DE GRUYTER OPEN Open Math. 2017; 15: 77–91

Open Mathematics Open Access

Research Article

Fatma Kanca*

Determination of a diffusion coefficient in a quasilinear parabolic equation

DOI 10.1515/math-2017-0003
Received August 2, 2016; accepted October 21, 2016.

Abstract: This paper investigates the inverse problem of finding the time-dependent diffusion coefficient in a quasilinear parabolic equation with the nonlocal boundary and integral overdetermination conditions. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the solution are shown. Finally, some numerical experiments are presented.

Keywords: Heat equation, Inverse problem, Nonlocal boundary condition, Integral overdetermination condition, Time-dependent diffusion coefficient

MSC: 35K59, 35R30

1 Introduction

In this paper, an inverse problem of determining of the diffusion coefficient $a(t)$ has been considered with extra integral condition $\int_0^1 u(x,t)dx$ which has appeared in various applications in industry and engineering [1]. The mathematical model of this problem is as follows:

$$u_t = a(t)u_{xx} + f(x,t,u),\ (x,t) \in D_T := (0,1) \times (0,T) \tag{1}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1], \tag{2}$$

$$u(0,t) = u(1,t),\ u_x(1,t) = 0,\ t \in [0,T], \tag{3}$$

$$E(t) = \int_0^1 u(x,t)dx, 0 \le t \le T, \tag{4}$$

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions.

The problem of a coefficient identification in nonlinear parabolic equation is an interesting problem for many scientists [2–5]. In [6] the nature of (3)-type conditions is demonstrated.

In this study, we consider the inverse problem (1)-(4) with nonlocal boundary conditions and integral overdetermination condition. We prove the existence, uniqueness and continuous dependence on the data of the solution by applying the generalized Fourier method and we construct an iteration algorithm for the numerical solution of this problem.

The plan of this paper is as follows: In Section 2, the existence and uniqueness of the solution of inverse problem (1)-(4) is proved by using the Fourier method and iteration method. In Section 3, the continuous dependence upon the

***Corresponding Author: Fatma Kanca:** Department of Management Information Systems, Kadir Has University, 34083, Istanbul, Turkey, E-mail: fatma.kanca@khas.edu.tr

data of the inverse problem is shown. In Section 4, the numerical procedure for the solution of the inverse problem is given.

2 Existence and uniqueness of the solution of the inverse problem

We have the following assumptions on the data of the problem (1)-(4).
(A_1) $E(t) \in C^1[0,T]$, $E'(t) \leq 0$,

(A_2)

(1) $\varphi(x) \in C^4[0,1], \varphi(0) = \varphi(1), \varphi'(1) = 0, \varphi''(0) = \varphi''(1)$,

(2) $\varphi_{2k} \geq 0, k = 1, 2, ...$

(A_3)

(1) Let the function $f(x,t,u)$ be continuous with respect to all arguments in $\bar{D}_T \times (-\infty, \infty)$ and satisfy the following condition

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \leq b(x,t)\,|u - \tilde{u}|\ , n = 0, 1, 2,$$

where $b(x,t) \in L_2(D_T)$, $b(x,t) \geq 0$,

(2) $f(x,t,u) \in C^4[0,1], t \in [0,T]$, $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=1}$, $f_x(x,t,u)|_{x=1} = 0$, $f_{xx}(x,t,u)|_{x=0} = f_{xx}(x,t,u)|_{x=1}$,

(3) $f_{2k}(t) \geq 0$, $f_0(t) > 0$, $\forall t \in [0,T]$, where

$$\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx,\ f_k(t) = \int_0^1 f(x,t,u) Y_k(x) dx, k = 0, 1, 2, ...$$

$$X_0(x) = 2,\ X_{2k-1}(x) = 4\cos 2\pi kx,\ X_{2k}(x) = 4(1-x)\sin 2\pi kx,\ k = 1, 2, ...$$

$$Y_0(x) = x,\ Y_{2k-1}(x) = x\cos 2\pi kx,\ Y_{2k}(x) = \sin 2\pi kx,\ k = 1, 2, ...$$

The systems of functions $X_k(x)$ and $Y_k(x)$, $k = 0, 1, 2, ...$ are biorthonormal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ (see [7]).

We obtain the following representation for the solution of (1)-(3) for arbitrary $a(t)$ by using the Fourier method:

$$\begin{aligned} u(x,t) = & \left[\varphi_0 + \int_0^t f_0(\tau) d\tau \right] X_0(x) \\ & + \sum_{k=1}^{\infty} \left[\varphi_{2k}\, e^{-(2\pi k)^2 \int_0^t a(s)ds} + \int_0^t f_{2k}(\tau) d\tau\, e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right] X_{2k}(x) \\ & + \sum_{k=1}^{\infty} \left[(\varphi_{2k-1} - 4\pi k \varphi_{2k} t)\, e^{-(2\pi k)^2 \int_0^t a(s)ds} \right] X_{2k-1}(x) \\ & + \sum_{k=1}^{\infty} \left[\int_0^t (f_{2k-1}(\tau) - 4\pi k f_{2k}(\tau)(t-\tau))\, e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right] X_{2k-1}(x) \end{aligned} \tag{5}$$

Differentiating (5) we obtain

$$\int_0^1 u_t(x,t) dx = E'(t), 0 \leq t \leq T. \tag{6}$$

(5) and (6) yield

$$a(t)=\frac{-E'(t)+2f_0(t)+\sum\limits_{k=1}^{\infty}\frac{2}{\pi k}f_{2k}(t)}{\sum\limits_{k=1}^{\infty}8\pi k\left[\varphi_{2k}e^{-(2\pi k)^2\int\limits_0^t a(s)ds}+\int\limits_0^t f_{2k}(\tau)e^{-(2\pi k)^2\int\limits_\tau^t a(s)ds}d\tau\right]} \tag{7}$$

Definition 2.1. $\{u(t)\}=\{u_0(t),u_{2k}(t),u_{2k-1}(t),k=1,...,n\}$ *,are continuous functions on* $[0,T]$ *and satisfying the condition* $\max\limits_{0\le t\le T}|u_0(t)|+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{2k}(t)|+\max\limits_{0\le t\le T}|u_{2k-1}(t)|\right)<\infty$. *The set of these functions is denoted by* $\mathbf{B}_1$ *and the norm in* $\mathbf{B}_1$ *is* $\|u(t)\|=\max\limits_{0\le t\le T}|u_0(t)|+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|u_{2k}(t)|+\max\limits_{0\le t\le T}|u_{2k-1}(t)|\right)$. *It can be shown that* $\mathbf{B}_1$ *is the Banach space.*

Theorem 2.2. *If the assumptions* $(A_1)-(A_3)$ *are satisfied, then the inverse coefficient problem* (1)-(4) *has at most one solution for small T.*

Proof. We define an iteration for Fourier coefficient of (5) as follows:

$$\begin{aligned}
u_0^{(N+1)}(t)&=u_0^{(0)}(t)+\int\limits_0^t\int\limits_0^1 f(\xi,\tau,u^{(N)}(\xi,\tau))\xi d\xi d\tau\\
u_{2k}^{(N+1)}(t)&=u_{2k}^{(0)}(t)+\int\limits_0^t\int\limits_0^1 f(\xi,\tau,u^{(N)}(\xi,\tau))\sin 2\pi k\xi\, e^{-(2\pi k)^2\int\limits_\tau^t a^{(N)}(s)ds}d\xi d\tau\\
u_{2k-1}^{(N+1)}(t)&=u_{2k-1}^{(0)}(t)+\int\limits_0^t\int\limits_0^1 f(\xi,\tau,u^{(N)}(\xi,\tau))\xi\cos 2\pi k\xi\, e^{-(2\pi k)^2\int\limits_\tau^t a^{(N)}(s)ds}d\xi d\tau\\
&\quad-4\pi k\int\limits_0^t\int\limits_0^1(t-\tau)f(\xi,\tau,u^{(N)}(\xi,\tau))\sin 2\pi k\xi\, e^{-(2\pi k)^2\int\limits_\tau^t a^{(N)}(s)ds}d\xi d\tau
\end{aligned} \tag{8}$$

where $N=0,1,2,...$ and

$$u_0^{(0)}(t)=\varphi_0,u_{2k}^{(0)}(t)=\varphi_{2k}e^{-(2\pi k)^2\int\limits_0^t a(s)ds},u_{2k-1}^{(0)}(t)=(\varphi_{2k}-4\pi k t\varphi_{2k-1})\,e^{-(2\pi k)^2\int\limits_0^t a(s)ds}.$$

It is obvious that $u^{(0)}(t)\in\mathbf{B}_1$ and $a^{(0)}\in C[0,T]$.

For $N=0$,

$$u_0^{(1)}(t)=u_0^{(0)}(t)+\int\limits_0^t\int\limits_0^1[f(\xi,\tau,u^{(0)}(\xi,\tau))-f(\xi,\tau,0)]\xi d\xi d\tau+\int\limits_0^t\int\limits_0^1 f(\xi,\tau,0)\xi d\xi d\tau.$$

Let us apply Cauchy inequality,

$$\begin{aligned}
\left|u_0^{(1)}(t)\right|&\le|\varphi_0|+\left(\int\limits_0^t d\tau\right)^{\frac12}\left(\int\limits_0^t\left\{\int\limits_0^1[f(\xi,\tau,u^{(0)}(\xi,\tau))-f(\xi,\tau,0)]\xi d\xi\right\}^2 d\tau\right)^{\frac12}\\
&\quad+\left(\int\limits_0^t d\tau\right)^{\frac12}\left(\int\limits_0^t\left\{\int\limits_0^1 f(\xi,\tau,0)\xi d\xi\right\}^2 d\tau\right)^{\frac12}.
\end{aligned}$$

and with Lipschitz condition we obtain

$$\left|u_0^{(1)}(t)\right| \leq |\varphi_0| + \sqrt{t}\left(\int_0^t \left\{\int_0^1 b(\xi,\tau)\left|u^{(0)}(\xi,\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{t}\left(\int_0^t \left\{\int_0^1 f(\xi,\tau,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

If we take the maximum of the last inequality, we get the following estimation for $u_0^{(1)}(t)$.

$$\max_{0\leq t\leq T}\left|u_0^{(1)}(t)\right| \leq |\varphi_0| + \sqrt{T}\,\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1} + \sqrt{T}\,\|f(x,t,0)\|_{L_2(D_T)}.$$

$$u_{2k}^{(1)}(t) = \varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds} + \int_0^t\int_0^1 [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]\sin 2\pi k\xi e^{-(2\pi k)^2\int_\tau^t a(s)ds} d\xi d\tau$$
$$+\int_0^t\int_0^1 f(\xi,\tau,0)\sin 2\pi k\xi e^{-(2\pi k)^2\int_\tau^t a(s)ds} d\xi d\tau.$$

Let us apply Cauchy inequality,

$$\left|u_{2k}^{(1)}(t)\right| \leq |\varphi_{2k}| + \left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\int_0^1 [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)]\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\int_0^1 f(\xi,\tau,0)\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

and take the sum of the last inequality and partial derivative of f with respect to ξ and apply Hölder inequality,

$$\sum_{k=1}^{\infty}\left|u_{2k}^{(1)}(t)\right| \leq \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{1}{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}$$
$$\cdot\left(\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^1 [f_\xi(\xi,\tau,u^{(0)}(\xi,\tau)) - f_\xi(\xi,\tau,0)]\cos 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\frac{1}{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^1 f_\xi(\xi,\tau,0)\cos 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

By applying Bessel inequality we obtain

$$\sum_{k=1}^{\infty}\left|u_{2k}^{(1)}(t)\right| \leq \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{\sqrt{6T}}{12}\left(\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^1 [f_\xi(\xi,\tau,u^{(0)}(\xi,\tau)) - f_\xi(\xi,\tau,0)]\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\frac{\sqrt{6T}}{12}\left(\int_0^t\sum_{k=1}^{\infty}\left\{\int_0^1 f_\xi(\xi,\tau,0)d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

If we use Lipschitzs condition and take the maximum of the last inequality, we get the following estimation for $\sum_{k=1}^{\infty}\left|u_{2k}^{(1)}(t)\right|$.

$$\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{2k}^{(1)}(t)\right| \leq \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{\sqrt{6T}}{12}\,\|b(x,t)\|_{L_2(D)}\left\|u^{(0)}(t)\right\| + \frac{\sqrt{6T}}{12}\,\|f_x(x,t,0)\|_{L_2(D)}.$$

$$\sum_{k=1}^{\infty} \max_{0\le t\le T} \left|u_{2k}^{(1)}(t)\right| \le \sum_{k=1}^{\infty} |\varphi_{2k}| + \frac{\sqrt{6T}}{12} \|b(x,t)\|_{L_2(D_T)} \left\|u^{(0)}(t)\right\|_{\mathbf{B}_1} + \frac{\sqrt{6T}}{12} M.$$

$$u_{2k-1}^{(1)}(t) = u_{2k-1}^{(0)}(t) + \int_0^t \int_0^1 f(\xi,\tau,u^{(0)}(\xi,\tau))\xi \cos 2\pi k\xi \, e^{-(2\pi k)^2 \int_\tau^t a^{(0)}(s)ds} d\xi d\tau$$
$$-4\pi k \int_0^t \int_0^1 (t-\tau) f(\xi,\tau,u^{(0)}(\xi,\tau)) \sin 2\pi k\xi \, e^{-(2\pi k)^2 \int_\tau^t a^{(0)}(s)ds} d\xi d\tau.$$

Similarly, let us apply Cauchy inequality,

$$\left|u_{2k-1}^{(1)}(t)\right| \le |\varphi_{2k-1}| + 4\pi k t\, |\varphi_{2k}| + \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)] \cos 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+ \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 f(\xi,\tau,0) \cos 2\pi k\xi d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+4\pi k t \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 [f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)] \sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+4\pi k t \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 f(\xi,\tau,0) \sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

and take the sum of the last inequality and partial derivative of f with respect to ξ and apply Hölder inequality and Bessel inequality,

$$\sum_{k=1}^{\infty} \left|u_{2k-1}^{(1)}(t)\right| \le \sum_{k=1}^{\infty} |\varphi_{2k-1}| + \frac{t}{\sqrt{6}} \sum_{k=1}^{\infty} \left|\varphi_{2k}''\right|$$
$$+ \sum_{k=1}^{\infty} \frac{\sqrt{t}}{2\pi k} \left(\int_0^t \left\{\int_0^1 [f_\xi(\xi,\tau,u^{(0)}(\xi,\tau)) - f_\xi(\xi,\tau,0)] \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+ \sum_{k=1}^{\infty} \frac{\sqrt{t}}{2\pi k} \left(\int_0^t \left\{\int_0^1 f_\xi(\xi,\tau,0) \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+ \sum_{k=1}^{\infty} \frac{4\pi k t\sqrt{t}}{(2\pi k)^2} \left(\int_0^t \left\{\int_0^1 [f_{\xi\xi}(\xi,\tau,u^{(0)}(\xi,\tau)) - f_{\xi\xi}(\xi,\tau,0)] \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+ \sum_{k=1}^{\infty} \frac{4\pi k t\sqrt{t}}{(2\pi k)^2} \left(\int_0^t \left\{\int_0^1 f_{\xi\xi}(\xi,\tau,0) \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

If we use Lipschitzs condition and take the maximum of the last inequality, we get the following estimation for $\sum_{k=1}^{\infty} \left|u_{2k-1}^{(1)}(t)\right|$.

$$\sum_{k=1}^{\infty} \max_{0\le t\le T} \left|u_{2k-1}^{(1)}(t)\right| \le \sum_{k=1}^{\infty} |\varphi_{2k-1}| + \frac{\sqrt{6}T}{6} \sum_{k=1}^{\infty} \left|\varphi_{2k}''\right|$$

$$+\left(\frac{\sqrt{6T}}{12}+\frac{\sqrt{6T}T}{6}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}$$
$$+\left(\frac{\sqrt{6T}\pi}{12}+\frac{\sqrt{6T}T}{6}\right)M.$$

Finally we obtain the following inequality:

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_{\mathbf{B}_1} &= \max_{0\le t\le T}\left|u_0^{(1)}(t)\right|+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(1)}(t)\right|+\max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t)\right|\right)\\ &\le \|\varphi\|+\frac{\sqrt{6}T}{6}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right|\\ &\quad+\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}\\ &\quad+\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)M.\end{aligned}$$

where $\|\varphi\|=|\varphi_0|+4\left[|\varphi_{2k}|+|\varphi_{2k-1}|\right]$. Hence $u^{(1)}(t)\in\mathbf{B}_1$. In the same way, for N we have

$$\begin{aligned}\left\|u^{(N)}(t)\right\|_{\mathbf{B}_1} &= \max_{0\le t\le T}\left|u_0^{(N)}(t)\right|+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{2k-1}^{(N)}(t)\right|\right)\\ &\le \|\varphi\|+\frac{\sqrt{6}T}{6}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right|\\ &\quad+\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(N-1)}(t)\right\|_{\mathbf{B}_1}\\ &\quad+\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)M.\end{aligned}$$

Since $u^{(N-1)}(t)\in\mathbf{B}_1$, we have $u^{(N)}(t)\in\mathbf{B}_1$,

$$\{u(t)\}=\{u_0(t),u_{2k}(t),u_{2k-1}(t),\ k=1,2,...\}\in\mathbf{B}_1.$$

We define an iteration for (7) as follows:

$$a^{(N+1)}(t)=\frac{-E'(t)+\int_0^1 f(\xi,\tau,u^{(N)})dx}{\sum_{k=1}^{\infty}8\pi k\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^t a^{(N)}(s)ds}+\int_0^t\int_0^1 f(\xi,\tau,u^{(N)})\sin 2\pi k\xi e^{-(2\pi k)^2\int_0^t a^{(N)}(s)ds}d\xi d\tau\right]}$$

It is clear that $\int_0^1 f(\xi,\tau,u)dx=2f_0(t)+\sum_{k=1}^{\infty}\frac{2}{\pi k}f_{2k}(t)$. For $N=0$,

$$a^{(1)}(t)=\frac{-E'(t)+\int_0^1 f(\xi,\tau,u^{(0)})dx}{\sum_{k=1}^{\infty}8\pi k\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^t a^{(0)}(s)ds}+\int_0^t\int_0^1 f(\xi,\tau,u^{(0)})\sin 2\pi k\xi e^{-(2\pi k)^2\int_0^t a^{(0)}(s)ds}d\xi d\tau\right]}$$

Let us add and subtract $\int_0^1 f(\xi,\tau,0)d\xi d\tau$ to the last equation and use the Cauchy inequality and take the maximum to obtain:

$$\left\|a^{(1)}(t)\right\|_{C[0,T]}\le\frac{\left|E'(t)\right|}{C_2}+\frac{1}{C_2}\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{\mathbf{B}_1}+\frac{1}{C_2}M$$

where

$$C_2 = E(T) - 2\varphi_0 - 2\int_0^T f_0(\tau)d\tau.$$

Hence $a^{(1)}(t) \in C[0,T]$. In the same way, for N, we have

$$\left\|a^{(N)}(t)\right\|_{C[0,T]} \leq \frac{\left|E'(t)\right|}{C_2} + \frac{1}{C_2}\|b(x,t)\|_{L_2(D_T)}\left\|u^{(N-1)}(t)\right\|_{\mathbf{B}_1} + \frac{1}{C_2}M$$

Since $u^{(N-1)}(t) \in \mathbf{B}_1$, we have $a^{(N)}(t) \in C[0,T]$.

Now let us prove that the iterations $u^{(N+1)}(t)$ and $a^{(N+1)}(t)$ converge in $\mathbf{B}_1$ and $C[0,T]$, respectively, as $N \to \infty$.

$$\begin{aligned}
u^{(1)}(t) - u^{(0)}(t) &= \left(u_0^{(1)}(t) - u_0^{(0)}(t)\right) + \sum_{k=1}^{\infty}[(u_{2k}^{(1)}(t) - u_{2k}^{(0)}(t)) + (u_{2k-1}^{(1)}(t) - u_{2k-1}^{(0)}(t))] \\
&= \left(\int_0^t\int_0^1 \left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right]\xi d\xi d\tau\right) + \int_0^t\int_0^1 f(\xi,\tau,0)\xi d\xi d\tau \\
&\quad + \sum_{k=1}^{\infty}\int_0^t\int_0^1 \left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\sin 2\pi k\xi d\xi d\tau \\
&\quad + \sum_{k=1}^{\infty}\int_0^t\int_0^1 f(\xi,\tau,0)e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\sin 2\pi k\xi d\xi d\tau \\
&\quad + \sum_{k=1}^{\infty}\int_0^t\int_0^1 \left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\xi\cos 2\pi k\xi d\xi d\tau \\
&\quad + \sum_{k=1}^{\infty}\int_0^t\int_0^1 f(\xi,\tau,0)e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\xi\cos 2\pi k\xi d\xi d\tau \\
&\quad -16\pi k\sum_{k=1}^{\infty}\int_0^t\int_0^1 (t-\tau)\left[f(\xi,\tau,u^{(0)}(\xi,\tau)) - f(\xi,\tau,0)\right] e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\sin 2\pi k\xi d\xi d\tau \\
&\quad +16\pi k\sum_{k=1}^{\infty}\int_0^t\int_0^1 (t-\tau)\, f(\xi,\tau,0)e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\sin 2\pi k\xi d\xi d\tau
\end{aligned}$$

Applying Cauchy inequality, Hölder inequality, Lipshitzs condition and Bessel inequality to the last equation, we obtain:

$$\left\|u^{(1)}(t) - u^{(0)}(t)\right\|_{B_1} \leq \left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{B_1} + \left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3}\right)M.$$

$$K = \left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(0)}(t)\right\|_{B_1} + \left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3}\right)M.$$

$$\begin{aligned}
u^{(2)}(t) - u^{(1)}(t) &= \left(u_0^{(2)}(t) - u_0^{(1)}(t)\right) + \sum_{k=1}^{\infty}[(u_{2k}^{(2)}(t) - u_{2k}^{(1)}(t)) + (u_{2k-1}^{(2)}(t) - u_{2k-1}^{(1)}(t))] \\
&= \left(\int_0^t\int_0^1 \left[f(\xi,\tau,u^{(1)}(\xi,\tau)) - f(\xi,\tau,u^{(0)}(\xi,\tau))\right]\xi d\xi d\tau\right)
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^1\left[f(\xi,\tau,u^{(1)}(\xi,\tau))-f(\xi,\tau,u^{(0)}(\xi,\tau))\right]e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}\sin 2\pi k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^1 f(\xi,\tau,u^{(0)}(\xi,\tau))\left(e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}-e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\right)\sin 2\pi k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^1\left[f(\xi,\tau,u^{(1)}(\xi,\tau))-f(\xi,\tau,u^{(0)}(\xi,\tau))\right]e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}\xi\cos 2\pi k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^1 f(\xi,\tau,u^{(0)}(\xi,\tau))\left(e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}-e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\right)\xi\cos 2\pi k\xi d\xi d\tau$$

$$-16\pi k\sum_{k=1}^{\infty}\int_0^t\int_0^1(t-\tau)\left[f(\xi,\tau,u^{(1)}(\xi,\tau))-f(\xi,\tau,u^{(0)}(\xi,\tau))\right]e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}\sin 2\pi k\xi d\xi d\tau$$

$$-16\pi k\sum_{k=1}^{\infty}\int_0^t\int_0^1(t-\tau)\,f(\xi,\tau,u^{(0)}(\xi,\tau))\left(e^{-(2\pi k)^2\int_\tau^t a^{(1)}(s)ds}-e^{-(2\pi k)^2\int_\tau^t a^{(0)}(s)ds}\right)\sin 2\pi k\xi d\xi d\tau$$

Applying the same estimations we obtain:

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_{B_1}\le\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(1)}-u^{(0)}\right\|_{B_1}$$
$$+\left(\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)TM\left\|a^{(1)}-a^{(0)}\right\|_{B_2}.$$

$$a^{(1)}-a^{(0)}=\frac{-E'(t)+\int_0^1 f(\xi,\tau,u^{(1)})d\xi}{\sum_{k=1}^{\infty}8\pi k\left[\varphi_{2k}e^{-(2\pi k)^2\int_0^t a^{(1)}(s)ds}+\int_0^t\int_0^1 f(\xi,\tau,u^{(1)})\sin 2\pi k\xi e^{-(2\pi k)^2\int_0^t a^{(1)}(s)ds}d\xi d\tau\right]}$$
$$-\frac{-E'(t)+\int_0^1 f(\xi,\tau,u^{(0)})d\xi}{\sum_{k=1}^{\infty}8\pi k\left[\varphi_{sk}e^{-(2\pi k)^2\int_0^t a^{(0)}(s)ds}+\int_0^t\int_0^1 f(\xi,\tau,u^{(0)})\sin 2\pi k\xi e^{-(2\pi k)^2\int_0^t a^{(0)}(s)ds}d\xi d\tau\right]}$$

If we apply the Cauchy inequality, the Hölder Inequality, the Lipschitz condition and the Bessel inequality to the last equation, we obtain:

$$\left\|a^{(1)}-a^{(0)}\right\|_{C[0,T]}\le\left(\frac{\left|E'(t)\right|\pi}{2\sqrt{6}C_2^2}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{(4)}\right|+\frac{\left|E'(t)\right|M}{2\sqrt{6}C_2^2}+\frac{M}{2\sqrt{6}C_2^2}\sum_{k=1}^{\infty}\left|\varphi_{2k}^{(4)}\right|+M^2\right)T\left\|a^{(1)}-a^{(0)}\right\|_{C[0,T]}$$
$$+\left(\frac{2\left|E'(t)\right|}{\sqrt{6}C_2^2}+\frac{2}{\sqrt{6}C_2^2}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right|+M\right)\|b(x,t)\|_{L_2(D_T)}\left\|u^{(1)}-u^{(0)}\right\|_{\mathbf{B_1}}$$

$$A=\left(\frac{2\left|E'(t)\right|}{\sqrt{6}C_2^2}+\frac{2}{\sqrt{6}C_2^2}\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right|+M\right),$$

$$B = \left(\frac{\left|E'(t)\right| \pi}{2\sqrt{6}C_2^2} \sum_{k=1}^{\infty} \left|\varphi_{2k}^{(4)}\right| + \frac{\left|E'(t)\right| M}{2\sqrt{6}C_2^2} + \frac{M}{2\sqrt{6}C_2^2} \sum_{k=1}^{\infty} \left|\varphi_{2k}^{(4)}\right| + M^2 \right)$$

$$\left\| a^{(1)} - a^{(0)} \right\|_{C[0,T]} \leq \frac{A}{1-BT} \left\| b(x,t) \right\|_{L_2(D_T)} \left\| u^{(1)} - u^{(0)} \right\|_{\mathbf{B}_1}$$

$$\left\| u^{(2)}(t) - u^{(1)}(t) \right\|_{\mathbf{B}_1} \leq \left[\left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3} \right) + \left(\frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3} \right) \frac{MAT}{1-BT} \right] \left\| b(x,t) \right\|_{L_2(D_T)} K$$

$$C = \left(\sqrt{T} + \frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3} \right)$$

$$D = \left(\frac{\sqrt{6T}}{6} + \frac{\sqrt{6T}T}{3} \right)$$

$$\left\| u^{(2)}(t) - u^{(1)}(t) \right\|_{\mathbf{B}_1} \leq \left(C + D \frac{MAT}{1-BT} \right) \left\| b(x,t) \right\|_{L_2(D_T)} K$$

If we use the same estimations, we get

$$\left\| u^{(3)}(t) - u^{(2)}(t) \right\|_{\mathbf{B}_1} \leq \frac{1}{\sqrt{2}} \left(C + D \frac{MAT}{1-BT} \right)^2 \left\| b(x,t) \right\|_{L_2(D_T)}^2 K$$

For N :

$$\left\| a^{(N+1)} - a^{(N)} \right\|_{C[0,T]} \leq \frac{A}{1-BT} \left\| b(x,t) \right\|_{L_2(D_T)} \left\| u^{(N+1)} - u^{(N)} \right\|_{\mathbf{B}_1}$$

$$\left\| u^{(N+1)}(t) - u^{(N)}(t) \right\|_{\mathbf{B}_1} \leq \frac{K}{\sqrt{N!}} \left(C + D \frac{MAT}{1-BT} \right)^N \left\| b(x,t) \right\|_{L_2(D_T)}^N \tag{9}$$

It is easy to see that if $u^{(N+1)} \to u^{(N)}$, $N \to \infty$, then $a^{(N+1)} \to a^{(N)}$, $N \to \infty$. Therefore $u^{(N+1)}(t)$ and $a^{(N+1)}(t)$ convergence in $\mathbf{B}_1$ and $C[0,T]$, respectively.

Now let us show that there exist u and a such that

$$\lim_{N\to\infty} u^{(N+1)}(t) = u(t), \quad \lim_{N\to\infty} a^{(N+1)}(t) = a(t).$$

If we apply the Cauchy inequality, the Hölder Inequality, the Lipshitzs condition and the Bessel inequality to $\left|u - u^{(N+1)}\right|$ and $\left|a - a^{(N)}\right|$

$$\begin{aligned}
\left|u - u^{(N+1)}\right| \leq\ & C \left(\int_0^t \int_0^1 b^2(x,t) \left|u(\tau) - u^{(N+1)}(\tau)\right|^2 d\xi d\tau \right)^{\frac{1}{2}} \\
& + C \left(\int_0^t \int_0^1 b^2(x,t) \left|u^{(N+1)}(\tau) - u^{(N)}(\tau)\right|^2 d\xi d\tau \right)^{\frac{1}{2}} \\
& + D \left(\int_0^t \int_0^1 \left|a(\tau) - a^{(N)}(\tau)\right|^2 d\xi d\tau \right)^{\frac{1}{2}}
\end{aligned}$$

$$\left|a - a^{(N)}\right| \leq \frac{A}{1-BT} \left(\int_0^t \int_0^1 b^2(x,t) \left|u(\tau) - u^{(N+1)}(\tau)\right|^2 d\xi d\tau \right)^{\frac{1}{2}}$$

$$+\frac{A}{1-BT}\left(\int_0^t\int_0^1 b^2(x,t)\left|u^{(N+1)}(\tau)-u^{(N)}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

and the Gronwall inequality to the last inequality and using inequality (9), we have

$$\left\|u(t)-u^{(N+1)}(t)\right\|_{B_1}^2 \leq 2\left(\left(C+\frac{DMAT}{1-BT}\right)^{N+1}\frac{K}{\sqrt{N!}}\|b(x,t)\|_{L_2(D_T)}^{N+1}\right)^2 \times$$
$$\exp 2\left\{C+\frac{DMAT}{1-BT}\right\}^2\|b(x,t)\|_{L_2(D_T)}$$

Then $N\to\infty$ we obtain $u^{(N+1)}\to u$, $a^{(N+1)}\to a$.

Let us prove the uniqueness of these solutions. Assume that problem (1)-(4) has two solution pair (a,u), (b,v). Applying the Cauchy inequality, the Hölder Inequality, the Lipshitzs condition and the Bessel inequality to $|u(t)-v(t)|$ and $|a(t)-b(t)|$, we obtain:

$$|u(t)-v(t)| \leq \left(\frac{\sqrt{6}}{3}\left\{\sum_{k=1}^{\infty}\left|\varphi_{2k}'''\right|+\left|\varphi_{2k-1}'''\right|\right\}+\frac{\sqrt{6}T}{6}M\sum_{k=1}^{\infty}\left|\varphi_{2k-1}^{iv}\right|+\frac{\sqrt{6T}M}{3}+\frac{2\sqrt{6T}M}{3}\right)\times$$
$$T\left(\int_0^t\int_0^{\pi}|a(\tau)-b(\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\left(\sqrt{T}+\frac{\sqrt{6T}}{6}+\frac{\sqrt{6T}T}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}},$$

$$|a(t)-b(t)| \leq \frac{A}{1-BT}\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,|u(\tau)-v(\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}},$$

and applying the Gronwall inequality to the last inequality we have $u(t)=v(t)$. Hence $a(t)=b(t)$, here $T<\frac{1}{B}$.

The theorem is proved. □

3 Continuous dependence of solution upon the data

Theorem 3.1. *If the assumptions* $(A_1)-(A_3)$ *are satisfied, the solution (a,u) of problem* (1)-(4) *depends continuously upon the data* φ, E.

Proof. Let $\Phi=\{\varphi,\ E,\ f\}$ and $\overline{\Phi}=\left\{\overline{\varphi},\ \overline{E},\ f\right\}$ be two sets of the data, which satisfy the assumptions $(A_1)-(A_3)$. Suppose that there exist positive constants M_i, $i=0,1,2$ such that

$$\|E\|_{C^1[0,T]}\leq M_1, \left\|\overline{E}\right\|_{C^1[0,T]}\leq M_1, \|\varphi\|_{C^4[0,1]}\leq M_2, \|\overline{\varphi}\|_{C^4[0,1]}\leq M_2.$$

Let us denote $\|\Phi\|=(\|E\|_{C^1[0,T]}+\|\varphi\|_{C^4[0,1]}+\|f\|_{C^{4,0}(\overline{D}_T)})$. Let (a,u) and $(\overline{a},\overline{u})$ be solutions of (1)-(4) corresponding to the data $\Phi=\{\varphi,\ E,\ f\}$ and $\overline{\Phi}=\left\{\overline{\varphi},\ \overline{E},\ f\right\}$ respectively. According to (5), we have

$$|u-\overline{u}| \leq \|\varphi-\overline{\varphi}\|_{C^4[0,1]}$$
$$+\left(\frac{2\sqrt{6T}}{3}\left(\sum_{k=1}^{\infty}\left|\varphi_{2k}^{(4)}\right|+\left|\varphi_{2k-1}^{(4)}\right|+4\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right|\right)+\frac{2\sqrt{6T}M}{3}+\frac{2\sqrt{6T}}{3}\right)$$
$$\cdot T\left(\int_0^t\int_0^1|a(\tau)-\overline{a}(\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$+\left(2\sqrt{T}+\frac{2\sqrt{6T}\pi}{3}+\frac{\sqrt{6T}T}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$|a-\overline{a}|\leq M_3\left\|\varphi-\overline{\varphi}\right\|_{C^4[0,1]}+M_4\left\|E(t)-E^{'}(t)\right\|_{C^1[0,T]}$$
$$+M_5\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}+M_6T\left|a-\overline{a}\right|$$

$$(1-TM_6)\left|a-\overline{a}\right|\leq M_7\left(\left\|\Phi-\overline{\Phi}\right\|+\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}\right)$$

$$|a-\overline{a}|\leq M_8\left(\left|\Phi-\overline{\Phi}\right|+\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)^{\frac{1}{2}}\right)$$

$$|u-\overline{u}|^2\leq 2M_9^2\left\|\Phi-\overline{\Phi}\right\|^2+2M_{10}^2\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau)-\overline{u}(\tau)\right|^2 d\xi d\tau\right)$$

where $M_7=\max(M_3,M_4,M_6)$, $M_8=\frac{M_7}{1-TM_6}$, $M_{10}=M_8+\left(2\sqrt{T}+\frac{2\sqrt{6T}\pi}{3}+\frac{\sqrt{6T}T}{3}\right)$, $T<\frac{1}{M_6}$.

Applying the Gronwall inequality,

$$\|u-\overline{u}\|_{B_1}^2\leq 2M_9^2\left\|\Phi-\overline{\Phi}\right\|^2\times\exp 2M_{10}^2\left(\int_0^t\int_0^1 b^2(\xi,\tau)d\xi d\tau\right)^2.$$

For $\Phi\to\overline{\Phi}$ then $u\to\overline{u}$. Hence $a\to\overline{a}$. □

4 Numerical method for the problem (1)-(4)

In order to solve problem (1)-(4) numerically, we need the linearization of the nonlinear terms:

$$u_t^{(n)}=a(t)u_{xx}^{(n)}+f(x,t,u^{(n-1)}),\qquad (x,t)\in D_T \tag{10}$$
$$u^{(n)}(0,t)=u^{(n)}(1,t),\qquad t\in[0,T] \tag{11}$$
$$u_x^{(n)}(1,t)=0,\qquad t\in[0,T] \tag{12}$$
$$u^{(n)}(x,0)=\varphi(x)\,,\qquad x\in[0,1]\,. \tag{13}$$

Let $u^{(n)}(x,t)=v(x,t)$ and $f(x,t,u^{(n-1)})=\widetilde{f}(x,t)$. Then we obtain a linear problem:

$$v_t^{(n)}=a(t)v_{xx}^{(n)}+\widetilde{f}(x,t)\qquad (x,t)\in D_T \tag{14}$$
$$v(0,t)=v(1,t),\qquad t\in[0,T] \tag{15}$$
$$v_x(1,t)=0,\qquad t\in[0,T] \tag{16}$$
$$v(x,0)=\varphi(x),\qquad x\in[0,1]\ . \tag{17}$$

In this step, we use the implicit finite difference approximation for the discretizing problem (14)-(17):

$$\frac{1}{\tau}\left(v_i^{j+1}-v_i^j\right)=a^{j+1}\frac{1}{h^2}\left(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}\right)+\widetilde{f}_i^{j+1},$$

$$v_i^0 = \phi_i, \tag{18}$$

$$v_0^j = v_{N_x+1}^j, \tag{19}$$

$$v_{N_x-1}^j = v_{N_x+1}^j, \tag{20}$$

where $x = ih$, $t = j\tau$, $1 \le i \le N_x$ and $1 \le j \le N_t$, $v_i^j = v(x_i, t_j)$, $\phi_i = \varphi(x_i)$, $\widetilde{f}_i^j = \widetilde{f}(x_i, t_j)$, $x_i = ih$, $t_j = j\tau$.

Let us integrate the equation (1) with respect to x and use (3) and (4) to obtain

$$a(t) = \frac{-E'(t) + \int_0^1 \widetilde{f}(x,t)dx}{v_x(0,t)}. \tag{21}$$

The finite difference approximation of (21) is

$$a^j = \frac{\left[\left(\left(E^{j+1} - E^j\right)/\tau\right) - (Fin)^j\right]h}{v_1^j - v_0^j},$$

where $E^j = E(t_j)$, $(Fin)^j = \int_0^1 \widetilde{f}(x, t_j)dx$, $j = 0, 1, ..., N_t$. We mention that $\int_0^1 \widetilde{f}(x, t_j)dx$ is numerically calculated using Simpson's rule of integration.

$a^{j(s)}$, $v_i^{j(s)}$ are the values of a^j, v_i^j at the s-th iteration step, respectively. At each $(s+1)$-th iteration step, $a^{j+1(s+1)}$ is as follows

$$a^{j+1(s+1)} = \frac{\left[\left(\left(E^{j+2} - E^{j+1}\right)/\tau\right) - (Fin)^{j+1}\right]h}{v_1^{j+1(s)} - v_0^{j+1(s)}}.$$

The iteration of (18)-(20) is

$$\frac{1}{\tau}\left(v_i^{j+1(s+1)} - v_i^{j+1(s)}\right) = \frac{1}{h^2}a^{j+1(s+1)}\left(v_{i-1}^{j+1(s+1)} - 2v_i^{j+1(s+1)} + v_{i+1}^{j+1(s+1)}\right) + \widetilde{f}_i^{j+1}, \tag{22}$$

$$v_0^{j+1(s)} = v_{N_x}^{j+1(s)}, \tag{23}$$

$$v_1^{j+1(s)} = v_{N_x+1}^{j+1(s)},\ s = 0, 1, 2, \dots . \tag{24}$$

The system of equations (22)-(24) is solved by the Gauss elimination method and $v_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $a^{j+1(s+1)}$, $v_i^{j+1(s+1)}(i = 1, 2, ..., N_x)$ as a^{j+1}, $v_i^{j+1}(i = 1, 2, ..., N_x)$, on the $(j+1)$-th time step, respectively.

Example 4.1 (smooth diffusion coefficient)**.** *The first example investigates finding the exact solution*

$$\{a(t),\ u(x,t)\} = \left\{(t^2+2),\ \left(x^3 - 2x^2 + x + 5\right)\exp(-t)\right\}.$$

for the given functions

$$\varphi(x) = x^3 - 2x^2 + x + 5,\ E(t) = \frac{61}{12}\exp(-t),$$
$$F(x,t) = -u - (t^2+2)(6x-4)\exp(-t).$$

The step sizes are $h = 0.01$, $\tau = 0.005$.

The comparisons between the exact solution and the numerical finite difference solution are shown in Figures 1 and 2 when $T = 2$.

Fig. 1. The analytical and numerical solutions of $a(t)$ when $T = 2$ The analytical solution is shown with dashed line.

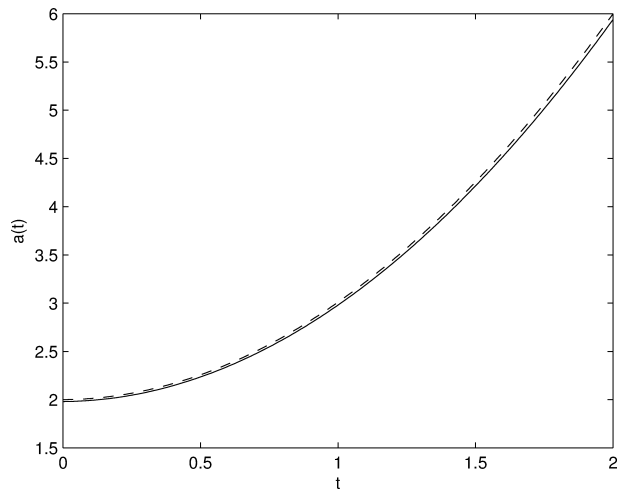
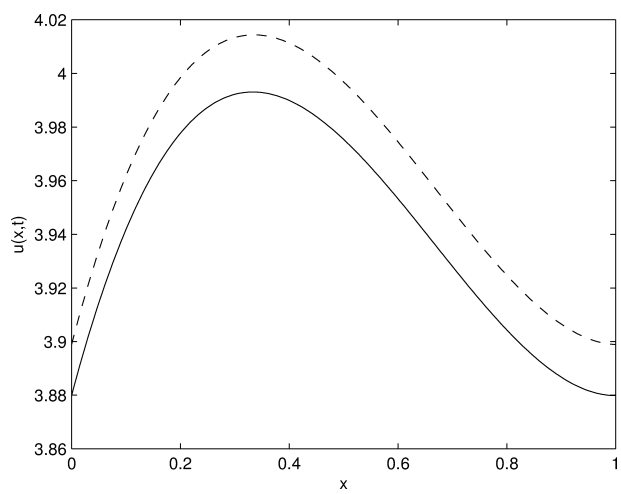


Fig. 2. The analytical and numerical solutions of $u(x,t)$ when $T = 2$ The analytical solution is shown with dashed line.



In order to investigate the stability of the numerical solution, noise is added to the overdetermination data (4) as follows

$$E_\gamma(t) = E(t)(1 + \gamma\theta), \tag{25}$$

where γ is the percentage of noise and θ are random variables generated from a uniform distribution in the interval $[-1, 1]$.

Figure 3 shows the exact and numerical solutions of $a(t)$ when the input data (4) are contaminated by $\gamma = 1\%$, 5% and 10% noise. From these figures it can be seen that the numerical solution becomes unstable as the input data is contaminated with noise. We use wavelet decomposition and thresholding to remove noise and we obtain Figure 4.

Example 4.2 (discontinuous diffusion coefficient)**.** *In the previous Example 4.1, a smooth function given by $a(t) = t^2 + 1$ is considered. In Example 4.2, a more severe discontinuous test function is given:*

$$a(t) = \begin{cases} -\left(t^2 + 2\right), t \in [0, 1) \\ \left(t^2 + 2\right), t \in [1, 2] \end{cases}$$

Let us apply the scheme above for the step sizes $h = 0.01$, $\tau = 0.005$. Figure 5 shows the exact and the numerical solutions of $a(t)$ when $T = 2$.

Fig. 3. The exact and approximate solutions of a(t),(a) for 1% noisy data, (b) for 5% noisy data, (c) for 10% noisy data. In figure (a)-(c) the exact solution is shown with dashed line.

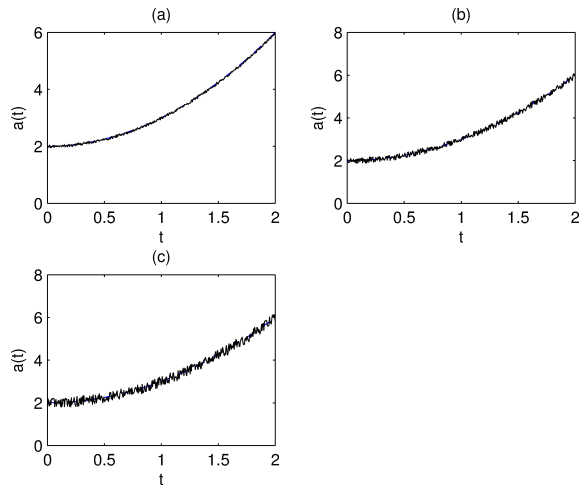


Fig. 4. The exact and approximate solutions of a(t), after thresholding, (a) for 3% noisy data, (b) for 5% noisy data, (c) for 10% noisy data. In figure (a)-(c) the exact solution is shown with dashed line.

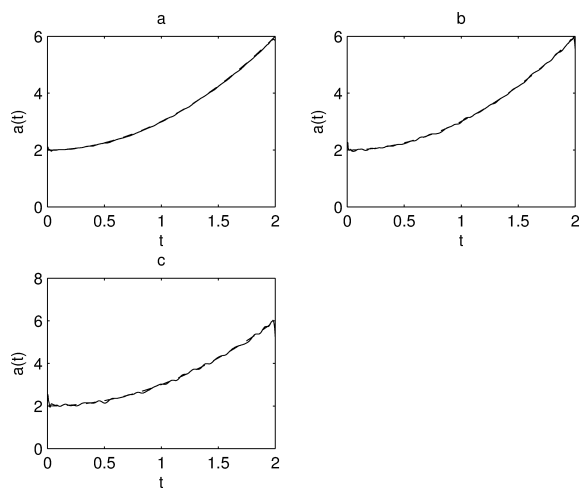
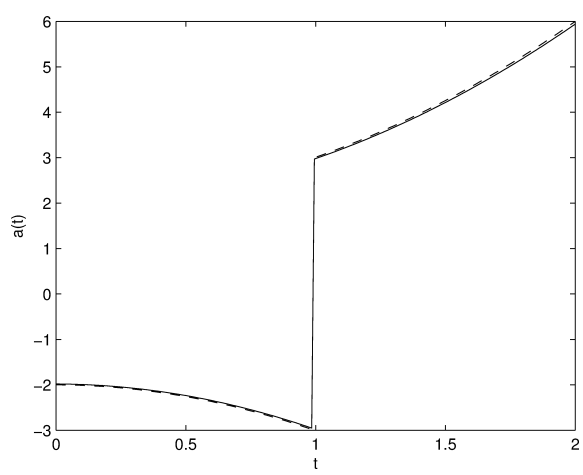


Fig. 5. The analytical and numerical solutions of $a(t)$ when $T = 2$ The analytical solution is shown with dashed line.



Some discussions

In the previous section, in Example 4.1, the man-made noise in the measured output data is added to show the stability of the numerical method. Unstable numerical solution is obtained and wavelet decomposition and thresholding are used to remove noise. Also in Example 4.2, discontinuous source function is given to show the efficiency of the present method. From Figure 5 it can be seen that the agreement between the numerical and exact solutions for $a(t)$ is excellent.

In future the fractional problem of this inverse problem can be studied [8–10].

References

[1] Ionkin NI., Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, Differential Equations, 1977, 13, 204-211.
[2] Cannon JR., Lin Y., Determination of parameter p(t) in Hölder classes for some semilinear parabolic equations, Inverse Problems, 1988, 4, 595-606.
[3] Pourgholia R, Rostamiana M and Emamjome M., A numerical method for solving a nonlinear inverse parabolic problem, Inverse Problems in Science and Engineering, 2010, 18(8), 1151-1164.
[4] Gatti S., An existence result for an inverse problem for a quasilinear parabolic equation, Inverse Problems, 1998;14: 53–65.
[5] Kanca F., Baglan I., An inverse coefficient problem for a quasilinear parabolic equation with nonlocal boundary conditions, Boundary Value Problems, 2013, 213.
[6] Nakhushev A. M., Equations of Mathematical Biology, Moscow, 1995 (in Russian).
[7] Ismailov M., Kanca F., An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions, Mathematical Methods in the Applied Science, 2011, 34, 692–702.
[8] Alkahtani Badr Saad T., Atangana A., Analysis of non-homogeneous heat model with new trend of derivative with fractional order, Chaos, Solitons & Fractals,. 2016, 89, 566-571.
[9] Alkahtani Badr Saad T, Atangana A., Modeling the potential energy field caused by mass density distribution with Eton approach, Open Physics, 2016, 14 (1), 106-113.
[10] Atangana A., On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation, Applied Mathematics and Computation, 2016, 273, 948-956.