On sectional curvatures of a Weyl manifold

By Abdülkadir Özdeğer

Kadir Has University, Faculty of Arts and Sciences, Cibali Campus,
34083, Cibali-Istanbul, Turkey

(Communicated by Shigefumi Mori, M.J.A., Oct. 12, 2006)

Abstract: In this paper, it is proved that if, at each point of a Weyl manifold, the sectional curvature is independent of the plane chosen, then the Weyl manifold is locally conformal to an Einstein manifold and that the scalar curvature of the Weyl manifold is prolonged covariant constant.

Key words: Weyl manifold; Einstein-Weyl manifold; Einstein manifold; sectional curvature; prolonged covariant constant.

1. Introduction. A differentiable manifold of dimension n having a conformal class C of metrics and a torsion- free connection ∇ preserving the conformal class C is called a Weyl manifold which will be denoted by $W_n(g,w)$ where $g \in C$ and w is a $1-$form satisfying the so-called compatibility condition

$$\nabla g = 2(g \otimes w). \tag{1}$$

Under the conformal re-scaling (renormalisation)

$$\bar{g} = \lambda^2 g \qquad (\lambda > 0) \tag{2}$$

of the representative metric tensor g, w is transformed by the law

$$\bar{w} = w + d \ln \lambda. \tag{3}$$

A quantity A defined on $W_n(g,w)$ is called a satellite of g of weight $\{p\}$ if it admits a transformation of the form

$$\bar{A} = \lambda^p A \tag{4}$$

under the conformal re-scaling (2) of g ([1–3]).

It can be easily seen that the pair $(\bar{g},\bar{w})$ generates the same Weyl manifold. The process of passing from (g,w) to $(\bar{g},\bar{w})$ is called a gauge transformation.

The curvature tensor , covariant curvature tensor, the Ricci tensor and the scalar curvature of $W_n(g,w)$ are respectively defined by

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) v^p = v^j W^p_{jkl}\,, \tag{5}$$

$$W_{hjkl} = g_{hp}\, W^p_{jkl}\,, \tag{6}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}\,, \tag{7}$$

$$W = g^{ij}\, W_{ij}\,. \tag{8}$$

Clearly, the curvature tensor and the Ricci tensor are gauge invariants while the covariant curvature tensor and the scalar curvature are satellites of g of weight $\{2\}$ and $\{-2\}$ respectively.

The curvature tensor, the covariant curvature tensor and the Ricci tensor of $W_n(g,w)$ satisfy the following properties ([4–6]):

$$W^p_{jkl} = -W^p_{jlk},\ W_{ijkl} = -W_{ijlk},\ W^k_{kij} = -2W_{[ij]}, \tag{9}$$

$$W_{ijkl} + W_{jikl} = 2g_{ij}(\nabla_l w_k - \nabla_k w_l) = 4g_{ij}\nabla_{[l} w_{k]}, \tag{10}$$

$$W_{[ij]} = n \nabla_{[i} w_{j]}. \tag{11}$$

The prolonged (extended) covariant derivative of the satellite A of weight $\{p\}$ in the direction of the vector X is defined by

$$\dot{\nabla}_X A = \nabla_X A - p w(X) A \tag{12}$$

from which it follows that

$$\dot{\nabla}_X\, g = 0 \tag{13}$$

for any X ([1–3]).

A satellite of g is called prolonged covariant constant if its prolonged covariant derivative vanishes identically.

A Riemannian manifold is called an Einstein manifold if its Ricci tensor is proportional to its metric.

2000 Mathematics Subject Classiffcation. Primary: 53A30, Secondary: 53A40.

A Weyl manifold is said to be an Einstein-Weyl manifold if the symmetric part of the Ricci tensor is proportional to the metric $g \in C([8, 9])$, i. e., if

$$W_{(ij)} = \lambda\, g_{ij} \tag{14}$$

where λ is a scalar function defined on $W_n(g,w)$.

By using the second Bianchi identity

$$\dot\nabla_l W_{mijk} + \dot\nabla_k W_{milj} + \dot\nabla_j R_{mikl} = 0,$$

proved in [5], for a Weyl manifold and the relations (9) and (10), the generalization of Einstein's tensor for a Riemannian manifold to a Weyl manifold is obtained, in [6], as

$$G^j_l = \frac{1}{2}\delta^j_l W - W^j_l + 2g^{jk}\nabla_{[k}w_{l]}, \quad W^j_l = g^{ij}W_{il} \tag{15}$$

satisfying the equation

$$\dot\nabla_j G^j_l = 0 \tag{16}$$

where we have called G^j_l the generalized Einstein's tensor for $W_n(g,w)$, and $\dot\nabla_j G^j_l$ the generalized divergence of G^j_l, since in the case of a Riemannian manifold they reduce to Einstein's tensor and its divergence respectively.

We note that for an Einstein-Weyl manifold we have from (15) and (16) that

$$G^j_l = \frac{n-2}{2}\left(\frac{W}{n}\delta^j_l - 2g^{jk}\nabla_{[k}w_{l]}\right),$$
$$\dot\nabla_j G^j_l = \frac{1}{2}(n-2)\left[\frac{1}{n}(\dot\nabla_j W)\delta^j_l - 2g^{jk}\dot\nabla_j(\nabla_{[k}w_{l]})\right] = 0$$

from which it follows for $n > 2$ that

$$\frac{1}{n}(\dot\nabla_j W)\delta^j_l - 2g^{jk}\dot\nabla_j(\nabla_{[k}w_{l]}) = 0. \tag{17}$$

We now state the following lemma which will be used in our subsequent work:

Lemma 1.1. *Suppose that S is any 4-covariant tensor and that X and Y are two arbitrary linearly independent vectors. If for all X and Y*

$$S_{\alpha\beta\lambda\mu}X^\alpha\, Y^\beta\, X^\lambda\, Y^\mu \;=\; 0,$$

then

$$S_{\alpha\beta\lambda\mu} + S_{\lambda\mu\alpha\beta} + S_{\alpha\mu\lambda\beta} + S_{\lambda\beta\alpha\mu} = 0, \tag{18}$$

where X^α and Y^β are respectively the components of X and Y ([7]).

Recently, there has been considerable interest in Weyl geometry, mainly in Einstein-Weyl manifolds ([9–11]).

In [9] it is proved that if, in a compact positive-definite Einstein-Weyl manifold, the scalar curvature W is everywhere strictly negative, then the manifold is conformal to an Einstein manifold.

In the present paper, we give a sufficient condition for a Weyl manifold to be locally conformal to an Einstein manifold by means of sectional curvatures (Theorem 2.1).

2. Sectional curvatures of a Weyl manifold. Let $P(x^k)$ be any point of $W_n(g,w)$ and let us denote by X^α, Y^α the components of two arbitrary linearly independent vectors $X, Y \in T_p(W_n)$. These vectors determine a two-dimensional subspace (plane) π of T_pM. The scalar defined by [7]

$$K(\pi) = K(x,X,Y) = \frac{W_{\alpha\beta\lambda\mu}X^\alpha Y^\beta X^\lambda Y^\mu}{(g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda})X^\alpha Y^\beta X^\lambda Y^\mu} \tag{19}$$

is called the sectional curvature of $W_n(g,w)$ at P with respect to the plane π.

From (19) it follows that

$$S_{\alpha\beta\lambda\mu}X^\alpha Y^\beta X^\lambda Y^\mu = 0 \tag{20}$$

where we have put

$$S_{\alpha\beta\lambda\mu} = [W_{\alpha\beta\lambda\mu} - K(\pi)(g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda})]X^\alpha Y^\beta X^\lambda Y^\mu. \tag{21}$$

Assume now that at the point $P \in W_n(g,w)$ the sectional curvature is the same for all planes in T_pM. The case of a 2-dimensional Weyl manifold need not be considered, for it has only one plane at each point. Then, according to Lemma 1.1, the condition (18) gives

$$W_{\alpha\beta\lambda\mu} + W_{\lambda\mu\alpha\beta} + W_{\alpha\mu\lambda\beta} + W_{\lambda\beta\alpha\mu} = 4Kg_{\alpha\lambda}g_{\beta\mu} - 2K(g_{\mu\alpha}g_{\lambda\beta} + g_{\alpha\beta}g_{\mu\lambda}). \tag{22}$$

Transvecting (22) by $g^{\alpha\gamma}$ and using (6) yields

$$W^\gamma_{\beta\lambda\mu} + W^\gamma_{\mu\lambda\beta} + g^{\alpha\gamma}W_{\lambda\mu\alpha\beta} + g^{\alpha\gamma}W_{\lambda\beta\alpha\mu} = 2K(2\delta^\gamma_\lambda g_{\beta\mu} - \delta^\gamma_\mu g_{\lambda\beta} - \delta^\gamma_\beta g_{\lambda\mu}). \tag{23}$$

Using the first Bianchi identity [5]

$$W_{\lambda\mu\alpha\beta} + W_{\lambda\alpha\beta\mu} + W_{\lambda\beta\mu\alpha} = 0 \tag{24}$$

and the relations (9) and (10) we find that

$$(25)\qquad \begin{aligned} &g^{\alpha\gamma}(W_{\lambda\mu\alpha\beta} + W_{\lambda\beta\alpha\mu}) \\ &\quad = g^{\alpha\gamma}[W_{\alpha\lambda\beta\mu} + 2W_{\beta\lambda\mu\alpha}] \\ &\qquad - 4\delta^{\gamma}_{\lambda}\nabla_{[\mu}w_{\beta]} - 8g^{\alpha\gamma}g_{\lambda\beta}\nabla_{[\alpha}w_{\mu]}. \end{aligned}$$

Inserting (25) into (23) and making the necessary arrangements we obtain

$$(26)\qquad \begin{aligned} &W^{\gamma}_{\beta\lambda\mu} + W^{\gamma}_{\mu\lambda\beta} + W^{\gamma}_{\lambda\beta\mu} + 2g^{\alpha\gamma}W_{\beta\lambda\mu\alpha} \\ &\quad = 4\delta^{\gamma}_{\lambda}\nabla_{[\mu}w_{\beta]} + 8g^{\alpha\gamma}g_{\lambda\beta}\nabla_{[\alpha}w_{\mu]} \\ &\qquad + 2K(2\delta^{\gamma}_{\lambda}g_{\beta\mu} - \delta^{\gamma}_{\mu}g_{\lambda\beta} - \delta^{\gamma}_{\beta}g_{\lambda\mu}). \end{aligned}$$

Contracting with respect to γ and β and making use of (8), the third relation in (9) and (11) we get

$$2W_{(\lambda\mu)} - 2W_{[\lambda\mu]} = 4\nabla_{[\lambda}w_{\mu]} + 2K(1-n)g_{\lambda\mu}$$

from which it follows, by (11), that

$$(27)\qquad W_{(\lambda\mu)} = K(1-n)g_{\lambda\mu},$$

$$(28)\qquad \nabla_{[\lambda}w_{\mu]} = 0.$$

(27) means that $W_n(g,w)$ is an Einstein-Weyl manifold while (28) implies that the 1- form w is locally a gradient and so can be removed by a conformal rescaling (2)-(3).

On the other hand, remembering that the scalar curvature W is a satellite of g of weight $\{-2\}$ we get from (17) and (28) that

$$(29)\qquad \dot{\nabla}_j W = \nabla_j W + 2Ww_j = 0,$$

showing that, unlike the Riemannian case, instead of being constant in general, W is prolonged covariant constant. However, under the condition (28) the scalar curvature W can be made constant by a conformal rescaling of the representative metric g.

Summing up what we have found above we can state

Theorem 2.1. *A sufficient condition for a Weyl manifold of dimension $n > 2$ to be locally conformal to an Einstein manifold is that the sectional curvature at each point be independent of the plane chosen.*

This theorem may be considered as an analogue of Schur's theorem for a Riemannian manifold which can be stated as follows:

If at each point of a Weyl manifold the sectional curvature is independent of the plane chosen, then the scalar curvature W is prolonged covariant constant throughout the manifold and that the manifold is locally conformal to an Einstein manifold.

Remark 2.1. By straightforward calculations it can be shown that every two dimensional Weyl manifold is an Einstein-Weyl manifold and that any two 2-dimensional Weyl manifolds can be locally mapped conformally upon each other.

References

[1] V. Hlavaty, Theorie d'immersion d'une W_m dans W_n, Ann. Soc. Polon. Math., **21** (1949), 196–206.

[2] A. Norden, *Affinely Connected Spaces*, Nauka, Moscow, 1976.

[3] G. Zlatanov and B. Tsareva, On the geometry of the nets in the n-dimensional space of Weyl, J. Geom. **38** (1990), no.1–2, 182–197.

[4] J. L. Synge and A. Schild, *Tensor Calculus*, Univ. Toronto Press, Toronto, Ont., 1949.

[5] E. Canfes and A. Özdeğer, Some applications of prolonged covariant differentiation in Weyl spaces, J. Geom., **60** (1997), no.1–2, 7–16.

[6] A. Özdeğer, Conformal mapping of Einstein-Weyl spaces and the generalized Einstein's tensor. (Submitted).

[7] D. Lovelock and H. Rund, Tensors, differential forms, and variational principles, Dover publ. Inc., New York, 1989.

[8] N. J. Hitchin, Complex manifolds and Einstein's equations, in *Twistor geometry and nonlinear systems* (*Primorsko*, 1980), 73–99, Lecture Notes in Math., 970, Springer, Berlin, 1982.

[9] H. Pedersen and K. P. Tod, Three-dimensional Einstein-Weyl geometry, Adv. Math. **97** (1993), no.1, 74–1089.

[10] H. Pedersen and A. Swann, Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature, J. Reine Angew. Math. **441** (1993), 99–113.

[11] M. Katagiri, On compact conformally flat Einstein-Weyl manifolds, Proc. Japan Acad. Ser. A Math. Sci. **74** (1998), no.6, 104–105.