frac{\nu g\gamma}{c}\big)\exp(\frac{\nu g^2\gamma}{c(g+c)})$. Since $\nu g^2 \to 0$ and $g \to 0$, the second term can be made arbitrarily close to 1 if K is sufficiently large. Therefore, for any $\epsilon > 0$, $\exp\Big(-\frac{\nu g\gamma}{g+c}\Big) \le \exp\big(-\frac{\nu g\gamma}{c}\big)(1+\epsilon)$ for sufficiently large K. Substituting into (13) we have a bound which is useful to prove both Theorem 1 and part (2) of Theorem 2:

$$\frac{\log P_{out}}{\nu} \le z + \frac{K}{\nu}\log\Big[\int_0^g f_{\eta_k}(x)dx + \exp\Big(-\frac{\nu g\gamma}{c}\Big)(1+\epsilon)\Big]. \quad (14)$$

For Theorem 1, we choose $\nu = K$ and $g = K^{-\delta}$ for $\frac{1}{2} < \delta < 1$. Substituting this choice, and taking the limit as $K \to \infty$, the right-hand side goes to $-\infty$ which implies that $-\log P_{tot}/K \to \infty$. This establishes the Theorem.

Intuitively, Theorem 1 maintains that P_{out} goes to zero faster than $\exp(-C_1K)$ for any fixed constant $C_1 > 0$. That is, the exponent of the outage grows faster than any linear function of K. In the next theorem, we show that the exponential rate cannot be faster than $K\log K$.

Theorem 2 *For any $\gamma > 0$ and $z > 0$, if $P_{tot} = KP_0$, then*

1.

$$\lim_{K\to\infty} -\frac{\log P_{out}}{K\log K} \le 1 , \quad (15)$$

2. and if $z < (\gamma P_0)/(\gamma\sigma_\theta^2+1)$, then

$$\lim_{K\to\infty} -\frac{\log P_{out}}{K\log K} \ge 1 - \frac{\gamma\sigma_\theta^2+1}{P_0\gamma}z. \quad (16)$$

Proof : We begin with a lower bound on the outage in (9). Since $\eta_k/(\eta_k+c) \le \eta_k/c$,

$$\begin{aligned} P_{out} &= P\left[\sum_{k=1}^K \frac{\eta_k\gamma}{\eta_k+c} \le z\right] \\ &\ge P\left[\sum_{k=1}^K \eta_k \le \frac{zc}{\gamma}\right] \\ &= \frac{1}{\zeta^K\Gamma(K)}\int_0^{\frac{zc}{\gamma}} x^{K-1}e^{-x/\zeta}dx , \end{aligned} \quad (17)$$

where the right-hand side follows since $\sum_{k=1}^K \eta_k$ is a χ^2 random variable with $2K$ degrees of freedom. We further lower bound (17):

$$\begin{aligned} \frac{1}{\zeta^K\Gamma(K)}\int_0^{\frac{zc}{\gamma}} x^{K-1}e^{-x/\zeta}dx &\ge \frac{e^{-\frac{zc}{\gamma\zeta}}}{\zeta^K\Gamma(K)}\int_0^{\frac{zc}{\gamma}} x^{K-1} \\ &= \frac{e^{-\frac{zc}{\gamma\zeta}}}{K\zeta^K\Gamma(K)}\Big(\frac{zc}{\gamma}\Big)^K, \end{aligned} \quad (18)$$

so that the right-hand side of (18) lower bounds P_{out}. Using this, taking logarithms of both sides, and normalizing with $-K\log K$ we have

$$-\frac{\log P_{out}}{K\log K} \le \frac{\log\Gamma(K)}{K\log K} + \frac{K\log\zeta}{K\log K} - \frac{-\log K - \frac{zc}{\gamma\zeta} + K\log\frac{zc}{\gamma}}{K\log K} . \quad (19)$$

Due to Stirling's formula, we can express the $\Gamma(\cdot)$ as

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\Big(\frac{x}{e}\Big)^x\left(1+\mathcal{O}\Big(\frac{1}{x}\Big)\right) . \quad (20)$$

Taking the limit as $K \to \infty$, the first term on the right-hand side of the inequality in (19) converges to 1 due to (20). The second and third terms in (19) clearly go to zero. This completes the proof of (15).

We refer to [10] for the proof of Part (2) of Theorem 2. Part(2) shows that $P_{out} < \exp[-(1-\frac{\gamma\sigma_\theta^2+1}{P_0\gamma}z)K\log K]$, for sufficiently large K. This is useful only when $z < (\gamma P_0)/(\gamma\sigma_\theta^2+1)$, which guarantees that the right-hand side of (16) is positive. This can be fulfilled if the per-sensor power P_0 is chosen sufficiently large.

Combining the two results of Theorem 2, we can roughly state that $P_{out} \sim \exp(-C_2K\log K)$ where the constant satisfies $1 - z(\gamma\sigma_\theta^2+1)/(P_0\gamma) \le C_2 \le 1$. If the lower bound on C_2 is negative, which might happen when P_0 is not large enough, then we cannot guarantee $P_{out} \sim \exp(-C_2K\log K)$, for a positive C_2. However, we are still assured by Theorem 1 that $P_{out} < \exp(-C_1K)$ for any constant C_1 if K is sufficiently large.

Note that the results in Theorem 1 and Part (1) of Theorem 2 do not depend on γ. In other words, the outage probability for a fixed $\gamma > 0$ goes to zero at least as fast as exponentially in K, and not faster than $\exp(-K \log K)$, independently of the value of γ.

5. NUMERICAL RESULTS

We consider the case where the power per sensor P_0 is fixed and show how the outage probability behaves as K increases. We assume that $P_0 = 10$ and $\gamma_k = \gamma$ is fixed and equal. Fig. 2 shows the values of $-\log P_{out}/K$ as a function of K for several cases with different choices of z and γ. From Fig. 2, we see that all curves continue to grow as K increases, which verifies Theorem 1. Fig. 3 plots $-\log P_{out}/K/\log K$ versus K when $z = 4$ and $\gamma = 1$. The upper bound and the lower bound are given by the right-hand side of (15) and (16), respectively. As expected, the simulation results fall between the upper bound and the lower bound as K increases. This verifies that $K \log K$ is an asymptotical tight bound for $\log P_{out}$ if $z < (\gamma P_0)/(\gamma\sigma_\theta^2 + 1)$.

6. CONCLUSION

We focus on the scaling law of the outage for distributed estimation over orthogonal fading channels where instantaneous channel SNRs are random and the sensing SNRs are deterministic. In this analysis, we considered a natural scenario where the per-sensor power is fixed with increasing the number of sensors. Our results show that the outage goes to zero more rapidly than exponentially in the number of sensors, and slower than $\exp(-K \log K)$.

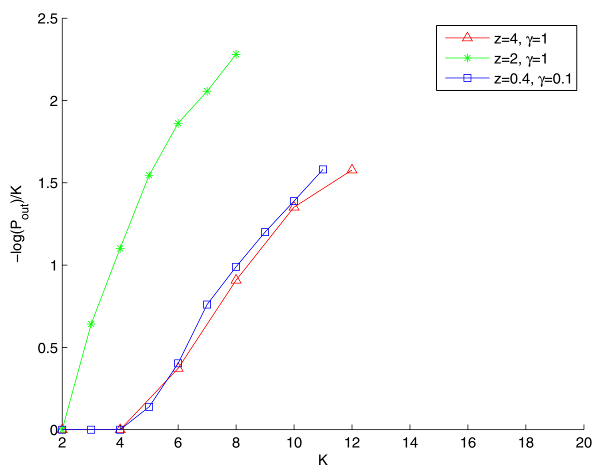

Fig. 2. Illustration of Theorem 1

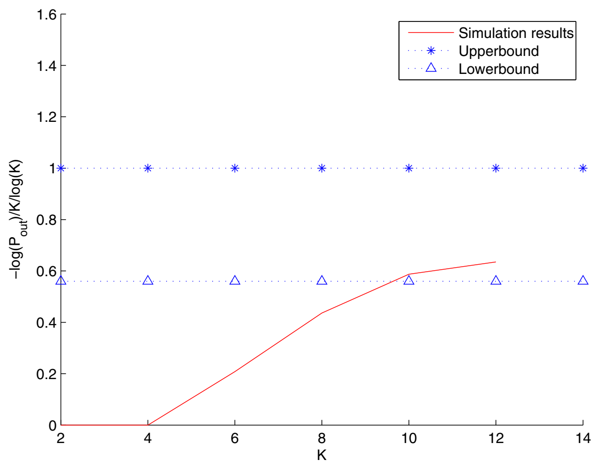

Fig. 3. Illustration of Theorem 2 ($z = 4, \gamma = 1, \zeta = 1$ and $P_0 = 10$)

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