Pilot-Aided Bayesian MMSE Channel Estimation for OFDM Systems: Algorithm and Performance Analysis

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Abstract— **This paper proposes a computationally efficient, pilot-aided minimum mean square error (MMSE) channel estimation algorithm for OFDM systems. The proposed approach employs a convenient representation of the discrete multipath fading channel based on the Karhunen-Loeve (KL) orthogonal expansion and estimates uncorrelated series expansion coefficients. Moreover, optimal rank reduction is achieved in the proposed approach by exploiting the optimal truncation property of the KL expansion resulting in a smaller computational load on the estimation algorithm. The performance of the proposed approach is studied through analytical and experimental results. We first consider the stochastic Cramer-Rao bound and derive the closed-form expression for the random KL coefficients. We then exploit the performance of the MMSE channel estimator based on the evaluation of minimum Bayesian MSE.**

I. INTRODUCTION

Traditional wireless technologies are not very well suited to meet the demanding requirements of providing very high data rates with the ubiquity and mobility. Given the scarcity and exorbitant cost of radio spectrum, such data rates dictate the need for extremely high spectral efficient modulation schemes [1]. Holding great promise to use the frequency resources as efficiently as possible, OFDM is a strong candidate to provide substantial capacity enhancement for future wireless systems [2]. OFDM is therefore currently being adopted and tested for many standards, including terrestrial digital broadcasting (DAB and DVB) in Europe, and high speed modems over Digital Subscriber Lines in the US. It has also been implemented for broadband indoor wireless systems including IEEE802.11a, MMAC and HIPERLAN/2.

An OFDM system operating over a wireless communication channel effectively forms a number of parallel frequency nonselective fading channels thereby reducing intersymbol interference (ISI) and obviating the need for complex equalization thus greatly simplifying channel estimation/equalization task. Moreover, OFDM is bandwidth efficient since the spectra of the neighboring subchannels overlap, yet channels can still be separated through the use of orthogonality of the carriers. Furthermore, its structure also allows efficient hardware implementations using fast Fourier transform (FFT) and polyphase filtering [2].

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Although the structure of OFDM signalling avoids ISI arising due to channel memory, fading multipath channel still introduces random attenuations on each tone. Furthermore, simple frequency domain equalization, which divides the FFT output by the corresponding channel frequency response, does not assure symbol recovery if the channel has nulls on some subcarriers. Hence, accurate channel estimation technique have to be used to improve the performance of the OFDM systems. Numerous pilot-aided channel estimation methods for OFDM have been developed [3], [4], [5]. In particular, a low-rank approximation is applied to linear MMSE estimator for the estimation of subcarrier channel attenuations by using the frequency correlation of the channel [3]. In [4], a MMSE channel estimator, which makes full use of the time and frequency correlation of the time-varying dispersive channel was proposed. Moreover, a low complexity MMSE based doubly channel estimation approaches were presented in [5].

In this paper, we develop a pilot-aided low-rank MMSE channel estimation method with the inverse FFT based interpolation. In contrast to [3], the proposed approach requires a convenient representation of the multipath channel parameters by the Karhunen-Loeve (KL) series expansion. With the application of KL expansion, rather than estimating correlated channel impulse response, the uncorrelated series expansion coefficients are estimated. Furthermore, optimal rank reduction is achieved in the proposed approach by exploiting the optimal truncation property of the KL expansion, resulting in a smaller computational load on the MMSE channel estimation algorithm.

II. SYSTEM MODEL

In order to eliminate ISI arising due to multipath channel and preserve orthogonality of the subcarrier frequencies (tones), conventional OFDM systems first take the IFFT of data symbols and then insert redundancy in the form of a Cyclic Prefix (CP) of length L_{CP} larger than the channel order L. CP is discarded at the receiver and remaining part of the OFDM symbol is FFT processed. Combination of IFFT and CP at the transmitter with the FFT at the receiver converts the frequency-selective channel to separate flat-fading subchannels. The block diagram in Fig. 1 describes the conventional OFDM system. We consider an OFDM system with K subcarriers for the transmission of K parallel data symbols. Thus,

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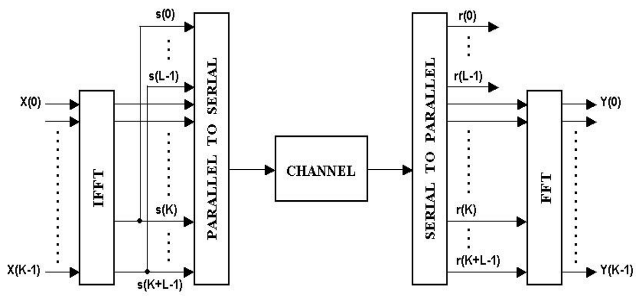

Fig. 1. OFDM System Block Diagram

the information stream $X(n)$ is parsed into K-long blocks: $\mathbf{X}_i = [X_i(0), X_i(1), \cdots, X_i(K-1)]^T$ where $i = 1, 2, \cdots$ is the block index and the superscript $(\cdot)^T$ indicates the vector transpose. The $K \times 1$ symbol block is then mapped to a $(K+L) \times 1$ vector by first taking the IFFT of $\mathbf{X}_i$ and then replicating the last L_{CP} elements as

$$\boldsymbol{s}_i = [s_i(0), s_i(1), \cdots, s_i(K + L_{CP} - 1)]^T \ . \tag{1}$$

$\boldsymbol{s}_i$ is serially transmitted over the channel. At the receiver, the CP of length L_{CP} is removed first and FFT is performed on the remaining $K \times 1$ vector. Therefore, we can write the output of the FFT unit in matrix form as

$$\mathbf{Y}_i = \mathbf{A}_i \mathbf{H}_i + \boldsymbol{\eta}_i \tag{2}$$

where $\mathbf{A}_i$ is the diagonal matrix $\mathbf{A}_i = \operatorname{diag}(\mathbf{X}_i)$ and $\mathbf{H}_i$ is the channel vector. The elements of $\mathbf{H}_i$ are values of the channel frequency response evaluated at the subcarriers. Therefore, we can write $\mathbf{H}_i = [H_i(0), H_i(\exp(j2\pi/K)), \cdots, H_i(\exp(j2\pi(K-1)/K))]^T$ as $\mathbf{H}_i = \mathcal{F}\boldsymbol{h}_i$ where $\mathcal{F}$ is the FFT matrix with (m,n) entry $\exp(-j2\pi mn/K)$ and $\boldsymbol{h}_i = [h_i(0), h_i(1), \cdots, h_i(L-1)]^T$ is the overall channel impulse during the ith OFDM block. Finally, $\boldsymbol{\eta}_i$ is an $K \times 1$ zero-mean, i.i.d complex Gaussian vector that models additive noise in the K sub-channels (tones). We have $E[\boldsymbol{\eta}_i \boldsymbol{\eta}_i^{\dagger}] = \sigma^2 \mathbf{I}_K$ where $\mathbf{I}_K$ represents an $K \times K$ identity matrix, σ^2 is the variance of the additive noise entering the system and the superscript $(\cdot)^{\dagger}$ indicates the Hermitian transpose.

Based on the model (2), our main objective in this paper is to develop a pilot-aided channel estimation algorithm according to MMSE criterion and then explore the performance of the estimator based on the evaluation of the Cramer-Rao bound and Bayesian MSE. An approach adapted herein explicitly model the channel parameters by the Karhunen-Loeve (KL) series representation since expansion allows one to tackle estimation of correlated parameters as a parameter estimation problem of the uncorrelated coefficients. Note that KL expansion is well known for its optimal truncation property [7]. That is, the KL expansion requires the minimum number of terms among all possible series expansions in representing a random channel for a given mean-squared error. Thus, the optimal truncation property of the KL expansion results in a smaller computational load on the channel estimation algorithm. We will therefore employ KL expansion of the multipath channel in the derivation of the MMSE estimator to further reduce the complexity.

III. MMSE Estimation of KL Coefficients

A low-rank approximation to the frequency-domain linear MMSE channel estimator is provided by [3] to reduce the complexity of the estimator. Optimal rank reduction is achieved in this approach by using the singular value decomposition (SVD) of the channel attenuations covariance matrix $\mathbf{C_H}$ of dimension $K \times K$. In contrast, we adapt the MMSE estimator for the estimation of multipath channel parameters $\boldsymbol{h}$ that uses covariance matrix of dimension $L \times L$. The proposed approach employs KL expansion of multipath channel parameters and reduces the complexity of the SVD used in *eigendecomposition* since L is usually much less than M. We will first develop MMSE estimator for pilot assisted OFDM system in the sequel.

A. MMSE Channel Estimation

Pilot symbol assisted techniques can provide information about a undersampled version of the channel that may be easier to identify. In this paper, we therefore address the problem of estimating multipath channel parameters by exploiting the distributed training symbols. Considering (2), and in order that the pilot symbols are included in the output vector for our estimation purposes, we focus on a under-sampled signal model. Assuming K_p pilot symbols are uniformly inserted at known locations of the i^{th} OFDM block, the $K_p \times 1$ vector corresponding the FFT output at the pilot locations becomes

$$\mathbf{Y} = \mathbf{A}\mathbf{F}\boldsymbol{h} + \boldsymbol{\eta} \tag{3}$$

where $\mathbf{A} = \operatorname{diag}[\mathbf{A}_i(0), \mathbf{A}_i(\Delta), \cdots, \mathbf{A}_i((K_p - 1)\Delta)]$ is a diagonal matrix with pilot symbol entries, Δ is pilot spacing interval, $\mathbf{F}$ is an $K_p \times L$ FFT matrix generated based on pilot indices, and similarly $\boldsymbol{\eta}$ is the under-sampled noise vector.

For the estimation of $\boldsymbol{h}$, the new linear signal model can be formed by premultiplying both sides of (3) by $\mathbf{A}^{\dagger}$ and assuming pilot symbols are taken from a PSK constellation, then the new form of (3) becomes

$$\begin{aligned} \mathbf{A}^{\dagger}\mathbf{Y} &= \mathbf{F}\boldsymbol{h} + \mathbf{A}^{\dagger}\boldsymbol{\eta} \\ \tilde{\mathbf{Y}} &= \mathbf{F}\boldsymbol{h} + \tilde{\boldsymbol{\eta}} \end{aligned} \tag{4}$$

where $\tilde{\mathbf{Y}}$ and $\tilde{\boldsymbol{\eta}}$ are related to $\mathbf{Y}$ and $\boldsymbol{\eta}$ by the linear transformation respectively. Furthermore, $\tilde{\boldsymbol{\eta}}$ is statistically equivalent to $\boldsymbol{\eta}$.

Equation (4) offers a Bayesian linear model representation. Based on this representation, the minimum variance estimator for the time-domain channel vector $\boldsymbol{h}$ for the ith OFDM block, i.e., conditional mean of $\boldsymbol{h}$ given $\tilde{\mathbf{Y}}$, can be obtained using MMSE estimator. We should clearly make the assumptions

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that $\boldsymbol{h} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\boldsymbol{h}})$, $\tilde{\boldsymbol{\eta}} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\tilde{\boldsymbol{\eta}}})$ and $\boldsymbol{h}$ is uncorrelated with $\tilde{\boldsymbol{\eta}}$. Therefore, MMSE estimate of $\boldsymbol{h}$ is given by [8]:

$$\hat{\boldsymbol{h}} = (\mathbf{F}^\dagger \mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1} \mathbf{F} + \mathbf{C}_{\boldsymbol{h}}^{-1})^{-1} \mathbf{F}^\dagger \mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1} \tilde{\mathbf{Y}} \,. \tag{5}$$

Due to PSK pilot symbol assumption, $\mathbf{C}_{\tilde{\boldsymbol{\eta}}} = E\left[\tilde{\boldsymbol{\eta}}\tilde{\boldsymbol{\eta}}^\dagger\right] = \sigma^2 \mathbf{I}_{K_p}$, therefore we can express (5) by

$$\hat{\boldsymbol{h}} = (\mathbf{F}^\dagger \mathbf{F} + \sigma^2 \mathbf{C}_{\boldsymbol{h}}^{-1})^{-1} \mathbf{F}^\dagger \tilde{\mathbf{Y}} \,. \tag{6}$$

Under the assumption that uniformly spaced pilot symbols are inserted with pilot spacing interval Δ and $K = \Delta \times K_p$, correspondingly, $\mathbf{F}^\dagger \mathbf{F}$ reduces to

$$\mathbf{F}^\dagger \mathbf{F} = K_p \mathbf{I}_L \tag{7}$$

Then according to (6) and (7), we arrive at the expression

$$\hat{\boldsymbol{h}} = (K_p \mathbf{I}_L + \sigma^2 \mathbf{C}_{\boldsymbol{h}}^{-1})^{-1} \mathbf{F}^\dagger \tilde{\mathbf{Y}} \,. \tag{8}$$

Since MMSE estimation still requires the inversion of $\mathbf{C}_{\boldsymbol{h}}^{-1}$, it therefore suffers from a high computational complexity. However, it is possible to reduce complexity of the MMSE algorithm by diagonalizing channel covariance matrix with an KL expansion.

B. KL Expansion

Channel impulse response $\boldsymbol{h}$ is a zero-mean Gaussian process with covariance matrix $\mathbf{C}_{\boldsymbol{h}}$. The KL transformation is therefore employed here to rotate the vector $\boldsymbol{h}$ so that all its components are uncorrelated. The vector $\boldsymbol{h}$, representing the channel impulse response during i^{th} OFDM block, can be expressed as a linear combination of the orthonormal basis vectors as follows:

$$\boldsymbol{h} = \sum_{l=0}^{L-1} g_l \boldsymbol{\psi}_l = \boldsymbol{\Psi} \boldsymbol{g} \tag{9}$$

where $\boldsymbol{\Psi} = [\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_{L-1}]$, $\boldsymbol{\psi}_l$'s are the orthonormal basis vectors, $\boldsymbol{g} = [g_0, g_1, \cdots g_{L-1}]^T$, and g_l is the weights of the expansion. If we form the covariance matrix $\mathbf{C}_{\boldsymbol{h}}$ as

$$\mathbf{C}_{\boldsymbol{h}} = \boldsymbol{\Psi} \boldsymbol{\Lambda}_{\boldsymbol{g}} \boldsymbol{\Psi}^\dagger \tag{10}$$

where $\boldsymbol{\Lambda}_{\boldsymbol{g}} = E\{\boldsymbol{g}\boldsymbol{g}^\dagger\}$, the KL expansion is the one in which $\boldsymbol{\Lambda}_{\boldsymbol{g}}$ of $\mathbf{C}_{\boldsymbol{h}}$ is a diagonal matrix (i.e., the coefficients are uncorrelated). If $\boldsymbol{\Lambda}_{\boldsymbol{g}}$ is diagonal, then the form $\boldsymbol{\Psi}\boldsymbol{\Lambda}_{\boldsymbol{g}}\boldsymbol{\Psi}^\dagger$ is called an *eigendecomposition* of $\mathbf{C}_{\boldsymbol{h}}$. The fact that only the eigenvectors diagonalize $\mathbf{C}_{\boldsymbol{h}}$ leads to the desirable property that the KL coefficients are uncorrelated. Furthermore, in Gaussian case, the uncorrelateness of the coefficients renders them independent as well, providing additional simplicity.

Thus, the channel estimation problem in this application is equivalent to estimating the iid complex Gaussian vector $\boldsymbol{g}$ KL expansion coefficients.

C. Estimation of KL Coefficients

In contrast to (4) in which only $\boldsymbol{h}$ is to be estimated, we now assume the KL coefficients $\boldsymbol{g}$ is unknown. Thus the data model (4) is rewritten for each OFDM block as

$$\tilde{\mathbf{Y}} = \mathbf{F}\boldsymbol{\Psi}\boldsymbol{g} + \tilde{\boldsymbol{\eta}} \tag{11}$$

which is also recognized as a Bayesian linear model, and recall that $\boldsymbol{g} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Lambda}_{\boldsymbol{g}})$. As a result, the MMSE estimator of $\boldsymbol{g}$ is

$$\begin{aligned} \hat{\boldsymbol{g}} &= \boldsymbol{\Lambda}_{\boldsymbol{g}} (K_p \boldsymbol{\Lambda}_{\boldsymbol{g}} + \sigma^2 \mathbf{I}_L)^{-1} \boldsymbol{\Psi}^\dagger \mathbf{F}^\dagger \tilde{\mathbf{Y}} \\ &= \boldsymbol{\Gamma}\, \boldsymbol{\Psi}^\dagger \mathbf{F}^\dagger \tilde{\mathbf{Y}} \end{aligned} \tag{12}$$

where

$$\begin{aligned} \boldsymbol{\Gamma} &= \boldsymbol{\Lambda}_{\boldsymbol{g}} (K_p \boldsymbol{\Lambda}_{\boldsymbol{g}} + \sigma^2 \mathbf{I}_L)^{-1} \\ &= \mathrm{diag}\left\{ \frac{\lambda_{g_0}}{\lambda_{g_0} K_p + \sigma^2}, \cdots, \frac{\lambda_{g_{L-1}}}{\lambda_{g_{L-1}} K_p + \sigma^2} \right\} \end{aligned} \tag{13}$$

and $\lambda_{g_0}, \lambda_{g_1}, \cdots, \lambda_{g_{L-1}}$ are the singular values of $\boldsymbol{\Lambda}_{\boldsymbol{g}}$.

It is clear that the complexity of the MMSE estimator in (8) is reduced by the application of KL expansion. However, the complexity of the $\hat{\boldsymbol{g}}$ can be further reduced by exploiting the optimal truncation property of the KL expansion [7].

D. Truncated KL Expansion

A truncated expansion $\boldsymbol{g}_r$ can be formed by selecting r orthonormal basis vectors among all basis vectors that satisfy $\mathbf{C}_{\boldsymbol{h}}\boldsymbol{\Psi} = \boldsymbol{\Psi}\boldsymbol{\Lambda}_{\boldsymbol{g}}$. The optimal one that yields the smallest average mean-squared truncation error $\frac{1}{L}\, E[\boldsymbol{\epsilon}_r^\dagger \boldsymbol{\epsilon}_r]$ is the one expanded with the orthonormal basis vectors associated with the first largest r eigenvalues as given by

$$\frac{1}{L}\, E[\boldsymbol{\epsilon}_r^\dagger \boldsymbol{\epsilon}_r] = \frac{1}{L} \sum_{i=r}^{L-1} \lambda_{g_i} \tag{14}$$

where $\boldsymbol{\epsilon}_r = \boldsymbol{g} - \boldsymbol{g}_r$. For the problem at hand, truncation property of the KL expansion results in a low-rank approximation as well. Thus, a rank-r approximation to $\boldsymbol{\Lambda}_{\boldsymbol{g}_r}$ is defined as

$$\boldsymbol{\Lambda}_{\boldsymbol{g}_r} = diag\left\{\lambda_{g_0}, \lambda_{g_1}, \cdots, \lambda_{g_{r-1}}, 0, \cdots, 0\right\} \,. \tag{15}$$

Since the trailing $L-r$ variances $\{\lambda_{g_l}\}_{l=r}^{L-1}$ are small compared to the leading r variances $\{\lambda_{g_l}\}_{l=0}^{r-1}$, then the trailing $L-r$ variances are set to zero to produce the approximation. However, typically the pattern of eigenvalues for $\boldsymbol{\Lambda}_{\boldsymbol{g}}$ splits the eigenvectors into dominant and subdominant sets. Then the choice of r is more or less obvious. The optimal truncated KL (rank-r) estimator of (12) now becomes

$$\hat{\boldsymbol{g}}_r = \boldsymbol{\Gamma}_r\, \boldsymbol{\Psi}^\dagger \mathbf{F}^\dagger \tilde{\mathbf{Y}} \tag{16}$$

where

$$\begin{aligned} \boldsymbol{\Gamma}_r &= \boldsymbol{\Lambda}_{\boldsymbol{g}_r} (K_p \boldsymbol{\Lambda}_{\boldsymbol{g}_r} + \sigma^2 \mathbf{I}_L)^{-1} \\ &= \mathrm{diag}\left\{ \frac{\lambda_{g_0}}{\lambda_{g_0} K_p + \sigma^2}, \cdots, \frac{\lambda_{g_{r-1}}}{\lambda_{g_{r-1}} K_p + \sigma^2}, 0, \cdots, 0 \right\} \,. \end{aligned} \tag{17}$$

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Since our ultimate goal is to obtain MMSE estimator for the channel frequency response $\mathbf{H}$, from the invariance property of the MMSE estimator, it follows that if $\hat{\boldsymbol{g}}$ is the estimate of $\boldsymbol{g}$, then the corresponding estimate of $\mathbf{H}$ can be obtained for the ith OFDM block as

$$\hat{\mathbf{H}} = \mathcal{F}\boldsymbol{\Psi}\hat{\boldsymbol{g}} \, . \tag{18}$$

IV. Performance Analysis

In this section, we turn our attention to analytical performance results. We first consider the CRB and derive the closed-form expression for the random KL coefficients. We then exploit the performance of the MMSE channel estimator based on the evaluation of minimum Bayesian MSE.

A. Cramer-Rao Bound for Random KL Coeficients

The mean-squared estimation error for any estimate of a non-random parameter has a lower bound, the *Cramer-Rao bound* (CRB), which defines the ultimate accuracy of any estimation procedure. Suppose $\hat{\boldsymbol{g}}$ is an unbiased estimator of a vector of unknown parameters $\boldsymbol{g}$ (i.e. $E\{\hat{\boldsymbol{g}}\} = \boldsymbol{g}$) then the mean-squared error matrix is lower bounded by a inverse of a Fisher information matrix (FIM):

$$E\left\{(\boldsymbol{g}-\hat{\boldsymbol{g}})(\boldsymbol{g}-\hat{\boldsymbol{g}})^{\dagger}\right\} \geq \mathbf{J}^{-1}(\boldsymbol{g}) \, . \tag{19}$$

Since we consider the estimation of unknown random parameters $\boldsymbol{g}$ via MMSE approach in this paper, the modified FIM needs to be taken into account in the derivation of stochastic CRB [9]. Fortunately, modified FIM can be obtained by a straightforward modification of the (19) as,

$$\mathbf{J}_M(\boldsymbol{g}) \triangleq \mathbf{J}(\boldsymbol{g}) + \mathbf{J}_P(\boldsymbol{g}) \tag{20}$$

where $\mathbf{J}_P(\boldsymbol{g})$ represents the a priori information.

Under the assumption that $\boldsymbol{g}$ and $\tilde{\boldsymbol{\eta}}$ are independent and $\tilde{\boldsymbol{\eta}}_p$ is a zero-mean, from [9] the conditional PDF is given by

$$p(\tilde{\mathbf{Y}}|\boldsymbol{g}) = \frac{1}{\pi^{K_p}|\mathbf{C}_{\tilde{\boldsymbol{\eta}}}|} \ \exp\{-(\tilde{\mathbf{Y}}-\mathbf{F}\boldsymbol{\Psi}\boldsymbol{g})^{\dagger}\mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1}(\tilde{\mathbf{Y}}-\mathbf{F}\boldsymbol{\Psi}\boldsymbol{g})\} \tag{21}$$

from which the derivatives follow as

$$\frac{\partial \ln p(\tilde{\mathbf{Y}}|\boldsymbol{g})}{\partial \boldsymbol{g}^T} = (\tilde{\mathbf{Y}}-\mathbf{F}\boldsymbol{\Psi}\boldsymbol{g})^{\dagger}\mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1}\mathbf{F}\boldsymbol{\Psi} \tag{22}$$

$$\frac{\partial^2 \ln p(\tilde{\mathbf{Y}}|\boldsymbol{g})}{\partial \boldsymbol{g}^* \partial \boldsymbol{g}^T} = -\boldsymbol{\Psi}^{\dagger}\mathbf{F}^{\dagger}\mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1}\mathbf{F}\boldsymbol{\Psi} \, . \tag{23}$$

Using $\mathbf{C}_{\tilde{\boldsymbol{\eta}}_p} = \sigma^2\mathbf{I}_{K_p}$, $\boldsymbol{\Psi}^H\boldsymbol{\Psi} = \mathbf{I}_L$ and $\mathbf{F}_p^H\mathbf{F}_p = K_p\mathbf{I}_L$, and taking the expected value yields the following simple form:

$$\begin{aligned} \mathbf{J}(\boldsymbol{g}) &= -E[\frac{\partial^2 \ln p(\tilde{\mathbf{Y}}_p|\boldsymbol{g})}{\partial \boldsymbol{g}^* \partial \boldsymbol{g}^T}] \\ &= -E[-\frac{K_p}{\sigma^2}\mathbf{I}_L] \\ &= \frac{K_p}{\sigma^2}\mathbf{I}_L \, . \end{aligned} \tag{24}$$

Second term in (20) is easily obtained as follows. Consider prior PDF of $\boldsymbol{g}$

$$p(\boldsymbol{g}) = \frac{1}{\pi^L|\boldsymbol{\Lambda}_{\boldsymbol{g}}|} \ \exp\{-\boldsymbol{g}^{\dagger}\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1}\boldsymbol{g}\} \, . \tag{25}$$

The derivatives are found as

$$\frac{\partial \ln p(\boldsymbol{g})}{\partial \boldsymbol{g}^T} = -\boldsymbol{g}^{\dagger}\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1} \tag{26}$$

$$\frac{\partial^2 \ln p(\boldsymbol{g})}{\partial \boldsymbol{g}^* \partial \boldsymbol{g}^T} = -\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1} \tag{27}$$

Upon taking the negative expectations, second term in (20) becomes

$$\begin{aligned} \mathbf{J}_P(\boldsymbol{g}) &= -E[\frac{\partial^2 \ln p(\boldsymbol{g})}{\partial \boldsymbol{g}^* \partial \boldsymbol{g}^T}] \\ &= -E[-\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1}] \\ &= \boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1} \end{aligned} \tag{28}$$

Substituting (24) and (28) in (20) produces for the modified FIM

$$\begin{aligned} \mathbf{J}_M(\boldsymbol{g}) &= \mathbf{J}(\boldsymbol{g}) + \mathbf{J}_P(\boldsymbol{g}) \\ &= \frac{K_p}{\sigma^2}\mathbf{I}_L + \boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1} \\ &= \frac{1}{\sigma^2}\left(K_p\mathbf{I}_L + \sigma^2\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1}\right) \\ &= \frac{1}{\sigma^2}\boldsymbol{\Gamma}^{-1} \, . \end{aligned} \tag{29}$$

Inverting the matrix $\mathbf{J}_M(\boldsymbol{g})$ yields

$$\begin{aligned} CRB(\hat{\boldsymbol{g}}) &= \mathbf{J}_M^{-1}(\boldsymbol{g}) \\ &= \sigma^2\,\boldsymbol{\Gamma} \, . \end{aligned} \tag{30}$$

B. Bayesian MSE

For the MMSE estimator $\hat{\boldsymbol{g}}$, the error is

$$\boldsymbol{\epsilon} = \boldsymbol{g} - \hat{\boldsymbol{g}} \, . \tag{31}$$

Since the diagonal entries of the covariance matrix of the error represent the minimum Bayesian MSE, we now derive covariance matrix of the error $\mathbf{C}_{\boldsymbol{\epsilon}}$. From *the Performance of the MMSE estimator for the Bayesian Linear model Theorem* [8], the error covariance matrix is obtained as

$$\begin{aligned} \mathbf{C}_{\boldsymbol{\epsilon}} &= \left(\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1} + (\mathbf{F}\boldsymbol{\Psi})^{\dagger}\mathbf{C}_{\tilde{\boldsymbol{\eta}}}^{-1}(\mathbf{F}\boldsymbol{\Psi})\right)^{-1} \\ &= \sigma^2\left(K_p\mathbf{I}_L + \sigma^2\boldsymbol{\Lambda}_{\boldsymbol{g}}^{-1}\right)^{-1} \\ &= \sigma^2\,\boldsymbol{\Gamma} \end{aligned} \tag{32}$$

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and the Bayesian MSE is

$$
\begin{aligned}
\mathbf{B}_{MSE}(\hat{\boldsymbol{g}}) &= \frac{1}{L}\,\mathrm{tr}\left(\mathbf{C}_{\boldsymbol{\epsilon}}\right) \\
&= \frac{1}{L}\mathrm{tr}\left(\sigma^2\boldsymbol{\Gamma}\right) \\
&= \frac{1}{L}\sum_{i=0}^{L-1}\frac{\lambda_{g_i}}{1+K_p\,\lambda_{g_i}SNR} \qquad (33)
\end{aligned}
$$

where $SNR = 1/\sigma^2$. Similarly, the Bayesian MSE for the low-rank case is

$$\mathbf{B}_{MSE}(\hat{\boldsymbol{g}_r}) = \frac{1}{L}\sum_{i=0}^{r-1}\frac{\lambda_{g_i}}{1+K_p\lambda_{g_i}SNR} + \frac{1}{L}\sum_{i=r}^{L-1}\lambda_{g_i}\,. \quad (34)$$

Comparing (30) with (32), the error covariance matrix of the MMSE estimator coincides with the stochastic CRB of the random vector estimator. Thus, the MMSE estimate of $\boldsymbol{g}$ achieves the stochastic CRB.

V. Simulations

In this section, we will illustrate the merits of our channel estimator through simulations. The figure of merit here is to average mean square error (MSE). In the simulation, number of subchannels(K),pilot space(Δ),number of channel taps(L), and rms value of path delays (τ_{rms}) are chosen as 1024, 20, 40, and 5 sample respectively.

The MSE at each SNR point is averaged over 500 realizations. We compare the experimental MSE performance and its theoretical Bayesian MSE of the proposed MMSE estimator with maximum likelihood (ML) estimator and its corresponding Cramer-Rao bound (CRB). Fig. 2 confirms that MMSE estimator performs better than ML estimator at low SNR. However, two approaches has comparable performance at high SNRs. To observe the performance, we also present the theoretical, MLE as well as MMSE estimated channel SER results in Fig. 3.

VI. Conclusion

We have developed a low complexity MMSE channel estimation scheme for OFDM systems. Modelling multipath channel as stochastic processes, KL expansion was employed to represent the correlated channel parameters with an i.i.d. Gaussian coefficients. Thus, KL representation allowed us to tackle the estimation of correlated multipath parameters as a parameter estimation problem of the uncorrelated coefficients resulting in reduced computational load in the MMSE channel estimation approach. Moreover, the performance of our proposed method was first studied through the derivation of stochastic CRB for Bayesian approach. Then the stochastic CRB result is compared with the MMSE estimator performance measure Bayesian MSE.

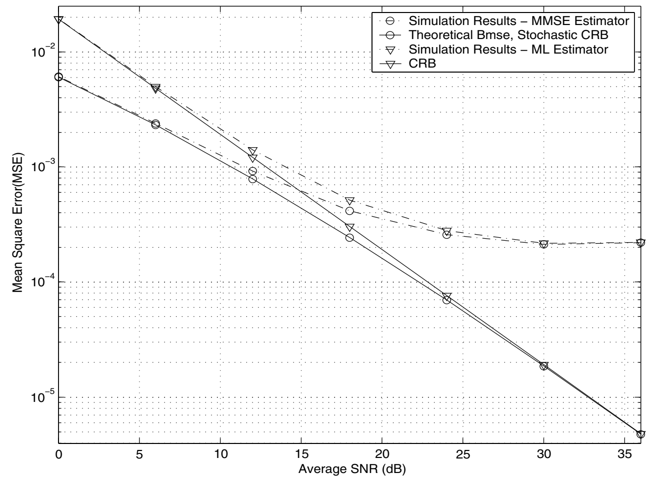

Fig. 2. Performance of Proposed MMSE and MLE together with Bmse and CRB

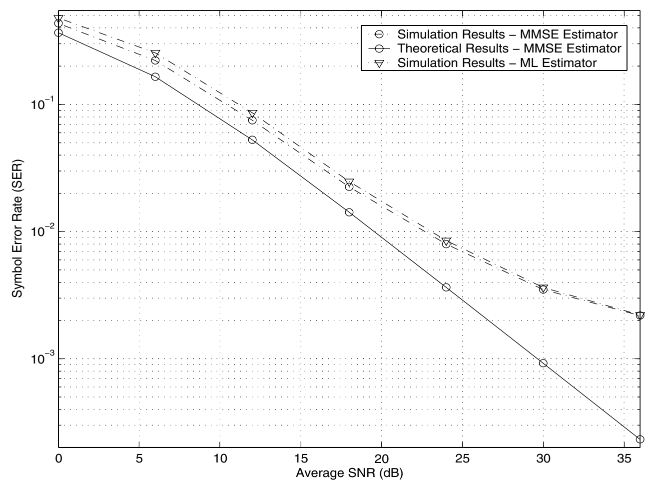

Fig. 3. Symbol Error Rate results

References

[1] R. Van Nee and R. Prasad, " OFDM Wireless Multimedia Communications", in *Artech House Publishers*, 2000.

[2] H. Sari, G. Karam, and I. Jeanclaude, "Transmission techniques for digital terrestrial TV broadcasting," *IEEE Commun. Mag.*, vol. 33, pp. 100-109, Feb., 1995.

[3] O. Edfords, M. Sandell, J.J. Van de Beek, S.K.. Wilson, and P.O. Borjesson, "OFDM Channel estimation by singular value decomposition," *IEEE Trans. on Commun.* vol. 46, pp. 931-938 july 1998.

[4] Y. (G.) Li, L. J. Cimini, and N. R. Sollenberger, "Robust channel estimation for OFDM systems with rapid dispersive fading channels," *IEEE Trans. Commun.*, Vol. 46, No.7, pages 902-914, July 1998.

[5] P. Schniter, Low-Complexity Estimation of Doubly-Selective Channels, IEEE workshop on Signal Processing Advances inWireless Communications, SPAWC-2003, Rome, Italy, 15-18 June 2003.

[6] W.C. Jakes, "Microwave Mobile Communications," *New York Plenum* 1974.

[7] K. Yip and T. Ng, "Karhunen-Loeve Expansion of the WSSUS Channel Output and its Application to Efficient Simulation," *IEEE Journal on Selected Areas in Communications,* vol. 15, no 4, pp.640-646, May 1997.

[8] S.M. Kay, "Fundamentals of Statistical Signal Processing: Estimation Theory," *Prentice Hall* 1993.

[9] H. L. Van Trees, "Detection, Estimation and Modulation Theory, Part I," *Wiley Interscience* 1993.