Searching Circulant Graphs

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1 Introduction

Edge searching (or graph searching) is an extensively studied graph theoretical problem. Its origins date back to the late 1960s in works of Parsons [10] and Breisch [3]. It was first faced by a group of spelunkers who were trying to find a person lost in a system of caves. They were interested in the minimum number of people they needed in the searching team and an optimal search strategy.

Assume that we want to secure a system of tunnels from a hidden intruder who is trying to avoid us and has unbounded speed. We model this system as a finite connected graph $G = (V, E)$ where junctions correspond to vertices and tunnels correspond to edges. We will launch a group of searchers into the system in order to catch the intruder. We assume that every edge of G is contaminated initially and our aim is to clean the whole graph by a sequence of steps. At each step we are allowed to do one of these moves: (1) Place a searcher at a vertex, (2) Remove a searcher from one vertex and place it on another vertex (a "jump"), (3) Slide a searcher from a vertex along an edge to an adjacent vertex. Note that placing multiple searchers on any vertex is allowed. We don't pose any restriction on the number of searchers used.

If a searcher slides along an edge $e = uv$ from u to v, then the edge e is *cleaned* if either (i) another searcher is stationed at u, or (ii) all other edges incident to u are already clean. An *edge search strategy* is a combination of the moves so that the state of all edges being simultaneously clean is achieved, in which case we say that the graph is *cleaned*. The least number of searchers needed to clean the graph is the *(edge) search number* of the graph and is denoted $s(G)$. The problem becomes cleaning (or searching) the graph using the fewest searchers. In this respect, we are interested in the optimal search strategies, those that use only $s(G)$ searchers.

For a given graph, it is a natural question to ask the following: *What is the smallest value of* $s(G) = k$ *with which we can clean the graph?* and *How can we clean the graph using the minimum possible number of searchers?*

Notice that even once an edge is cleaned, it may not necessarily be true that it will remain clean until the end of the search strategy. In other words, an edge can be cleaned at some step and at a later step it can get contaminated again. If a searcher is stationed at a vertex v, then we say that v is *guarded*. If a path does not contain any guarded vertex, then it is called an *unguarded path*. If there is an unguarded path that contains one endpoint of a contaminated edge and one endpoint of a cleaned edge e, then e gets *recontaminated*. Hence, a clean edge remains clean as long as every path from it to a contaminated edge is blocked by at least one searcher.

The edge search problem has many variants based on, for instance, how searchers move or how the edges are cleaned. Due to its closeness with the layout problems, the problem is related to widely utilized graph parameters such as pathwidth [5], bandwidth [6] and cutwidth of a graph which arises in VLSI design [4]. The problem and its variants are related to many applications such as network security [1] and robotics [8].

The NP-completeness of the Edge Searching problem and its variations invoked interest in solving these problems on special classes of graphs. In this note, we are going to consider edge searching of the Ciculant Graphs. This family of graphs play a significant role in many discrete optimization problems [2,9]. On a related pursuit evasion game, the cops and robber game, the cop number of the circulant graph with connection set of size s is fund to be at most $\lceil \frac{s+1}{2} \rceil$ in [7]. Here we give an upper bound on the edge search number of the circulant graphs of prime order and conjecture that this can be made small.

2 Searching Circulant Graphs

We consider edge searching of circulant graphs of prime order and state our conjecture. Let $(\mathcal{G}, +)$ be a finite group with identity element 0. Let $\mathcal{S} \subseteq (\mathcal{G} \backslash \{0\})$ such that $\mathcal{S} = -\mathcal{S}$, that is $a \in \mathcal{S}$ if and only if $-a \in \mathcal{S}$. Recall that $-a$ denotes the inverse of a in $(\mathcal{G}, +)$. The *Cayley graph* on a group $\mathcal{G}$ with *connection set (or generating set)* $\mathcal{S}$, denoted as $\mathrm{Cay}(\mathcal{G}, \mathcal{S})$, is the graph that is constructed as follows: (1) Each element of $\mathcal{G}$ corresponds to a vertex v_i, and, (2) There exists an edge joining v_i and v_j if and only if $v_i = v_j + a$ where $a \in \mathcal{S}$.

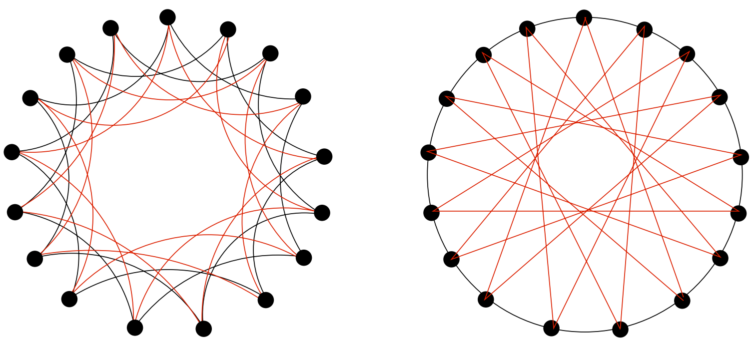

Fig. 1. The circulant graphs $\mathrm{circ}(17; 3, 4)$ and $\mathrm{circ}(17; 1, 7)$

A special class of Cayley graphs is those on cyclic groups. A *circulant graph*, denoted as $\mathrm{circ}(\mathrm{n}, \mathrm{S})$, is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_\mathrm{n}, \mathrm{S})$ where $\mathbb{Z}_n$ is the abelian group of integers modulo n.

Let p be a prime number and consider the circulant graph $\mathrm{circ}(p, S)$ where $\mathcal{S} \subseteq (\mathbb{Z}_p \backslash \{0\})$. Notice that $G = \mathrm{circ}(p, S)$ is a Hamiltonian cycle when $|S| = 2$ and $3 \leq p$; thus $s(G) = 2$. Nevertheless the calculation of the search number of $\mathrm{circ}(p, S)$ gets complicated rapidly as the size of the set S increases.

For brevity, we denote a circulant graph $G = \mathrm{circ}(p, S)$ with connection set $S = \{a, -a, b, -b\}$ as $\mathrm{circ}(p; a, b)$ where $1 \leq a < b \leq \frac{p-1}{2}$.

Theorem 2.1 If p is a prime number and $1 \leq a < b \leq \frac{p-1}{2}$, then

$$s(\mathrm{circ}(p; a, b)) \leq 2b + 1.$$

Proof (sketch) Place the $2b$ searchers on the following vertices: $v_1, v_2, \ldots, v_b$ and $v_{n-b+1}, v_{n-b+2}, \ldots, v_n$. Label these searchers as $\sigma_1, \sigma_2, \ldots, \sigma_{2b}$ respectively. Place σ_{2b+1} on v_1 and clean all the edges with end vetices in $\{v_{n-b+1}, v_{n-b+2}, \ldots, v_n, v_1, v_2, \ldots, v_b\}$. The only contaminated edge incident to v_1 is $v_1 v_{b+1}$, thus let σ_1 slide along this edge and clean v_1. Next let σ_1 clean all the edges with end vetices in $\{v_{n-b+1}, v_{n-b+2}, \ldots, v_n, v_2, \ldots, v_b, v_{b+1}\}$. Now σ_2 can slide along $v_2 v_{b+2}$ and clean v_2. We clean $v_1, v_2, \ldots, v_b$ in the same way. We repeat this shifting of searchers on $v_1, v_2, \ldots, v_b$ to $v_{b+1}, v_{b+2}, \ldots, v_{2b}$ for every group of b consequtive vertex and clean the whole graph. ∎

The upper bound in Theorem 2.1 is tight for some graphs (for instance, when $G = \mathrm{circ}(5; 1, 2)$). On the other hand, this bound will be big when b is large. We claim that this number can be made as small as twice the root of the order of the graph plus one.

In order to consider isomorphic circulant graphs, we use multiplication in $\mathbb{Z}_p$. Let $f : \{1, 2, \ldots, p\} \to \{1, 2, \ldots, p\}$ so that $f(n) = (n-1)a + 1$ where $1 \leq a < \frac{p-1}{2}$. It is a simple observation that f is an isomorphism between $\mathrm{circ}(p; a, b)$ and $\mathrm{circ}(p; 1, c)$ where $1 \leq a < b \leq \frac{p-1}{2}$ and $c = ba^{-1}$.

Furthermore, we can show that for any $k \geq 1$, $\mathrm{circ}(p; 1, c) \simeq \mathrm{circ}(p; k, ck)$.

The following conjecture gives a bound on the product of an element of $\mathbb{Z}_p$ and a positive integer less then the ceiling of the root of p.

Conjecture 4 For every prime p and every integer $i = 1, 2, \ldots, \frac{p-1}{2}$, there exists an integer j, $1 \leq j \leq \lceil \sqrt{p} \rceil$ such that either

$$ij \leq \lceil \sqrt{p} \rceil \pmod{p}, \text{ or, } p - ij \leq \lceil \sqrt{p} \rceil \pmod{p}.$$

An ongoing Maple code that mininimizes the maximum desired product shows that Conjecture 4 holds for up to the 6000th prime. Thus by Theorem 2.1 and the isomophisms we've given, the following is true for the first 6000 primes:

$$s(\mathrm{circ}(p, S)) \leq 2\lceil \sqrt{p} \rceil + 1 \tag{2.1}$$

for every circulant graph, $\mathrm{circ}(p, S)$, where $S \subseteq (\mathbb{Z}_p \backslash \{0\}), |S| \leq 4$.

This is a very good bound considering the size of the graph and the existing upper bounds on edge search number.

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