The Effects of Higher-Order Approximations in a Fluid-Filled Elastic Tube with Stenosis

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Treating arteries as thin-walled prestressed elastic tubes with a narrowing (stenosis) and blood as an inviscid fluid, we study the propagation of weakly nonlinear waves in such a fluid-filled elastic tube by employing the reductive perturbation method in the long wave approximation. It is shown that the evolution equation of the first-order term in the perturbation expansion may be described by the conventional Korteweg-de Vries (KdV) equation. The evolution equation for the second-order term is found to be the linearized KdV equation with a nonhomogeneous term, which contains the contribution of the stenosis. A progressive wave type solution is sought for the evolution equation, and it is observed that the wave speed is variable, which results from the stenosis. We study the variation of the wave speed with the distance parameter τ for various amplitude values of the stenosis. It is observed that near the center of the stenosis the wave speed decreases with increasing stenosis amplitude. However, sufficiently far from the center of the stenosis stenosis amplitude becomes negligibly small.

Key words: Progressive Waves; Elastic Tubes; Stenosed Tubes.

1. Introduction

Due to its application in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by Pedley [1] and Fung [2]. Most of the works on wave propagation in compliant tubes deal with small amplitude waves, ignoring the nonlinear effects, and focus on the dispersive character of waves (see Atabek and Lew [3], Rachev [4] and Demiray [5]). However, when the nonlinear terms, arising from the constitutive equations and kinematical relations, are introduced, one has to consider either finite amplitude, or small-but-finite amplitude waves, depending on the order of nonlinearity.

The propagation of finite amplitude waves in fluid-filled elastic or viscoelastic tubes has been examined, for instance by Rudinger [6], Ling and Atabek [7], Anliker et al. [8] and Tait and Moodie [9] by using the method of characteristics in studying the shock formation. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson [10], Hashizume [11], Yomosa [12] and Demiray [13, 14] by employing various asymptotic methods. In all these works [10 – 14], depending on the balance between the nonlinearity, dispersion and dissipation, the Korteweg-de Vries (KdV), Burgers' or KdV-Burgers' equations are obtained as the evolution equations. In obtaining such evolution equations, they treated the arteries as cylindrical long thin tubes of constant cross-section. The arteries have a variable radius along the axis of the tube.

In the present work, treating the arteries as thin-walled prestressed elastic tubes with narrowing (stenosis) and blood as an inviscid fluid, we study the propagation of weakly nonlinear waves in such a fluid-filled elastic tube by employing the reductive perturbation method in the long wave approximation. It is shown that the evolution equation of the first-order term in the perturbation expansion may be described by the conventional KdV equation. The evolution equation for the second-order term is found to be the linearized KdV equation with a nonhomogeneous term, which contains the contribution of the stenosis. A progressive wave type solution is sought for the evolution equation, and it is observed that the wave speed is variable, which results from the stenosis. We study the variation of the wave speed with the distance parameter τ for various amplitude values of the stenosis; the result is depicted in Figs. 1 – 3. It is observed that near the center of the stenosis the wave speed decreases with increasing stenosis amplitude. However, sufficiently far from the center of the stenosis the stenosis amplitude becomes negligibly small.

2. Basic Equations and Theoretical Preliminaries

2.1. Equations of the Tube

In this section we shall derive the basic equations governing the motion of a prestressed thin elastic tube with an axially symmetric bump (stenosis), and filled with a viscous fluid. For that purpose we consider a circularly cylindrical tube of radius R_0. It is assumed that such a tube is subjected to an axial stretch λ_z and a static pressure $P_0(Z^*)$. Under the effect of such a variable pressure, the position vector of a generic point on the tube is assumed to be described by

$$\boldsymbol{r_0} = [r_0 - f^*(z^*)]\boldsymbol{e_r} + z^*\boldsymbol{e_z}, \quad z^* = \lambda_z Z^*, \tag{1}$$

where $\boldsymbol{e_r}$, $\boldsymbol{e_\theta}$ and $\boldsymbol{e_z}$ are the unit base vectors in the cylindrical polar coordinates, r_0 is the deformed radius at the origin of the coordinate system, Z^* the axial coordinate before the deformation, z^* the axial coordinate after static deformation, and $f^*(z^*)$ is a function that characterizes the axially symmetric bump on the surface of the arterial wall and will be specified later.

Upon this initial static deformation we shall superimpose a dynamical radial displacement $u^*(z^*,t^*)$, where t^* is the time parameter, but, in view of the external tethering, the axial displacement is assumed to be negligible. Then, the position vector $\boldsymbol{r}$ of a generic point on the tube may be described by

$$\boldsymbol{r} = [r_0 - f^*(z^*) + u^*]\boldsymbol{e_r} + z^*\boldsymbol{e_z}. \tag{2}$$

The arc lengths along the meridional and circumferential curves are, respectively, given by

$$\begin{aligned} \mathrm{d}s_z &= [1 + (-f^{*'} + \frac{\partial u^*}{\partial z^*})^2]^{1/2}\mathrm{d}z^*, \\ \mathrm{d}s_\theta &= (r_0 - f^* + u^*)\mathrm{d}\theta. \end{aligned} \tag{3}$$

Then, the stretch ratios along the meridional and circumferential curves may, respectively, be given by

$$\begin{aligned} \lambda_1 &= \lambda_z[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}, \\ \lambda_2 &= \frac{1}{R_0}(r_0 - f^* + u^*), \end{aligned} \tag{4}$$

where the prime denotes the differentiation of the corresponding quantity with respect to z^*. The unit tangent vector $\boldsymbol{t}$ along the deformed meridional curve and the unit exterior normal vector $\boldsymbol{n}$ to the deformed tube are given by

$$\begin{aligned} \boldsymbol{t} &= \frac{(-f^{*'} + \partial u^*/\partial z^*)\boldsymbol{e_r} + \boldsymbol{e_z}}{[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}, \\ \boldsymbol{n} &= \frac{\boldsymbol{e_r} - (-f^{*'} + \partial u^*/\partial z^*)\boldsymbol{e_z}}{[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}. \end{aligned} \tag{5}$$

Let T_1 and T_2 be the membrane forces along the meridional and circumferential curves, respectively. Then, the equation of the radial motion of a small tube element placed between the planes $z^* = \text{const}$, $z^* + dz^* = \text{const}$, $\theta = \text{const}$ and $\theta + \mathrm{d}\theta = \text{const}$ may be given by

$$\begin{aligned} &-T_2\left[1 + (-f^{*'} + \frac{\partial u^*}{\partial z^*})^2\right]^{1/2} \\ &+\frac{\partial}{\partial z^*}\left\{\frac{(r_0 - f^* + u^*)(-f^{*'} + \partial u^*/\partial z^*)}{[1 + (f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}T_1\right\} \\ &+P_\mathrm{r}^*(r_0 - f^* + u^*)\left[1 + \left(-f^* + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} \\ &\quad = \rho_0\frac{HR_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}, \end{aligned} \tag{6}$$

where ρ_0 is the mass density of the tube, H the thickness in the undeformed configuration and P_r^* the fluid reaction force to be specified later.

Let $\mu\Sigma$ be the strain energy density function of the membrane, where μ is the shear modulus of the tube material. Then, the membrane forces may be expressed in terms of the stretch ratios as

$$T_1 = \frac{\mu H}{\lambda_2}\frac{\partial\Sigma}{\partial\lambda_1}, \quad T_2 = \frac{\mu H}{\lambda_1}\frac{\partial\Sigma}{\partial\lambda_2}. \tag{7}$$

Introducing (7) into (6), the equation of motion of the tube in the radial direction takes the form

$$\begin{aligned} &-\frac{\mu}{\lambda_z}\frac{\partial\Sigma}{\partial\lambda_2} \\ &+\mu R_0\frac{\partial}{\partial z^*}\left\{\frac{(-f^{*'} + \partial u^*/\partial z^*)}{[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2}}\frac{\partial\Sigma}{\partial\lambda_1}\right\} \\ &+\frac{P_\mathrm{r}^*}{H}(r_0 - f^* + u^*)[1 + (-f^{*'} + \partial u^*/\partial z^*)^2]^{1/2} \\ &\quad = \rho_0\frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}. \end{aligned} \tag{8}$$

2.2. Equations of the Fluid

In general, blood is an incompressible non-Newtonian fluid. The main factor for blood to behave like a non-Newtonian fluid is the level of cell concentration (hematocrit ratio) and the deformability of red blood cells. In the course of blood flow in arteries, the red blood cells move to the central region of the artery and, thus, the hematocrit ratio is reduced near the arterial wall, where the shear rate is quite high, as can be seen from Poiseuille flow. Experimental studies indicate that, when the hematocrit ratio is low and the shear rate is high, blood behaves like a Newtonian fluid [2]. Moreover, as pointed out by Rudinger [6], for flows in large blood vessels the viscosity of blood may be neglected as a first approximation. Thus, the equations of axially symmetric motion of an incompressible nonviscous fluid may be given in cylindrical polar coordinates by

$$\frac{\partial V_r^*}{\partial r} + \frac{V_r^*}{r} + \frac{\partial V_z^*}{\partial z^*} = 0 \text{ (incompressibility)}, \tag{9}$$

$$\frac{\partial V_r^*}{\partial t^*} + V_r^* \frac{\partial V_r^*}{\partial r} + V_z^* \frac{\partial V_r^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial r} = 0, \tag{10}$$

$$\frac{\partial V_z^*}{\partial t^*} + V_r^* \frac{\partial V_z^*}{\partial r} + V_z^* \frac{\partial V_z^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial z^*} = 0, \tag{11}$$

where V_r^*, V_z^* are the fluid velocity components in the radial and axial directions, respectively, $\bar{P}$ is the fluid pressure function and ρ_f the density of the fluid.

In general it is quite difficult to deal with these exact equations of motion of a nonviscous fluid. Therefore we shall make some simplifying assumptions, the so-called "the hydraulic approximations". In this approximations it is assumed that the axial velocity is much larger than the radial one and than averaging procedure with respect to the cross-sectional area is permissible. Applying the averaging procedure to (9) – (11) we have

$$\frac{\partial A}{\partial t^*} + \frac{\partial}{\partial z^*}(Aw^*) = 0, \tag{12}$$

$$\frac{\partial w^*}{\partial t^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial P^*}{\partial z^*} = 0, \tag{13}$$

where A denotes the inner cross-sectional area, i. e., $A = \pi r_f^2$, where $r_f = r_0 - f^* + u^*$ is the radius of the tube after final deformation, and other quantities are defined by

$$Aw^* = 2\pi \int_0^{r_f} r V_z^* \mathrm{d}r, \quad AP^* = 2\pi \int_0^{r_f} r\bar{P} \mathrm{d}r. \tag{14}$$

Here w^* is the averaged axial fluid velocity and P^* is the averaged fluid pressure. In obtaining (14) we have made use of the following assumption (Prandtl and Tietjens [15]):

$$A(w^*)^2 = 2\pi \int_0^{r_f} r V_z^2 \mathrm{d}r. \tag{15}$$

Noting the relation between the cross-sectional area and the final radius, i. e., $A = \pi(r_0 - f^* + u^*)^2$, (12) reads

$$2\frac{\partial u^*}{\partial t^*} + 2w^* \frac{\partial u^*}{\partial z^*} + (r_0 - f^* + u^*)\frac{\partial w^*}{\partial z^*} = 0. \tag{16}$$

For the present problem the fluid reaction force P_r^* takes the form

$$P_r^* = \frac{P^*}{[1 + (-f^{*\prime} + \partial u^*/\partial z^*)^2]^{1/2}}. \tag{17}$$

At this stage it is convenient to introduce the following dimensionless quantities:

$$\begin{aligned} t^* &= \left(\frac{R_0}{c_0}\right) t, \quad z^* = R_0 z, \quad u^* = R_0 u, \\ m &= \frac{\rho_0 H}{\rho_f R_0}, \quad w^* = c_0 w, \quad f^* = R_0 f, \\ r_0 &= R_0 \lambda_\theta, \quad P^* = \rho_f c_0^2 p, \quad c_0^2 = \frac{\mu H}{\rho_f R_0}. \end{aligned} \tag{18}$$

Introducing (18) into (8), (13) and (16) we obtain the following dimensionless equations:

$$2\frac{\partial u}{\partial t} + 2\left(-f' + \frac{\partial u}{\partial z}\right) w + (\lambda_\theta - f + u)\frac{\partial w}{\partial z} = 0, \tag{19}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0, \tag{20}$$

$$\begin{aligned} p = {}& \frac{m}{\lambda_z(\lambda_\theta - f + u)}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta - f + u)}\frac{\partial \Sigma}{\partial \lambda_2} \\ & - \frac{1}{(\lambda_\theta - f + u)}\frac{\partial}{\partial z}\left\{\frac{(-f' + \partial u/\partial z)}{[1 + (-f' + \partial u/\partial z)^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\}. \end{aligned} \tag{21}$$

These equations give sufficient relations to determine the field quantities u, w and p completely.

3. Long Wave Approximation

In this section we shall examine the propagation of small-but-finite amplitude waves in a fluid-filled thin elastic tube with a stenosis, whose dimensionless governing equations are given in (19)–(21). For this we adopt the long wave approximation and employ the modified reductive perturbation method, the details of which are given by Demiray [16].

The nature of the problem suggests to consider it as a boundary value problem. For such problems, the frequency is specified and the wave number is calculated through the use of the dispersion relation. Thus, it is convenient to introduce the following stretched coordinates

$$\xi=\varepsilon^{1/2}(z-ct),\quad \int_0^\tau \frac{\mathrm{d}s}{g(s)}=\varepsilon^{3/2}z, \tag{22}$$

where ε is a small parameter measuring the weakness of the nonlinearity and dispersion, c is a constant and $g(\tau)$ the scale function to be determined from the solution.

In the present work we shall assume that the geometry of the stenosis is of the form

$$f(z)=\Delta\,\mathrm{sech}Kz, \tag{23}$$

where Δ and K are two constants to be specified later. We shall further assume that the scale function $g(\tau)$ and the field variables u, w and p may be expressed as an asymptotic series of the form

$$\begin{aligned}
&g(\tau)=1+\varepsilon g_1(\tau)+\varepsilon^2 g_2(\tau)+\dots,\\
&u=\varepsilon u_1(\xi,\tau)+\varepsilon^2 u_2(\xi,\tau)+\varepsilon^3 u_3(\xi,\tau)+\dots,\\
&w=\varepsilon w_1(\xi,\tau)+\varepsilon^2 w_2(\xi,\tau)+\varepsilon^3 w_3(\xi,\tau)+\dots,\\
&p=p_0+\varepsilon p_1(\xi,\tau)+\varepsilon^2 p_2(\xi,\tau)+\varepsilon^3 p_3(\xi,\tau)+\dots.
\end{aligned} \tag{24}$$

Solving z in terms of τ and using the expansion of $g(\tau)$ we obtain

$$z=\varepsilon^{-3/2}\int_0^\tau\{1-\varepsilon g_1(s)+\varepsilon^2[g_1(s)^2-g_2(s)]+\dots\}\mathrm{d}s. \tag{25}$$

Inserting (25) into (23) we have

$$\begin{aligned}
f(z)=\Delta\,\mathrm{sech}K\varepsilon^{-3/2}\int_0^\tau\big\{&1-\varepsilon g_1(s)\\
&+\varepsilon^2[g_1(s)^2-g_2(s)]+\dots\big\}\mathrm{d}s.
\end{aligned} \tag{26}$$

In order to take the effect of the stenosis into account we shall assume that $\Delta=\varepsilon^2\sigma$ and $K=\varepsilon^{3/2}\kappa$. In this case, the function $f(z)$ may be approximated, to the order of ε^4, as

$$f(z)=\varepsilon^2 h_0(\tau)+\varepsilon^3 h_1(\tau)+O(\varepsilon^4), \tag{27}$$

where $h_0(\tau)$ and $h_1(\tau)$ are defined by

$$\begin{aligned}
&h_0(\tau)=\sigma\ \mathrm{sech}\kappa\tau,\\
&h_1(\tau)=\sigma\kappa\ \mathrm{sech}\kappa\tau\ \tanh\kappa\tau\int_0^\tau g_1(s)\mathrm{d}s.
\end{aligned} \tag{28}$$

Noting the differential relations

$$\frac{\partial}{\partial t}\to-\varepsilon^{1/2}c\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial z}\to\varepsilon^{1/2}\frac{\partial}{\partial\xi}+g(\tau)\varepsilon^{3/2}\frac{\partial}{\partial\tau} \tag{29}$$

and introducing the expansion (24) into (19)–(21), the following sets of differential equations are obtained:

$O(\varepsilon)$ equations:

$$-2c\frac{\partial u_1}{\partial\xi}+\lambda_\theta\frac{\partial w_1}{\partial\xi}=0,\quad -c\frac{\partial w_1}{\partial\xi}+\frac{\partial p_1}{\partial\xi}=0. \tag{30}$$

$O(\varepsilon^2)$ equations:

$$\begin{aligned}
&-2c\frac{\partial u_2}{\partial\xi}+\lambda_\theta\frac{\partial w_2}{\partial\xi}+\lambda_\theta\frac{\partial w_1}{\partial\tau}+2w_1\frac{\partial u_1}{\partial\xi}+u_1\frac{\partial w_1}{\partial\xi}=0,\\
&-c\frac{\partial w_2}{\partial\xi}+\frac{\partial p_2}{\partial\xi}+\frac{\partial p_1}{\partial\tau}+w_1\frac{\partial w_1}{\partial\xi}=0.
\end{aligned} \tag{31}$$

$O(\varepsilon^3)$ equations:

$$\begin{aligned}
&-2c\frac{\partial u_3}{\partial\xi}+\lambda_\theta\frac{\partial w_3}{\partial\xi}+2w_2\frac{\partial u_1}{\partial\xi}+2w_1\frac{\partial u_2}{\partial\xi}\\
&+\lambda_\theta\frac{\partial w_2}{\partial\tau}+u_1\frac{\partial w_2}{\partial\xi}+2w_1\frac{\partial u_1}{\partial\tau}+\lambda_\theta g_1(\tau)\frac{\partial w_1}{\partial\tau}\\
&+u_1\frac{\partial w_1}{\partial\tau}+[u_2-h_0(\tau)]\frac{\partial w_1}{\partial\xi}=0,\\
&-c\frac{\partial w_3}{\partial\xi}+\frac{\partial p_3}{\partial\xi}+\frac{\partial}{\partial\xi}(w_1w_2)+\frac{\partial p_2}{\partial\tau}\\
&+w_1\frac{\partial w_1}{\partial\tau}+g_1(\tau)\frac{\partial p_1}{\partial\tau}=0.
\end{aligned} \tag{32}$$

In these equations, the functions p_1, p_2 and p_3 are yet to be determined in terms of the radial displacement u, from (21). Introducing the transformation (22) and the expansion (24) into (21) we obtain

$$\begin{aligned}
&p_0=\beta_0,\quad p_1=\beta_1u_1,\\
&p_2=\beta_2u_1^2+\beta_1[u_2-h_0(\tau)]+\left(\frac{mc^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2u_1}{\partial\xi^2},\\
&p_3=\beta_3u_1^3+2\beta_2u_1[u_2-h_0(\tau)]+\beta_1[u_3-h_1(\tau)]\\
&\quad+\left(\frac{mc^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2u_2}{\partial\xi^2}-\alpha_1\left(\frac{\partial u_1}{\partial\xi}\right)^2\\
&\quad+\left(\frac{\alpha_0}{\lambda_\theta}-\frac{mc^2}{\lambda_z\lambda_\theta^2}-2\alpha_1\right)u_1\frac{\partial^2u_1}{\partial\xi^2}-2\alpha_0\frac{\partial^2u_1}{\partial\xi\partial\tau}.
\end{aligned} \tag{33}$$

In obtaining the relations given in (32) we have made use of the following expansions:

$$
\begin{aligned}
&\lambda_1 = \lambda_z \left[1+\varepsilon^3 \frac{1}{2}\left(\frac{\partial u_1}{\partial \xi}\right)^2\right],\\
&\lambda^{-1} = \left[1-\varepsilon^3 \frac{1}{2}\left(\frac{\partial u_1}{\partial \xi}\right)^2\right]/\lambda_z,\\
&\lambda_2 = \lambda_\theta + \varepsilon u_1 + \varepsilon^2[u_2 - h_0(\tau)] + \varepsilon^3[u_3 - h_1(\tau)],\\
&(\lambda_\theta - f + u)^{-1} = \frac{1}{\lambda_\theta} - \varepsilon\frac{u_1}{\lambda_\theta^2} + \varepsilon^2\left\{\frac{u_1^2}{\lambda_\theta^3} - \frac{[u_2-h_0(\tau)]}{\lambda_\theta^2}\right\}\\
&+\varepsilon^3\left\{-\frac{u_1^3}{\lambda_\theta^4} + \frac{2}{\lambda_\theta^3}u_1[u_2-h_0(\tau)] - \frac{[u_3-h_1(\tau)]}{\lambda_\theta^2}\right\} + \dots\\
&\frac{\partial\Sigma}{\partial\lambda_2} = \lambda_\theta\lambda_z\Big(\beta_0 + \varepsilon\bar{\beta}_1 u_1 + \varepsilon^2\{\bar{\beta}_2 u_1^2 + \bar{\beta}_1[u_2-h_0(\tau)]\}\\
&+\varepsilon^3\Big\{\bar{\beta}_3 u_1^3 + 2\bar{\beta}_2 u_1[u_2-h_0(\tau)]\\
&\quad + \bar{\beta}_1[u_3-h_1(\tau)] + \alpha_1\left(\frac{\partial u_1}{\partial\xi}\right)^2\Big\}\Big).
\end{aligned}
\tag{34}
$$

Here we have defined

$$
\beta_1 = \bar{\beta}_1 - \frac{\beta_0}{\lambda_\theta}, \quad \beta_2 = \bar{\beta}_2 - \frac{\beta_1}{\lambda_\theta}, \quad \beta_3 = \bar{\beta}_3 - \frac{\beta_2}{\lambda_\theta}, \tag{35}
$$

where the coefficients α_0, α_1, β_0, $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\beta}_3$ are given by

$$
\begin{aligned}
&\alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial\Sigma}{\partial\lambda_z}|_{u=0}, \quad \alpha_1 = \frac{1}{2\lambda_\theta}\frac{\partial^2\Sigma}{\partial\lambda_z\partial\lambda_\theta}|_{u=0},\\
&\beta_0 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial\Sigma}{\partial\lambda_\theta}|_{u=0}, \quad \bar{\beta}_1 = \frac{1}{\lambda_\theta\lambda_z}\frac{\partial^2\Sigma}{\partial\lambda_\theta^2}|_{u=0},\\
&\bar{\beta}_2 = \frac{1}{2\lambda_\theta\lambda_z}\frac{\partial^3\Sigma}{\partial\lambda_\theta^3}|_{u=0}, \quad \bar{\beta}_3 = \frac{1}{6}\frac{\partial^4\Sigma}{\partial\lambda_\theta^4}|_{u=0}.
\end{aligned}
\tag{36}
$$

3.1. Solution of the Field Equations

From the solution of the sets (30) and $(33)_1$ we obtain

$$
\begin{aligned}
&u_1 = U(\xi,\tau), \quad w_1 = \frac{2c}{\lambda_\theta}U(\xi,\tau),\\
&p_1 = \frac{2c^2}{\lambda_\theta}U(\xi,\tau),
\end{aligned}
\tag{37}
$$

provided that c satisfies the relation

$$
c^2 = \lambda_\theta\beta_1/2, \tag{38}
$$

where $U(\xi,\tau)$ is an unknown function whose governing equation will be obtained later, and c is the phase velocity in the long wave approximation.

Introducing the solution given in (37) into (31) and $(33)_2$ we have

$$
\begin{aligned}
&-2c\frac{\partial u_2}{\partial\xi} + \lambda_\theta\frac{\partial w_2}{\partial\xi} + 2c\frac{\partial U}{\partial\tau} + \frac{6c}{\lambda_\theta}U\frac{\partial U}{\partial\xi} = 0,\\
&-c\frac{\partial w_2}{\partial\xi} + \frac{\partial p_2}{\partial\xi} + \frac{2c^2}{\lambda_\theta}\frac{\partial U}{\partial\tau} + \frac{4c^2}{\lambda_\theta^2}U\frac{\partial U}{\partial\xi} = 0
\end{aligned}
\tag{39}
$$

with

$$
p_2 = \left(\frac{mc^2}{\lambda_\theta\lambda_z} - \alpha_0\right)\frac{\partial^2 U}{\partial\xi^2} + \beta_2 U^2 + \beta_1[u_2 - h_0(\tau)]. \tag{40}
$$

Eliminating p_2 and w_2 between (39) and (40) we obtain the conventional Korteweg-de Vries equation

$$
\frac{\partial U}{\partial\tau} + \mu_1 U\frac{\partial U}{\partial\xi} + \mu_2\frac{\partial^3 U}{\partial\xi^3} = 0, \tag{41}
$$

where the coefficients μ_1 and μ_2 are defined by

$$
\mu_1 = \frac{5}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}, \quad \mu_2 = \frac{m}{4\lambda_z} - \frac{\alpha_0}{2\lambda_\theta\lambda_z\beta_1}. \tag{42}
$$

Here the coefficients μ_1 and μ_2 characterize the nonlinearity and dispersion, respectively.

For our future purposes we need the expression of w_2. From the solution of $(39)_1$ we have

$$
\begin{aligned}
w_2 = &\frac{2c}{\lambda_\theta}[u_2 + \bar{w}_2(\tau)] + \frac{c}{\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)U^2\\
&+ \frac{2c}{\lambda_\theta}\mu_2\frac{\partial^2 U}{\partial\xi^2},
\end{aligned}
\tag{43}
$$

where $\bar{w}_2(\tau)$ is an unknown function characterizing the steady flow of ε^2-order.

To obtain the solution for $O(\varepsilon^3)$ equations we introduce the solutions given in (37) and (43) into (32) and (33):

$$
\begin{aligned}
&-\beta_1\frac{\partial u_3}{\partial\xi} + c\frac{\partial w_3}{\partial\xi} + 6\frac{c^2}{\lambda_\theta^2}\frac{\partial}{\partial\xi}(u_2 U)\\
&+2\frac{c^2}{\lambda_\theta^2}(2\bar{w}_2 - h_0)\frac{\partial U}{\partial\xi} + 4\frac{c^2}{\lambda_\theta^2}(\mu_1 - \frac{3}{\lambda_\theta})U^2\frac{\partial U}{\partial\xi}\\
&+4\frac{c^2}{\lambda_\theta^2}\mu_2\frac{\partial U}{\partial\xi}\frac{\partial^2 U}{\partial\xi^2} + 2\frac{c^2}{\lambda_\theta}\mu_1 U\frac{\partial U}{\partial\tau} + 2\frac{c^2}{\lambda_\theta^2}\mu_2 U\frac{\partial^3 U}{\partial\xi^3}\\
&+2\frac{c^2}{\lambda_\theta}\mu_2\frac{\partial^3 U}{\partial\tau\partial\xi^2} + \beta_1\frac{\partial u_2}{\partial\tau} + \beta_1\frac{\mathrm{d}\bar{w}_2}{\partial\tau} + \beta_1 g_1(\tau)\frac{\partial U}{\partial\tau} = 0,
\end{aligned}
\tag{44}
$$

$$\beta_1 \frac{\partial u_3}{\partial \xi} - c\frac{\partial w_3}{\partial \xi} + \left(4\frac{c^2}{\lambda_\theta^2} + 2\beta_2\right)\frac{\partial}{\partial \xi}(u_2 U)$$
$$+ \left[\frac{c^2}{\lambda_\theta^2}\bar{w}_2 - 2\beta_2 h_0(\tau)\right]\frac{\partial U}{\partial \xi}$$
$$+ \left[3\beta_3 + 6\frac{c^2}{\lambda_\theta^2}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right]U^2\frac{\partial U}{\partial \xi}$$
$$+ \left(4\frac{c^2}{\lambda_\theta^2}\mu_2 + \frac{\alpha_0}{\lambda_\theta} - \frac{mc^2}{\lambda_z\lambda_\theta^2} - 4\alpha_1\right)\frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial \xi^2}$$
$$+ \left(4\frac{c^2}{\lambda_\theta^2} + 2\beta_2\right)U\frac{\partial U}{\partial \tau}$$
$$+ \left(4\frac{c^2}{\lambda_\theta^2}\mu_2 + \frac{\alpha_0}{\lambda_\theta} - \frac{mc^2}{\lambda_z\lambda_\theta^2} - 2\alpha_1\right)U\frac{\partial^3 U}{\partial \xi^3}$$
$$+ \left(\frac{mc^2}{\lambda_z\lambda_\theta} - 3\alpha_0\right)\mu_2\frac{\partial^3 U}{\partial \tau \partial \xi^2}$$
$$+ \left(\frac{mc^2}{\lambda_z\lambda_\theta} - \alpha_0\right)\frac{\partial^3 u_2}{\partial \xi^3} + \beta_1\frac{\partial u_2}{\partial \tau}$$
$$+ \beta_1 g_1(\tau)\frac{\partial U}{\partial \tau} - \beta_1\frac{\mathrm{d}h_0(\tau)}{\mathrm{d}\tau} = 0. \quad (45)$$

Eliminating u_3 and w_3 between (44) and (45) we obtain the following evolution equation for u_2:

$$\frac{\partial u_2}{\partial \tau} + \mu_1\frac{\partial}{\partial \xi}(u_2 U) + \mu_2\frac{\partial^3 u_2}{\partial \xi^3}$$
$$+ \left[\frac{2}{\lambda_\theta}\bar{w}_2 - \left(\frac{1}{2\lambda_\theta} + \frac{\beta_2}{\beta_1}\right)h_0(\tau)\right]\frac{\partial U}{\partial \xi}$$
$$+ \left[\frac{3\beta_3}{2\beta_1} + \frac{5}{2\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right]U^2\frac{\partial U}{\partial \xi}$$
$$+ \left(\frac{\mu_2}{\lambda_\theta} - 2\frac{\alpha_1}{\beta_1}\right)\frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial \xi^2} + \left(\frac{\mu_2}{2\lambda_\theta} - \frac{\alpha_1}{\beta_1}\right)U\frac{\partial^3 U}{\partial \xi^3}$$
$$+ \left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\beta_1}\right)\frac{\partial^3 U}{\partial \tau \partial \xi^2} + \left(\frac{\mu_1}{2} + \frac{1}{\lambda_\theta} + \frac{\beta_2}{\beta_1}\right)U\frac{\partial U}{\partial \tau}$$
$$+ g_1(\tau)\frac{\partial U}{\partial \tau} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}[\bar{w}_2 - h_0(\tau)] = 0. \quad (46)$$

Equation (46) must even be valid when $u_2 = U = 0$, which results in

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\bar{w}_2 - h_0(\tau)] = 0. \quad (47)$$

The solution of (47) yields $\bar{w}_2 = h_0(\tau)$. Thus, the evolution (46) reduces to the following linearized Korteweg-de Vries equation with a nonhomogeneous term:

$$\frac{\partial u_2}{\partial \tau} + \mu_1\frac{\partial}{\partial \xi}(u_2 U) + \mu_2\frac{\partial^3 u_2}{\partial \xi^3} + S(U) = 0, \quad (48)$$

where the function $S(U)$ is defined by

$$S(U) = \left(\frac{3}{2\lambda_\theta} - \frac{\beta_2}{\beta_1}\right)h_0(\tau)\frac{\partial U}{\partial \xi}$$
$$+ \left[\frac{3\beta_3}{2\beta_1} + \frac{5}{2\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right]U^2\frac{\partial U}{\partial \xi}$$
$$+ \left(\frac{\mu_2}{\lambda_\theta} - 2\frac{\alpha_1}{\beta_1}\right)\frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial \xi^2}$$
$$+ \left(\frac{\mu_2}{2\lambda_\theta} - \frac{\alpha_1}{\beta_1}\right)U\frac{\partial^3 U}{\partial \xi^3}$$
$$+ \left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\beta_1}\right)\frac{\partial^3 U}{\partial \tau \partial \xi^2}$$
$$+ \left(\frac{\mu_1}{2} + \frac{1}{\lambda_\theta} + \frac{\beta_2}{\beta_1}\right)U\frac{\partial U}{\partial \tau} + g_1(\tau)\frac{\partial U}{\partial \tau}. \quad (49)$$

3.2. Progressive Wave Solution

In this sub-section we shall study the localized travelling wave solution to the field equations given in (41) and (48). For that purpose we introduce

$$U = U(\zeta), \quad u_2 = V(\zeta), \quad \zeta = k(\xi - v_0\tau), \quad (50)$$

where k and v_0 are two constants to be determined from the solution of the field equations. Introducing (50) into (41) we obtain

$$-v_0 U' + \mu_1 U U' + \mu_2 k^2 U''' = 0, \quad (51)$$

where a prime denotes the derivative of the corresponding quantity with respect to ζ. Integrating (51) with respect to ζ and using the localization condition, i. e., $U \to 0$ as $\zeta \to \pm\infty$, we have

$$U'' - \frac{v_0}{\mu_2 k^2}U + \frac{\mu_1}{2\mu_2 k^2}U^2 = 0. \quad (52)$$

It is a common practice to employ the hyperbolic tangent method in solving this type of wave equations. For this purpose we introduce the coordinate transformation

$$y = \tanh\zeta. \quad (53)$$

The finite power series solution of (52) in the variable y, which satisfies the regularity conditions $U \to 0$ as $y \to \pm 1$, can be expressed as

$$U = a(1 - y^2), \quad (54)$$

where a is the constant wave amplitude. Noting the differential relation $\mathrm{d}/\mathrm{d}\zeta = (1-y^2)\mathrm{d}/\mathrm{d}y$, we have

$$U'' = a(-2+8y^2-6y^4). \tag{55}$$

Introducing (54) and (55) into (52) and setting the coefficients of like powers of y equal to zero, we obtain

$$k = \left(\frac{\mu_1 a}{12\mu_2}\right)^{1/2}, \quad v_0 = \frac{\mu_1 a}{3}. \tag{56}$$

Thus, the solution for the first-order equation is given by

$$U = a\,\mathrm{sech}^2\zeta, \quad \zeta = \left(\frac{\mu_1 a}{12\mu_2}\right)^{1/2}\left(\xi - \frac{\mu_1 a}{3}\tau\right). \tag{57}$$

To obtain the solution for the second-order term, we first introduce (50) into (48), which results in

$$-kv_0V' + k\mu_1(UV)' + \mu_2 k^3 V''' + S(U) = 0. \tag{58}$$

Integrating (58) with respect to ζ and using the localization conditions, we have

$$-v_0V + \mu_1(UV) + \mu_2 k^2 V'' + T(U) = 0, \tag{59}$$

where the function $T(U)$ is defined by

$$\begin{aligned} T(U) = &\left[\left(\frac{3}{2\lambda_\theta} - \frac{\beta_2}{\beta_1}\right)h_0(\tau) - v_0 g_1(\tau)\right]U \\ &+ \frac{1}{3}\left[\frac{3\beta_3}{2\beta_1} + \frac{5}{2\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right]U^3 \\ &+ k^2\left(\frac{\mu_2}{2\lambda_\theta} - \frac{\alpha_1}{\beta_1}\right)\left[\frac{1}{2}(U')^2 + UU''\right] \\ &- v_0k^2\left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\lambda_\theta\lambda_z\beta_1}\right)U'' \\ &- \frac{v_0}{2}\left(\frac{\mu_1}{2} + \frac{1}{\lambda_\theta} + \frac{\beta_2}{\beta_1}\right)U^2. \end{aligned} \tag{60}$$

For the solution of (60) we shall again employ the hyperbolic tangent method and introduce the following finite power series as the particular solution of the differential equation (59) for the function V, which satisfies the localization condition

$$V = (1-y^2)(a_0 + a_2y^2), \tag{61}$$

where a_0, a_2 are some constants to be determined by introducing (61) into (59). Noting the derivative of V

$$\begin{aligned} V'' = &\,2(a_2 - a_0) + (8a_0 - 20a_2)y^2 \\ &+ (38a_2 - 6a_0)y^4 - 20a_2y^6, \end{aligned} \tag{62}$$

introducing this expression into (59) and setting the coefficients of like powers of y equal to zero we obtain

$$\begin{aligned} a_0 &= \delta_1 a^2, \\ \delta_1 &= -\frac{1}{2\mu_1}\left[\frac{3\beta_3}{2\beta_1} + \frac{5}{2\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right] \\ &\quad - \frac{\mu_1}{3\mu_2}\left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\beta_1}\right) + \frac{1}{3}\left(\frac{\mu_1}{2} + \frac{1}{\lambda_\theta} + \frac{\beta_2}{\beta_1}\right), \\ a_2 &= \delta_2 a^2, \\ \delta_2 &= -\frac{1}{2\mu_1}\left[\frac{3\beta_3}{2\beta_1} + \frac{5}{2\lambda_\theta}\left(\mu_1 - \frac{3}{\lambda_\theta}\right)\right] \\ &\quad + \frac{1}{\mu_2}\left(\frac{\mu_2}{2\lambda_\theta} - \frac{\alpha_1}{\beta_1}\right), \\ g_1(\tau) &= \frac{3}{\mu_1 a}\left(\frac{3}{2\lambda_\theta} - \frac{\beta_2}{\beta_1}\right)h_0(\tau) \\ &\quad - \frac{\mu_1}{3\mu_2}\left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\beta_1}\right)a. \end{aligned} \tag{63}$$

Thus, the particular solution may be expressed as

$$u_2 = V = a^2(\delta_1 + \delta_2\tanh^2\zeta)\mathrm{sech}^2\zeta. \tag{64}$$

Here, it can be shown that $a_1\,\mathrm{sech}^2\zeta\,\tanh\zeta$ is the homogeneous solution of the differential equation (59). As stated before, the requirement of localized travelling wave solution made it possible to determine the scaling function $g_1(\tau)$. As had been pointed out in [16], without introducing the scaling function, the study of higher-order terms in perturbation expansion leads to some secularities in the solution. By use of the concept of scaling function these secularities are removed. As can be seen from $(63)_3$ the scaling function $g_1(\tau)$ also depends on the change of the tube radius.

The total solution up to ε^2-order terms may be given by

$$\begin{aligned} u &= \varepsilon\, a\,\mathrm{sech}^2\zeta[1 + \varepsilon\, a(\delta_1 + \delta_2\tanh^2\zeta)] + O(\varepsilon^3), \\ \zeta &= \left(\frac{\mu_1 a}{12\mu_2}\right)^{1/2}\left[\varepsilon^{1/2}(z-ct) - \frac{\mu_1 a}{3}\tau\right], \end{aligned} \tag{65}$$

where z and τ are related to each other by

$$\begin{aligned} \varepsilon^{3/2}z = &\left[1 + \varepsilon\frac{\mu_1}{2\mu_2}\left(\frac{3}{2}\mu_2 - \frac{\alpha_0}{\beta_1}\right)a\right]\tau \\ &- \varepsilon\frac{3\sigma}{\mu_1\kappa a}\left(\frac{3}{2\lambda_\theta} - \frac{\beta_2}{\beta_1}\right)\left[\tan^{-1}(\mathrm{sech}\,\kappa\tau) - \frac{\pi}{4}\right]. \end{aligned} \tag{66}$$

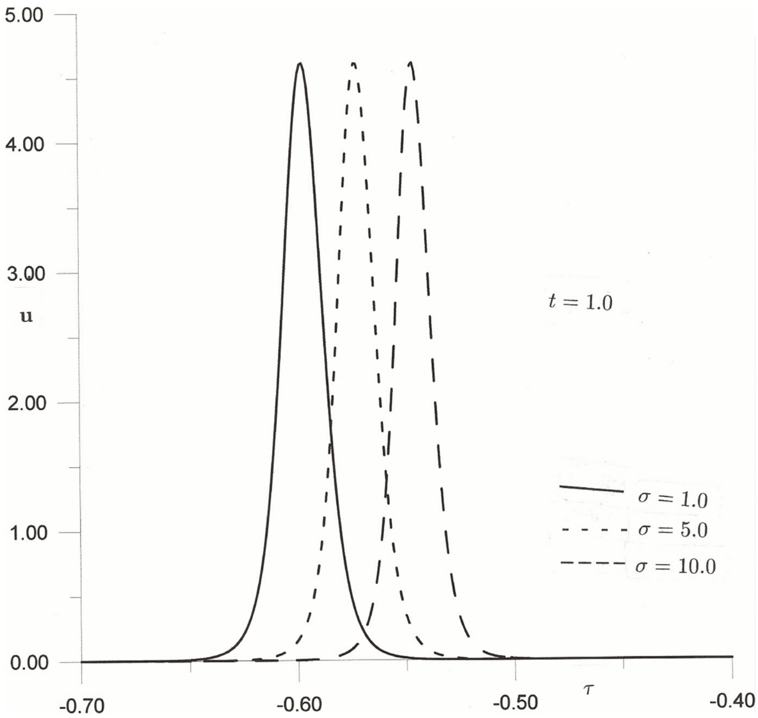

Fig. 1. The variation of the radial displacement with the space parameter τ for three values of the stenosis parameter σ.

Similar expressions may be given for the axial velocity w and the fluid pressure p.

As can be seen from the definition of the phase function $\zeta = k[\varepsilon^{1/2}(z-ct) - \mu_1 a\tau(z)/3]$, the wave front is not a plane anymore, it is rather a cylindrical surface in the (z,t) plane. This is of course the result of the stenosis in the tube. Noting the differential relation $\mathrm{d}\tau = \varepsilon^{3/2} g(\tau)\mathrm{d}z$, which can be obtained from (22), the speed v_p of the propagation may be defined by

$$v_\mathrm{p} = \frac{c}{1 - \varepsilon \dfrac{\mu_1 a}{3} g(\tau)}. \tag{67}$$

Recalling the perturbation expansion of $g(\tau)$, up to the $O(\varepsilon)$ approximation it reads

$$v_\mathrm{p} = \frac{c}{1 - \frac{\mu_1 a}{3}[\varepsilon + \varepsilon^2 g_1(\tau)]}, \tag{68}$$

where $g_1(\tau)$ is given in $(63)_3$.

4. Numerical Results and Discussion

In order to see the effects of a stenosis on the wave speed one has to know the numerical values of the coefficients α_0, β_1, β_2, μ_1 and μ_2. For that reason one must know the constitutive relation of the tube material. In this work we shall utilize the constitutive relation proposed by Demiray [17] for soft biological tissues. Following Demiray [17], the strain energy density function may be expressed as

$$\Sigma = \frac{1}{2\alpha}\{\exp[\alpha(I_1 - 3)] - 1\}, \tag{69}$$

where α is a material constant and I_1 is the first invariant of the Finger deformation tensor and defined by $I_1 = \lambda_z^2 + \lambda_\theta^2 + 1/\lambda_z^2\lambda_\theta^2$. Introducing (69) into (35) and (36), the coefficients α_0, β_0, β_1 and β_2 an β_3 are obtained as

$$\alpha_0 = \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right) G(\lambda_\theta, \lambda_z),$$

$$\beta_0 = \left[\frac{1}{\lambda_\theta^4\lambda_z^3} + \alpha\left(1 - \frac{1}{\lambda_\theta^4\lambda_z^2}\right)\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\right] \cdot G(\lambda_\theta, \lambda_z),$$

$$\beta_1 = \left[\frac{4}{\lambda_\theta^5\lambda_z^3} + 2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right] G(\lambda_\theta, \lambda_z),$$

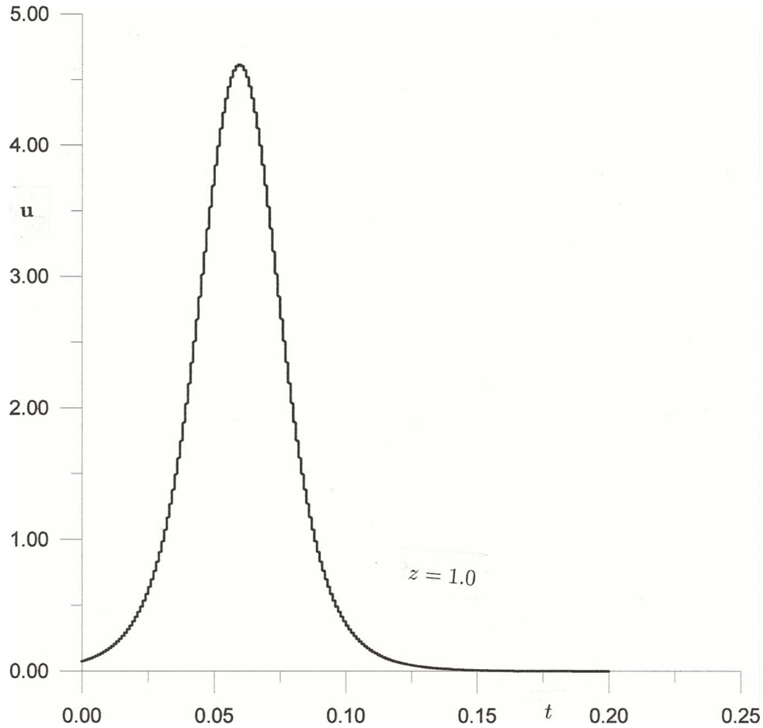

Fig. 2. The variation of the radial displacement u with time t for $z = 1.0$.

$$\beta_2 = \left[-\frac{10}{\lambda_\theta^6\lambda_z^3} + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1+\frac{11}{\lambda_\theta^4\lambda_z^2}\right) + 2\frac{\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]G(\lambda_\theta,\lambda_z),$$

$$\beta_3 = \left[\frac{20}{\lambda_\theta^7\lambda_z^5} + \frac{4\alpha}{\lambda_\theta^5\lambda_z^3}\left(-5+\frac{9}{\lambda_\theta^4\lambda_z^2}\right) + \frac{2\alpha^2}{\lambda_\theta\lambda_z}\left(1+\frac{7}{\lambda_\theta^4\lambda_z^2}\right)\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2 + \frac{4}{3}\frac{\alpha^3}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^4\right]G(\lambda_\theta,\lambda_z),$$

where the function G is defined by

$$G(\lambda_\theta,\lambda_\theta) = \exp\left[\alpha\left(\lambda_\theta^2+\lambda_z^2+\frac{1}{\lambda_\theta^2\lambda_z^2}-3\right)\right]. \quad (70)$$

For the static case, the present model was compared by Demiray [18] with the measurements by Simon et al. [19] on canine abdominal artery with the characteristics $R_\mathrm{i} = 0.31$ cm, $R_0 = 0.38$ cm and $\lambda_z = 1.53$, and the value of the material constant α was found to be $\alpha = 1.948$. Using this numerical value of the coefficient α, the values of μ_1, μ_2, α_0/β_1, α_1/β_1, β_2/β_1, β_3/β_1, δ_1, δ_2 and c are calculated numerically for $\lambda_\theta = \lambda_z = 1.6$, $m = 0.1$, and the result is found to be

$$\mu_1 = 4.911,\ \mu_2 = -0.0363,\ \alpha_0/\beta_1 = 0.266,\ \alpha_1/\beta_1 = 0.540,\ \beta_2/\beta_1 = 3.348,\ \beta_3/\beta_1 = 8.071,\ \delta_1 = -16.456,\quad \delta_2 = 13.183,\quad c = 15.39. \quad (71)$$

Here we note that the numerical value of the coefficient μ_2 is negative. In order to have a real k, given in (56), the sign of the amplitude a must be negative.

Using these values in the expression of $g_1(\tau)$ we have

$$g_1(\tau) = -\frac{1.473}{a}h_0(\tau) - 14.45a. \quad (72)$$

Introducing this expression of $g_1(\tau)$ into the equation (68), the displacement u and the wave speed v_p for the smallness parameter $\varepsilon = 0.5$ and the wave amplitude $a = -1$ take the following form:

$$u = \operatorname{sech}^2\zeta(-4.614 + 3.296\tanh^2\zeta),$$

$$v_\mathrm{p} = \frac{15.39}{7.733+0.603h_0(\tau)} = \frac{1.99}{1+0.08\,\sigma\operatorname{sech}\kappa\tau}. \quad (73)$$

For the numerical calculations we also need the expression of z relating to the variable of τ, which follows

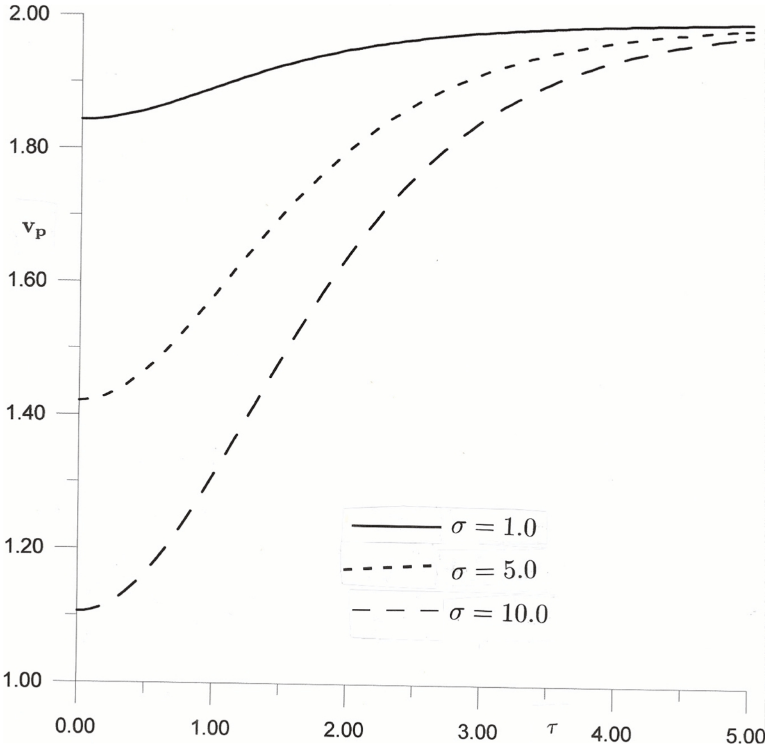

Fig. 3. The variation of the wave speed v_p with the space parameter τ for three values of the stenosis parameter σ.

from (66) as

$$z = -27.79\tau - 2.08\frac{\sigma}{\kappa}\left[\tan^{-1}(\operatorname{sech}\kappa\tau) - \frac{\pi}{4}\right]. \quad (74)$$

The radial displacement, up to $O(\varepsilon^3)$, is calculated for various parameters and the results are depicted in Figs. 1 and 2. Figure 1 shows the variation of the radial displacement with the space parameter τ for three values of the stenosis parameter σ at a fixed time, i.e. $t = 1.0$ and $\kappa = 1.0$. It is seen from this graph that the wave profile moves to the right for larger values of the stenosis parameter σ. Figure 2 explains the variation of the radial displacement u with the time parameter t, for a fixed space variable z, i. e. $z = 1.0$. This figure shows that the variation of of the radial displacement with time is not so sensitive to the stenosis parameter σ. For the values of $\sigma = 1.0, 5.0, 10.0$ the variations of the radial displacement with the time parameter t are almost the same.

The variation of wave speed with the distance parameter τ for various values of σ, which characterizes the amplitude of the stenosis, and for $\kappa = 1$ is depicted in Figure 3. As the figure reveals, at the center of the stenosis the wave speed decreases with increasing amplitude of the stenosis. As can be seen from the figure, the effect of stenosis to the wave speed at moderately far distances, e. g. $\tau = 5$ units, from the center of stenosis is negligibly small. As a matter of fact, for $\varepsilon = 0.5$ and for an arterial radius of 0.5 cm this distance is about 6 cm.

Acknowledgements

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