Frequenz
62 (2008)
1-2

Explicit Synthesis Formulae for Cascaded Lossless Commensurate Lines

By Metin Şengül

Abstract – In literature, synthesis of cascaded lossless commensurate lines have been realized via some iterative methods. So to be able to obtain the value of an element which is not the first one, the designer has to obtain all the values of the elements connected before the desired one. But in this paper, explicit synthesis formulae of the networks containing cascaded lossless commensurate lines up to three have been derived analytically, and all the element values can be calculated independently.

Index Terms – Synthesis, Lossless networks, Commensurate lines.

1. Introduction

Some synthesis techniques for cascaded lossless commensurate lines have been proposed in literature [1-5]. All this techniques employs iterative methods. So synthesis process is realized by step by step. To be able to obtain the value of an element, the designer has to synthesize the network section before the desired element. In this case, numerical errors accumulate, and get bigger. So to be able to obtain error-free element values, explicit synthesis formulae must be used. In this paper, these formulae have been derived for the networks containing up to three cascaded commensurate lines. Each element value in a network can be obtained independently without any error.

In the derivation, the network is described by scattering parameters in Belevitch form [6-7]. So in the following section, this form is explained briefly. Then synthesis formulae have been given, and finally an example has been solved.

2. Characterization of Commensurate Line Networks

Darlington representation of a transmission line network is given in Fig. 1. Here, $S_{11}(\lambda)$ represents the bounded real (BR) input reflectance function, $\lambda = \Sigma + j\Omega$ is the conventional Richards variable associated with the equal-length transmission lines (Unit elements, UEs), or so-called commensurate transmission lines [8]. In detail, $\lambda = \tanh p\tau$, where $p = \sigma + j\omega$ is the complex frequency and τ is the commensurate delay of the transmission line. Specifically on the imaginary axis ($\Sigma = 0$), the transformation takes the form ($\lambda = j\Omega = j\tan\omega\tau$).

A compact representation of scattering matrix in terms of three canonic polynomials is represented by Belevitch [6]. For a lossless two-port, the canonic forms of the scattering matrix is given by [6,7]

$$S(\lambda) = \begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{bmatrix} = \frac{1}{g(\lambda)} \begin{bmatrix} h(\lambda) & \mu f(-\lambda) \\ f(\lambda) & -\mu h(-\lambda) \end{bmatrix}, \tag{1}$$

where $\mu = f(-\lambda)/f(\lambda) = \pm 1$. For a lossless two-port with resistive termination, energy conversation requires that

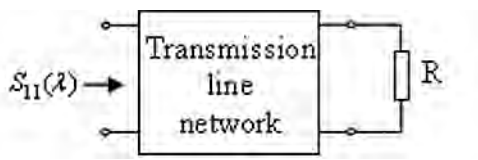

Fig. 1: Darlington representation of a transmission line network.

$$S(\lambda) S^T(-\lambda) = I, \tag{2a}$$

where I is the identity matrix and "T" designated the transpose of the matrix. The explicit form of (2a) is known as the Feldtkeller equation and given as

$$g(\lambda)g(-\lambda) = h(\lambda)h(-\lambda) + f(\lambda)f(-\lambda). \tag{2b}$$

In (1) and (2b), $g(\lambda)$ is the strictly Hurwitz polynomial of n^{th} degree with real coefficients, and $h(\lambda)$ is a polynomial of n^{th} degree with real coefficients. The polynomial function $f(\lambda)$ includes all transmission zeros of the two-port; its general form is given by

$$f(\lambda) = f_0(\lambda)(1-\lambda^2)^{n_\lambda/2}, \tag{3}$$

where n_λ specifies the number of cascaded equal-length transmission lines (Unit elements, UEs) contained in the two-port, and $f_0(\lambda)$ is an arbitrary real polynomial. According to (3), there may be a finite number of transmission zeros in the right half of the $\lambda-$ plane. Realization of transmission line network functions having such factors require, in general, complicated structures like coupled lines, Ikeno loops et cetera, which are difficult to implement and, therefore, undesirable [9, 10].

A powerful class of networks contains simple, series or shunt, stubs and equal-length transmission lines only. Series-short stubs and shunt-open stubs produce transmission zeros for $\lambda = \infty$, corresponding to the frequency $\omega = \pi/2\tau$ and odd multiples thereof. Series-open stubs and shunt-short stubs produce transmission zeros for $\lambda = 0$ (i.e., $\omega = 0$). For such networks, the polynomial function $f(\lambda)$ takes the more practical form

$$f(\lambda) = \lambda^k (1-\lambda^2)^{n_\lambda/2} \tag{4}$$

where n_λ is the number of equal-length transmission lines in cascade, k is the total number of series-open and shunt-short stubs, and the difference $n-(n_\lambda + k)$ gives the number of series-short and shunt-open stubs. Here, n denotes the degree of the two-port, which is also the degree of $g(\lambda)$. The synthesis of the input impedance, $Z_{in}(\lambda) = (1+S_{11}(\lambda))/(1-S_{11}(\lambda))$, for this case, is accomplished by extracting poles at 0 and ∞, corresponding to stubs, while equal-length transmission lines are extracted by employing Richards extraction method [1]. Alternatively, the synthesis can be carried out in a more general fashion using the cascade decomposition technique by Fettweis, which is based on the factorization of transfer matrices [2]. Also, the algorithm proposed in [5] can be used to synthesize the cascaded commensurate transmission lines.

3. Explicit Synthesis Formulae

The three canonic polynomials $g(\lambda), h(\lambda)$ and $f(\lambda)$ are in the following form for cascaded commensurate transmission lines;

$$g(\lambda) = g_n\lambda^n + g_{n-1}\lambda^{n-1} \ldots + g_1\lambda + g_0,$$

$$h(\lambda) = h_n \lambda^n + h_{n-1}\lambda^{n-1} \ldots h_1\lambda + h_0 \,,$$

$$f(\lambda) = (1-\lambda^2)^{n/2} \,.$$

Firstly, by using the coefficients of these polynomials, the following dummy parameters and the constant K must be calculated as,

$$a_i = \frac{g_i + h_i}{2}, \quad \begin{array}{l} i = 0,2,4\ldots,(n-1) \quad if\ n\ is\ odd \\ i = 0,2,4\ldots,n \quad if\ n\ is\ even \end{array} \tag{5a}$$

$$b_j = \frac{g_j + h_j}{2}, \quad \begin{array}{l} j = 1,3,5\ldots,n \quad if\ n\ is\ odd \\ j = 1,3,5\ldots,(n-1) \quad if\ n\ is\ even \end{array} \tag{5b}$$

$$c_k = \frac{g_k - h_k}{2}, \quad \begin{array}{l} k = 1,3,5\ldots,n \quad if\ n\ is\ odd \\ k = 1,3,5\ldots,(n-1) \quad if\ n\ is\ even \end{array} \tag{5d}$$

$$d_l = \frac{g_l - h_l}{2}, \quad \begin{array}{l} k = 0,2,4\ldots,(n-1) \quad if\ n\ is\ odd \\ i = 0,2,4\ldots,n \quad if\ n\ is\ even \end{array} \tag{5e}$$

$$K_{(n=1)} = \frac{a_0}{c_1}\left(\frac{a_0}{d_0} + 1\right) \tag{6a}$$

$$K_{(n=2)} = \frac{a_0 b_1}{d_0 c_1}\left(\frac{a_0}{d_0} + 1\right) \tag{6b}$$

$$K_{(n=3)} = \frac{b_3}{d_0}\left(\frac{a_0(b_1+b_3)}{a_0 d_0 + c_1 b_3}\right)^2\left(\frac{a_0}{d_0} + 1\right) \tag{6c}$$

Then obtain the modified polynomials g and h via the following equations,

$$g_m(\lambda) = g(\lambda)\cdot K \,, \tag{7a}$$

$$h_m(\lambda) = h(\lambda)\cdot K \,. \tag{7b}$$

Now calculate the new dummy parameters defined by (5) by using the modified polynomials $g_m(\lambda)$ and $h_m(\lambda)$. Then element values of the network seen in Fig. 2 can be calculated via the equations seen in Table 1.

4. Example

Let us synthesize the network described by the following polynomials via the proposed procedure explained in the previous section and the method given in [5], and then compare the results.

$$h(\lambda) = -1.3438\lambda^3 - 0.875\lambda^2 - 0.625\lambda - 0.25 \,,$$

$$g(\lambda) = 1.6563\lambda^3 + 3.5\lambda^2 + 3.125\lambda + 1 \,,$$

$$f(\lambda) = (1-\lambda^2)^{3/2} .$$

If we calculate the dummy parameters and the constant K, we found the following results,

$a_0 = 0.375$, $a_2 = 1.3125$, $b_1 = 1.25$, $b_3 = 0.1563$, $d_0 = 0.625$, $d_2 = 2.1875$, $c_1 = 1.8750$, $c_3 = 1.5001$, $K_{(n=3)} = 0.4$

After multiplying the polynomials $g(\lambda)$ and $h(\lambda)$ by $K_{(n=3)} = 0.4$, the following new parameters are obtained using the coefficients of the polynomials $g_m(\lambda)$ and $h_m(\lambda)$,

$a_0 = 0.15$, $a_2 = 0.525$, $b_1 = 0.55$, $b_3 = 0.0625$, $d_0 = 0.25$, $d_2 = 0.875$, $c_1 = 0.75$, $c_3 = 0.6$.

Then via Table 1, the element values are calculated as,

$$Z_3 = 0.5, Z_2 = 1, Z_1 = 0.5, R = 0.6 \,.$$

If we solve the same problem via the method in [5], the following results are obtained,

$$Z_3 = 0.5, Z_2 = 1, Z_1 = 0.49999, R = 0.6 \,.$$

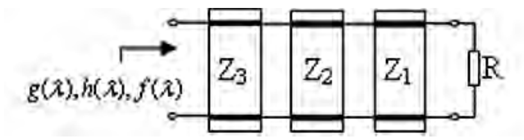

Fig. 2. Cascaded lossless commensurate line network.

Table 1: Explicit synthesis formulae.

Number of Lines	1	2	3
Z_1	$\frac{a_0+b_1}{c_1+d_0}$	$\frac{a_0 b_1}{d_0 c_1}\frac{c_1+d_0+d_2}{a_0+a_2+b_1}$	$\sqrt{b_3}\frac{c_1+c_3+d_0+d_2}{a_0+a_2+b_1+b_3}$
Z_2		$\frac{a_0+a_2+b_1}{c_1+d_0+d_2}$	$\frac{a_0(b_1+b_3)}{a_0 d_0+c_1 b_3}$
Z_3			$\frac{a_0+a_2+b_1+b_3}{c_1+c_3+d_0+d_2}$
R	$\frac{a_0}{d_0}$	$\frac{a_0}{d_0}$	$\frac{a_0}{d_0}$

As can be seen from the above results, explicit formulae give the exact results, but the other method has a very small error. So if the number of commensurate lines is three or less, the proposed synthesis procedure can be used precisely.

5. Conclusion

In this work, explicit formulae for the synthesis of cascaded lossless commensurate lines have been derived analytically. It is shown that if the number of lines is three or less, the derived formulae give exact results. Also to be able to find an element value, there is no need to synthesize the network up to this element. Each element value can be calculated independently.

6. References

[1] H.J. Carlin, "Distributed circuit design with transmission line elements", Proc. IEEE, vol.59, no.7, pp.1059-1081, July 1971.
[2] A. Fettweis, "Cascade synthesis of lossless two ports by transfer matrix factorization", in R. Boite, Network Theory, Gordon&Breach, 1972.
[3] J. Komiak and H.J. Carlin, „Improved accuracy for commensurate line synthesis", IEEE Trans MTT, vol: 24, no: 4, pp.212-215, Apr. 1976.
[4] M. Hyder Ali and R. Yarlagadda, "A note on the unit element synthesis", IEEE Trans Circuit and Systems, vol.cas-25, no.3, pp.172-174, March 1978.
[5] M. Şengül, "Synthesis of cascaded lossless commensurate lines", IEEE Trans CAS-II, to be published.
[6] V. Belevitch, *Classical network theory.* San Francisco, CA: Holden Day, 1968.
[7] A. Aksen, "Design of lossless two-ports with mixed lumped and distributed elements for broadband matching", Ph.D. dissertation, Bochum, Ruhr University, 1994.
[8] P.I. Richards, "Resistor-termination-line circuits", Proc. IRE, vol.36, pp.217-220, Feb. 1948.
[9] N, Ikeno, "A design theory of distributed constant filters", J Elec Comm Eng Japan, vol.35, pp.544-549, 1952.
[10] H.J. Carlin and R.A. Friedenson, "Gain bandwidth properties of a distributed parameter load", IEEE Trans. Circuit Theory, vol.15, pp.455-464, 1968.

Metin Şengül
Kadir Has University
Engineering Faculty
Cibali Campus
34083 Cibali-Fatih, Ýstanbul
Turkey
Fax: (+90) 212 533 57 53
E-mail: msengul@khas.edu.tr