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A Pseudo-Pareto Distribution and Concomitants of Its Order Statistics

Omer L. Gebizlioglu[1] · **Serap Yörübulut**[2]

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Abstract Pareto distributions are very flexible probability models with various forms and kinds. In this paper, a new bivariate Pseudo-Pareto distribution and its properties are presented and discussed. Main variables, order statistics and concomitants of this distribution are studied and their importance for risk and reliability analysis is explained. Joint and marginal distributions, complementing cumulative distributions and hazard functions of the variables are derived. Numerical illustrations, graphical displays and interpretations for the obtained distributions and derived functions are provided. An implementation example on defaultable bonds is performed.

Keywords Pseudo-Pareto distribution · Order statistics · Concomitants · Complementing cumulative distribution function · Hazard function · Defaultable bonds

Mathematics Subject Classification (2010) 60E05 · 62E15 · 62H10 · 91G70

1 Introduction

Pareto distributions have been the subject matter of many theoretical and applied studies in probability and statistics. These distributions are used in a wide perspective from social and natural sciences to engineering and medicine. Distribution of income between income groups, lifetimes of individuals, remaining life of a physical system, claim amounts in insurance and loss amounts in finance are the particular topics that Pareto distributions are used

✉ Omer L. Gebizlioglu
omer.gebizlioglu@khas.edu.tr

[1] Department of International Trade and Finance, Faculty of Economics, Administrative and Social Sciences, Kadir Has University, Istanbul, Turkey

[2] Department of Statistics, Faculty of Science and Letters, Kırıkkale University, Kırıkkale, Turkey

for probability modeling. Mardia (1962) introduced one of the oldest families of Pareto distributions which is now known as the multivariate Pareto distribution of the first kind. Different forms and families of univariate and multivariate Pareto distribution have been developed afterwards. Kotz et al. (2000) presented an almost complete account of these developments. A relatively recent review of the Pareto and generalized Pareto distributions was given by Arnold (2008). To date, research on various Pareto disributions with their application aspects in many actual life matters have been in continual progress. Application of the Pareto distributions in financial and actuarial risk areas is of particular concern for the present paper. In this regard, Papadakis and Tsionas (2010), Asimit et al. (2010) and Ahn et al. (2012) presented some new theoretical and application oriented results.

We introduce in this paper a new bivariate Pseudo-Pareto distribution that has the basic properties of the Pareto distribution of the first kind. Variables of this distribution, (X, Y), can be described as the main variable of interest X, and the concomitant variable Y which is associated to X depending upon a particular study goal. A sample of size n, $\{(X_i, Y_\mathrm{i}), i = 1, ..., n\}$, from this bivariate Psedo-Pareto distribution may come from a cross-sectional empirical study over an observation period or from a study over a sequence of periods in an observation time interval. Values in such a sample can be ordered in terms of the magnitudes of X_i's. The ordered n-tuple $(X_{1:n}, X_{2:n}, ..., X_{n:n})$,obtained in this way, defines order statistics $X_{r:n}$, $1 \leq r \leq n$, such that $X_{i:n} \leq X_{j:n}$, $i < j$, $j = 1, ..., n$. Then, each Y_i associated to an X_i in the sample becomes connected to the corresponding $X_{r:n}$ in the ordered n-tuple and we symbolize it with $Y_{[r:n]}$. The variable pair $(X_{r;n}, Y_{[r;n]})$ formed by such connections is named as the pair of r-th order satistic and r-th order concomitant and can be used as decision variables in applied studies along with (X, Y) pair.

Samples of variable-concomitant variable pairs can be observed in several contexts. For instance, variable X may represent a household income in a certain location of a country while Y is defined as a household income in a nearby location. Similarly, X may stand for loss quantities on returns from a stock of a corporation in a given industrial sector and Y may stand for a stock return loss of another corporation in the same capital market, both stocks being in trade simultaneously. It may also be the case that X and Y are monthly precipitation amounts in excess of certain levels at several sites in two distant but climotologically comparable areas of a geographical region.

Pseudo distribution of a random pair (X, Y) is obtained as a compound distribution. In the same line, Pseudo-Pareto distribution $F_{X,Y}\ (x, y)$ of $(X,\ Y)$ is also determined as a compound distribution. Distribution of the related $(X_{r;n}, Y_{[r;n]})$ pair is then obtained from the parent distribution $F_{X,Y}\ (x, y)$. Explicit computational procedures in this regard will be apparent in the following sections of the paper. There is a sound conceptual and methodological bacground for these procedures, and we refer to the works of David and Nagaraja (1998, 2003) for a comprehensive review on the essential theory and methodology in all aspects of order statistics and their concomitants.

Developments on the pseudo distributions are relatively new in the vast literature on satistical distributions. The main literature on the pseudo distributions and order statistics has grown up in the last two decades. Among them, the following are the relevant works for the subject matter of our paper. Diaz-Garcia et al. (1997) presented a pioneering work on Preudo-Wishart distributions, Filus and Filus (2006) applied their new class of pseudo-affine transformations on a set of independent random variables in two cases as Weibull and Gamma distributed variables and investigated the joint probability densities of the outcoming random vectors, Shahbaz and Ahmad (2009) proposed a bivariate Pseudo-Weibull distribution, Shahbaz et al. (2009) discussed the concomitants of order statistics for a bivariate Pseudo-Exponential distribution, an important paper by Filus et al. (2010) considered pseudo-Normal multivariate distributions with one coordinate variable having a Normal distribution and the others being determined by a specific triangular transformation model and obtained in this way a remarkably flexible family of distribution models, Ahsanullah et al. (2010) gave new results on the concomitants of upper record statistics for a bivariate Pseudo-Weibull distribution, Shahbaz and Ahmad (2011) studied a Pseudo-Rayleigh distribution, Shahbaz et al. (2011) defined afterwards a new bivariate Pseudo-Weibull distribution, and Yörübulut and Gebizlioglu (2013, 2014) introduced a bivariate Pseudo-Gompertz distribution and focused on its order statistics, upper record statistics and their concomitants. Lately, another influential paper by Filus and Filus (2013) presented a method of probability distribution constructions through their parameter dependence models that determine the conditional density of a random variable in a bivariate setting given any realization on the other. They discussed the method first for a bivariate Normal case with pseudo Normal extension and indicated that their paradigm can be generalized for many cases beyond the class of bivariate normal distributions.

A bivariate Pseudo-Pareto distribution for a random pair (X, Y) was proposed by Mohsin et al. (2012) with demonstrations of the moment-, maximum likelihood, and Bayesian methods for parameter estimations including a case analysis on a drought phenomenon. Their work has a parallesim to our work since we also introduce here a new bivariate Psedo-Pareto distribution. However, there are big differences between the two works. They defined the marginal density function of X as $f(x;\alpha,b) = \alpha b^{\alpha}(b+x)^{-\alpha-1}$ for $x>0$ with shape and scale parameters $\alpha>0$ and $b>0$, respectively. Their designation for the marginal distribution of Y contains a parametrization with function $\Phi(x) = \eta x^{\delta}$ on the scale parameter. On the contrary, we propose a bivariate Psedo-Pareto distribution with marginal density function of X defined as $f(x;\alpha,b) = \alpha b^{\alpha} x^{-\alpha-1}$, $\alpha>0$, $b>0$, $x>b$, and designate the shape parameter, not the scale parameter, of the marginal density function of Y with $\phi(x) = \ln(x)$ specification. Note also that the marginal densities of X and Y in our study come form the original Pareto density function of Mardia (1962). Furthermore, we focus deeply on the order statistic-concomitant pair, $(X_{r;n}, Y_{[r;n]})$, along with main variables (X, Y), derive new bivariate Pseudo-Pareto distributions for each of them, and treat the order statistic-concomitant pair as an extremely important decision making instrument for risk analysis. Our Pseudo-Pareto distribution proposal allows X and Y assume only those values that are larger than the scale parameters of their own marginal densities. This admits of a shape parameter for Y that depends on X, so the shape of the distribution of Y complies with that of X and makes X and Y positively correlated with each other. Obviously, all these features are analytically absorbed into the distributions that we obtain for the order statistic-concomitant variable pair.

The write up of our study is organized in five sections following the bacground, motivation and aim statements of this introduction. Section two introduces our main bivariate Pseudo-Pareto distribution and presents its distributional properties. Thereafter, section three shows the derivation of the distribution of order statistics and their concomitants under our Pseudo-Pareto distribution model. Section four elaborates on the complementing cumulative distribution functions (complementing CDF), or survival funcions, and hazard functions for (X, Y) and $(X_{r;n}, Y_{[r;n]})$. An implementation example on defaultable bonds is provided in section five that not only illustrates how the results of the paper can be applied but also gives a novel example for the finance area. Section six in the sequel concludes the paper.

2 The Bivariate Pseudo-Pareto Distribution

A bivariate Pseudo-Pareto distribution is a compound distribution of a pair of random variables (X, Y) whose marginals are Pareto type distributions. In resemblence to the known bivariate Pareto distribution of the first kind (Mardia 1962), we specify the following marginal Pareto density function for random variable X with shape parameter a_1 and scale parameter b_1

$$f_X(x; a_1, b_1) = a_1 b_1^{a_1} x^{-a_1-1}, \quad a_1 > 0,\ b_1 > 0,\ x > b_1$$

and write the conditional Pareto density function below for random variable Y with shape and scale parameters $\phi(x)$ and b_2 , respectively,

$$f_{Y|X=x}(y; \phi(x), b_2 | x) = \phi(x) b_2^{\phi(x)} y^{-\phi(x)-1}, \quad \phi(x) > 0,\ \ b_2 > 0, \qquad y > b_2$$

where $\phi(x)$ is a positive real valued function of X. The bivariate Pseudo-Pareto probability density function of (X, Y) is obtained from $f_{X,Y}(x, y) = f_X(x; a_1, b_1) f_{Y|X=x}(y; \phi(x), b_2 | x)$ as

$$f_{X,Y}(x, y) = a_1 b_1^{a_1} x^{-a_1-1} \phi(x) b_2^{\phi(x)} y^{-\phi(x)-1}, \quad \phi(x) > 0,\ \ a_1 > 0, x > b_1, \qquad y > b_2$$

implying that X and Y are in parameter dependence due to the existence of $\phi(x)$ function in the model for the purpose of shape parameter designations. Several bivariate Pseudo-Pareto type distributions can be obtained from this density structure by asserting various appropriate shape parameter functions for the conditional probability density of Y given a value of X. Distributions with this kind of flexibility can be used for many probability modeling attempts in bivariate analysis.

We have to refer to some other functions in the literature that are pertinent to the subject of this paper. Among them, we focus on the cumulative distribution function of the concomitant of an order statistics which is defined as

$$F_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} F(y|x) f_{X_{r:n}}(x) dx \quad , \quad f_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} f(y|x) f_{X_{r:n}}(x) dx$$

where $F(x|y)$ and $f(y|x)$ are the conditional distribution and density functions of (X, Y). The probability density function $f_{X_{r;n}}(x)$ of an r-th order statistic, $X_{r;n}$, is computed as

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x) \left[F_X(x)\right]^{r-1} \left[1 - F_X(x)\right]^{n-r}.$$

Furthermore, determination of the joint distribution of two order statistics, $X_{r:n} \le X_{s:n}$, follows from

$$\begin{aligned} f_{X_{r:n}, X_{s:n}}(x_1, x_2) \ = \ & \frac{n!}{(r-1)!\,(s-r)!\,(n-s)!} f(x_1) f(x_2) \left[F(x_1)\right]^{r-1} \\ & \times \left[F(x_2) - F(x_1)\right]^{s-r-1} \left[1 - F(x_2)\right]^{n-s}. \end{aligned}$$

The joint density function of $(X_{r:n}, X_{s:n})$ is obtained from

$$f_{Y_{[r:n]}, Y_{[s:n]}}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{X_{r:n}, X_{s:n}}(x_1, x_2) dx_1 dx_2.$$

Given all these prescriptions, we can present now a bivariate Pseudo-Pareto probability distribution for Y with the shape parameter specification $a_2 \equiv \phi(x) = ln(x)$. Under this specification, the bivariate probability density and cumulative distribution functions of (X, Y) are obtained as:

$$f_{X,Y}(x,y) = a_1 b_1^{a_1} x^{-a_1-1} ln(x) b_2^{ln(x)} y^{-ln(x)-1}, \quad a_1 > 0, x > b_1, \qquad y > b_2, \tag{1}$$

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{b_2}^{y}\int_{b_1}^{x} a_1 b_1^{a_1} x^{-a_1-1} ln(x) b_2^{ln(x)} y^{-ln(x)-1} dx dy \\ &= a_1 b_1^{a_1}\left(\frac{b_1^{-a_1} - (e^{a_1})^{-ln(x)}}{a_1} + \frac{-b_1^{-ln\left(\frac{e^{a_1}y}{b_2}\right)} + x^{-ln\left(\frac{e^{a_1}y}{b_2}\right)}}{ln\left(\frac{e^{a_1}y}{b_2}\right)}\right) \end{aligned} \tag{2}$$

where $F_{X,Y}(x, y)$ satisfies all the conditions to be a proper cumulative distribution function, including the most important property that $\lim_{x,y\to\infty} F_{X,Y}(x, y) = 1$.

The marginal Pseudo-Pareto distribution functions of X and Y follow from equation (2):

$$F_X(x) = \lim_{y\to b_2} F_{X,Y}(x,y) = 1 - b_1^{a_1} x^{-a_1}, \tag{3}$$

$$F_Y(y) = \lim_{x\to b_1} F_{X,Y}(x,y) = a_1 b_1 \left(\frac{b_2^{-a_1}}{a_1} - \frac{b_2^{-ln\left(\frac{ye^{a_1}}{b_2}\right)}}{ln\left(\frac{ye^{a_1}}{b_2}\right)}\right). \tag{4}$$

The respective marginal probability density functions for $X$and Y are found from these equations as:

$$f_X(x) = a_1 b_1^{a_1} x^{-a_1-1}, \quad a_1 > 0,\ b_1 > 0,\ x > b_1, \tag{5}$$

$$\begin{aligned} f_Y(y) &= \int_{b_1}^{\infty} a_1 b_1^{a_1} x^{-a_1-1} ln(x) b_2^{ln(x)} y^{-ln(x)-1} dx \\ &= \frac{a_1 b_1 b_2^{-ln\left(\frac{ye^{a_1}}{b_2}\right)}\left(1 + ln(b_2)\, ln\left(\frac{ye^{a_1}}{b_2}\right)\right)}{y\left(ln\left(\frac{ye^{a_1}}{b_2}\right)\right)^2}, \end{aligned} \tag{6}$$

where $f_Y(y)$ exists only when $\ln(ye^{a_1}/b_2) > 0, \ln(b_2) > 0, a_1 > 0, b_1 > 0, x > b_1$.

Values of the shape parameter a_1 in equation (2) has a direct effect on the right tails of $F_{X,Y}(x, y)$, $F_X(x)$ and $F_Y(y)$, shown above. That is to say that $a_1$plays the tail index role for our Pseudo-Pareto distribution. A probability statement for a random event defined on the right tail of $F_{X,Y}(x, y)$ can be better expressed with the complementing CDF of (X, Y):

$$S_{X,Y}(x,y) = \Pr\{X > x, Y > y\} = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y).$$

Note that complementing CDFs are named also as survival functions in the rich literature on life sciences and systems reliability analysis. A complementing CDF with a heavy right tail implies that the probability of large value realizations on its underlying variables is considerably high. The complementing CDF of (X, Y) obtains the explicit expression below on the basis of distributions given in (2), (3) and (4) above:

$$S_{X,Y}(x, y) = b_1^{a_1} x^{-a_1} - a_1 b_1 \left(\frac{b_2^{-a_1}}{a_1} - \frac{b_2^{-ln\left(\frac{ye^{a_1}}{b_2}\right)}}{ln\left(\frac{ye^{a_1}}{b_2}\right)} \right) + a_1 b_1^{a_1} \left(\frac{b_1^{-a_1} - (e^{a_1})^{-ln(x)}}{a_1} + \frac{-b_1^{-ln\left(\frac{e^{a_1} y}{b_2}\right)} + x^{-ln\left(\frac{e^{a_1} y}{b_2}\right)}}{ln\left(\frac{e^{a_1} y}{b_2}\right)} \right). \quad (7)$$

An essential theory and methodology on the complementing CDFs can be found in the text of London (1997), among others, where the matters of estimation for complementing CDFs are shown lucidly with several applications.

Figure 1 below display the plots of $f_{X,Y}(x, y)$ and $S_{X,Y}(x, y)$ functions for different values of the shape parameter. It is seen in the plots that the complementing CDF function converges to the (x,y) surface at a faster rate as the value of the shape parameter a_1 gets larger. If the rate of this convergence is faster on X values in comparison to Y, then Y is considered to be more risky. This implies that, if (X, Y) here stands for two joint loss amounts, the risk of loss attributable to Y will be more than what it can be for X, because of more likely large Y values.

This section completes the derivations of cumulative distribution function, density function and complementing CDF of (X, Y). These results enable us next to derive the bivariate Pseudo-Pareto distribution for the $(X_{r;n}, Y_{[r;n]})$ pair.

$a_1 = 1, \quad b_1 = 1, \quad b_2 = 1$ $\qquad$ $a_1 = 3, \quad b_1 = 1, \quad b_2 = 1$

a Probability density function $f_{X,Y}(x,y)$

$a_1 = 1, \quad b_1 = 1, \quad b_2 = 1$ $\qquad$ $a_1 = 3, \quad b_1 = 1, \quad b_2 = 1$

b Complementing cumulative distribution function $S_{X,Y}(x,y)$

Fig. 1 Plots of the density function and cumulative CDF of the Pseudo-Pareto distribution

3 Probability Functions for the r-th Order Statistic and its Concomitant

We devote this section to the derivation of the complementing CDFs and density functions of r-th order statistics and their $r-$th order concomitants. For this purpose, values of the scale parameters b_1 and b_2 have to be set larger than 1 in order to ensure a positive $\phi(x) = ln(x)$ under the condition that $x > b_1$.

Density function of the r-th order statistic $X_{r;n}$, $f_{r;n}(x)$, and the conditional density function

$$f_{Y|X=x}(y|x) = ln(x) b_2^{ln(x)} y^{-ln(x)-1} \tag{8}$$

have to be used together in order to derive the density function of r-th concomitant:

$$\begin{aligned} f_{Y_{[r:n]}}(y) &= \frac{a_1 n!}{(r-1)!\,(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \\ &\quad \times \int_{b_1}^{\infty} b_1^{a_1(n-r+h+1)} x^{-a_1(n-r+h+1)-1} ln(x) b_2^{ln(x)} y^{-ln(x)-1} dx \\ &= \frac{a_1 n!}{(r-1)!\,(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{1}{y\left(a_1(n-r+h+1)+ln(y)\right)^2} \\ &= \frac{n!}{a_1 y (r-1)!\,(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{1}{\left(h+n-r+1+\frac{ln(y)}{a_1}\right)^2}. \end{aligned} \tag{9}$$

Computation of this function requires lenghty summations. So, a computationally easy form of the function can be written by using the results and methods given in Connon (2007) and Gradshteyn and Ryzhik (2007). On this pupose, we let a part of the divisor in the summation component of (9) be denoted by $a \equiv n - r + 1(\ln(y)/a_1)$ and obtain the following result for the concerned sum:

$$\sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{1}{\left(h+n-r+1+\frac{ln(y)}{a_1}\right)^2} = \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{1}{(h+a)^2}.$$

This result can be put even in a more simplified form. To do this, we let

$$\begin{aligned} \sum_{h=0}^{r-1} \binom{r-1}{h} \frac{(-1)^h}{h+a} &= \frac{(r-1)!}{a(1+a)...(r-1+a)} \equiv \omega(a) \\ &= a^{-1} \binom{r-1+a}{r-1}^{-1}, \qquad a \notin (0, -1, ..., -(r-1)), \end{aligned}$$

and notice that function $\omega(a)$ has a connection with the well known Gamma function

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt = \frac{\Gamma(a+1)}{a} = \frac{\Gamma(a+2)}{a(a+1)} = ... = \frac{\Gamma(a+r)}{a(a+1)...(a+r-1))}, \qquad a > 0,$$

so we proceed with the following expression for the summation:

$$\omega(a) = \sum_{h=0}^{r-1} \binom{r-1}{h} \frac{(-1)^h}{h+a} = \frac{(r-1)!}{a(a+1)...(a+r-1))} = \frac{(r-1)!\Gamma(a)}{\Gamma(a+r)} \quad .$$

This function contains a harmonic number of order n that we denote by $H_{(n)}$ and define as:

$$H_{(n)} = \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \frac{1}{k}.$$

There exists a particular relation between the differences of $H_{(a+r-1)}$ and $H_{(n)}$quantities:

$$H_{(a+r-1)} - H_{(a-1)} = \{\psi(a+r) - \psi(a)\} = \sum_{h=a}^{a+r-1} \frac{1}{h},$$

where $\psi(\cdot)$'s are the so called digamma functions defined as the derivatives of $\log\Gamma(\cdot)$:

$$\psi(a) = \frac{d}{da}\log\Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}.$$

Hence, we can express the summation component of equation (9) in a much better way with

$$\begin{aligned}\sum_{h=0}^{r-1} \binom{r-1}{h} \frac{(-1)^h}{(h+a)^2} &= \omega(a)\{\psi(a+r) - \psi(a)\} \\ &= \frac{(r-1)!\Gamma(a)}{\Gamma(a+r)} \left(H_{(a+r-1)} - H_{(a-1)}\right)\end{aligned}$$

and reach to the following re-expression for the density function of $Y_{[r;n]}$:

$$f_{Y_{[r;n]}}(y) = \frac{\Gamma(n+1)}{a_1 y \Gamma(n-r+1)} \frac{\Gamma\left(\frac{ln(y)}{a_1} + n - r + 1\right)}{\Gamma\left(\frac{ln(y)}{a_1} + n + 1\right)} \left(H_{\left(\frac{ln(y)}{a_1}+n\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r\right)}\right). \tag{10}$$

The complementing CDF of r-th concomitant is then obtained as shown below:

$$\begin{aligned}F_{Y_{[r:n]}}(y) &= \int_1^y f_{Y_{[r:n]}}(t)dt \\ &= \frac{a_1 n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \int_1^y \frac{1}{t\left(a_1(n-r+h+1) + ln(t)\right)^2} dt \\ &= \frac{a_1 n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \int_0^{ln(y)} \frac{1}{\left(a_1(n-r+h+1) + u\right)^2} du \\ &= \frac{a_1 n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \\ &\quad \times \frac{ln(y)}{a_1(n-r+h+1)\left(a_1(n-r+h+1) + ln(y)\right)}.\end{aligned} \tag{11}$$

This function can be finally put in a more handy form in terms of the Gamma functions. Thus, utilizing the equality

$$\sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} \frac{1}{(h+c)} \left(\frac{b}{(h+c+b)}\right) = \frac{\Gamma(c)\Gamma(r)}{\Gamma(c+r)} - \frac{\Gamma(c+b)\Gamma(r)}{\Gamma(c+b+r)},$$

with $c \equiv n - r + 1$ and $b \equiv \ln(y)/a_1$ for notational simplicity, we reach to the following computationally tidy re-expression for (11):

$$F_{Y_{[r:n]}}(y) = 1 - \frac{\Gamma(n+1)\Gamma\left(n - r + (ln(y)/a_1) + 1\right)}{\Gamma(n-r+1)\Gamma\left(n + (ln(y)/a_1) + 1\right)}. \tag{12}$$

The cumulative distribution and probability density functions that we have produced in this section are fundamental for further distributional derivations on ($X_{r:n}$,$Y_{[r;n]}$). So, we pass on the results of this section to the next one where the other essential equations are derived especially for applications.

4 Complementing CDF and Hazard Function for the r-th Order Statistic and its Concomitant

Complementing CDFs and hazard functions are the most critical functions for risk analysis and reliability studies since they can describe with their right tail behaviors the occurrence probabilities for large and risky amounts. Hazard functions are actually derived from the complementing CDFs and used to express the conditional intensity of loss generating risk events. Considering a sample on loss amounts (X, Y), order r of $X_{r;n}$ in the sample specifies an r/n-th sample percentile for $F_X(x)$ as a bencmarking loss level in risk assessments. So, using an r-th order concomitant $Y_{[r;n]}$ corresponding to a specific r-th order statistic $X_{r;n}$ allow us to carry out risk analysis on loss amount Y in conjunction with a benchmark value of order rank r. In this regard, if $F_Y(y)$ bears a heavier right tail in comparison to a heavy right tailed $F_X(x)$ and a loss amount on $Y_{[r;n]}$ is considered along with a large r value choice in the analysis, it should mean that the risk of loss with regard to concomitant variable Y is overwhelmingly threatening.

Complementing CDF of the r-th order concomitant can be computed for any r as follows:

$$S_{Y_{[r:n]}}(y) = 1 - F_{Y_{[r;n]}} = \frac{\Gamma(n+1)\Gamma\left(n - r + (ln(y)/a_1) + 1\right)}{\Gamma(n-r+1)\Gamma\left(n + (ln(y)/a_1) + 1\right)}. \tag{13}$$

Since the hazard function for a random variable W is defined as $h(w) = f(w)[S(w)]^{-1}$, we can obtain the hazard function of r-th concomitant accordingly:

$$h_{Y_{[r:n]}}(y) = \frac{f_{Y_{[r:n]}}(y)}{S_{Y_{[r:n]}}(y)} = \frac{f_{Y_{[r:n]}}(y)}{1 - F_{Y_{[r:n]}}(y)} \quad = (a_1 y)^{-1}\left(H_{\left(\frac{ln(y)}{a_1}+n\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r\right)}\right). \tag{14}$$

Behavior of the complementing CDF and hazard function of $Y_{[r:n]}$ is important for risk evaluations and we can show an analytical study on this matter by several partial derivatives. The most important partial derivatives in this regard are given below:

$$\frac{\partial}{\partial y} S_{Y_{[r:n]}}(y) = -\frac{\Gamma(n+1)\Gamma\left(n-r+(ln(y)/a_1)+1\right)\left[\psi\left(n+(ln(y)/a_1)+1\right) - \psi\left(n-r+(ln(y)/a_1)+1\right)\right]}{a_1 y \Gamma(n-r+1)\Gamma\left(n+(ln(y)/a_1)+1\right)},$$

$$\frac{\partial}{\partial y} h_{Y_{[r:n]}}(y) = -\frac{\left(H_{\left(\frac{ln(y)}{a_1}+n\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r\right)}\right)}{a_1 y^2} - \frac{\left(H_{\left(\frac{ln(y)}{a_1}+n,2\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)}\right)}{(a_1 y)^2}$$

where $\psi(\cdot)$ is a digamma function and $H_{(\cdot)}$ is a harmonic number which we have already mentioned for the derivation of the probability density function of r-th concomitant. Our numerical assessments reveal that the instantaneous rate of change (IRC) in the complementing CDF under concern obtains increasing values at each ascending y value, whereas

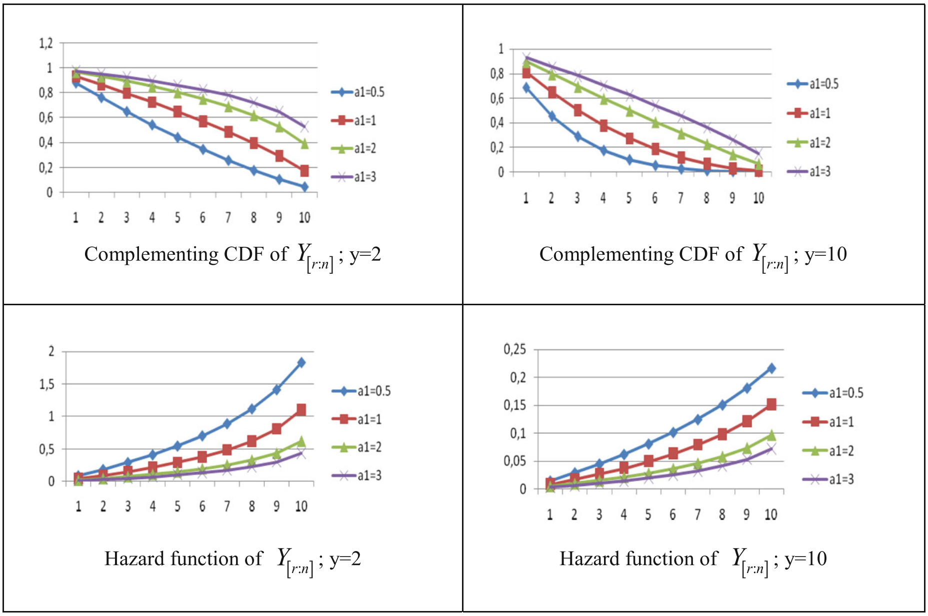

Fig. 2 Plots for the complementing CDF and hazard function of r-th concomitant $Y_{[r:n]}$

the instantaneous rate of change in the hazard function acquires decreasing values at each inreasing value of y.

It is obvious that r and a_1 have similarly strong influences on the instantaneous rate of changes in the complementing CDF and the hazard function of $Y_{[r;n]}$. In this regard, partial derivatives of these functions with respect to r, a_1 and y are demonstrated in Appendix 1. Some graphical displays of the marginal situations given there are presented in Fig. 2, below. The vertical axes of the plots in this Figure bear the values of complementing CDFs and hazard functions of $Y_{[r;n]}$, while r values are shown on the horizontal axis of each plot.

It is seen in Fig. 2 that the complementing CDFs of $Y_{[r;n]}$ with different parameter value assignments decline faster as the values of a_1 and r increase, and they tend to converge on each other at large y values. On the other hand, the hazard function for $Y_{[r;n]}$ assumes large values for large r and shows a faster increment as r grows larger. We observe that, compared to the large values of a_1, smaller a_1 values have a stronger effect on the right tail thicknesses of the complementing CDFs. Terefore, it is important to recognize that inclusion of a small valued a_1 and a large valued r in the complementing CDF of $Y_{[r;n]}$, as well as $X_{\mathrm{r;n}}$, is actually a sign of high loss amount anticipation for a risk bearing engagement.

Now, we derive the joint complementing CDF and joint hazard function for the pair $(X_{r;n}, Y_{[r;n]})$. In order to perform these derivations, we first need the complementing CDF and hazard function of $X_{r;n}$. Up to this aim, the distribution function of $X_{r:n}$ is obtained:

$$F_{X_{r:n}}(x) = \int_1^x f_{X_{[r:n]}}(x)dx = 1 - \frac{x^{-a_1(n-r+1)} n!{}_2F_1\left[n-r+1, 1-r, n-r+2, x^{-a_1}\right]}{(n-r+1)(n-r)!(r-1)!}. \tag{15}$$

Then, the complementing CDF of r-th order statistic follows:

$$S_{X_{r:n}}(x) \equiv 1 - F_{X_{r:n}}(x) = \frac{x^{-a_1(n-r+1)} n! {}_2F_1\left[n-r+1, 1-r, n-r+2, x^{-a_1}\right]}{(n-r+1)(n-r)!(r-1)!} \quad (16)$$

where ${}_2F_1[\cdot]$ is a Gauss Hypergeometric function (Gradshteyn and Ryzhik 2007) which has the following general form:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

The hazard function of$X_{r:n}$ follows from definion $h_{X_{r:n}}(x) = f_{X_{r:n}}(x) / S_{X_{r:n}}(x)$ and it is obtained as:

$$h_{X_{r:n}}(x) = \frac{\frac{a_1 n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} x^{-a_1(n-r+h+1)-1}}{\frac{x^{-a_1(n-r+1)} n! {}_2F_1\left(n-r+1, 1-r, n-r+2, x^{-a_1}\right)}{(n-r+1)(n-r)!(r-1)!}}. \quad (17)$$

Derivation of the bivariate complementing CDF and hazard function for $(X_{r;n}, Y_{[r;n]})$ requires the use of$S_{X_{r:n},Y_{[r:n]}}(x, y) = S(y|x) S_{X_{r:n}}(x)$ and $h_{X_{r:n},Y_{[r:n]}}(x, y) = h(y|x) h_{X_{r:n}}(x)$ functions. On this purpose we have to make use of the joint probability density functions $f(x, y) = a_1 x^{-a_1-1} ln(x) y^{-ln(x)-1}$, $f_{X_{r:n},Y_{[r:n]}}(x, y) = f(y|x) f_{X_{r:n}}(x)$, and the conditional probability density function $f_{Y_{[r:n]}}(y \,|X_{r:n} = x) = f(y|x)$ for the concerned variables, given that $a_1 > 0$, $x > 1$, $y > 1$. Then, using the following complementing CDF expressions:

$$S(y|x) = \frac{S(x, y)}{S(x)}, \quad S(x, y) = x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)} \quad , \quad S(x) = x^{-a_1} \quad ,$$

we obtain below the conditional complementing CDF for (X, Y):

$$S(y|x) = \frac{x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)}}{x^{-a_1}}, \quad (18)$$

and using the formal definition $S_{X_{r:n},Y_{[r:n]}}(x, y) = S(y|x) S_{X_{r:n}}(x)$, we come up with the following joint complementing CDF expression for $(X_{r;n}, Y_{[r;n]})$:

$$S_{X_{r;n},Y_{[r;n]}}(x, y) = \frac{x^{-a_1(n-r)} n! {}_2F_1\left[n-r+1, 1-r, n-r+2, x^{-a_1}\right]}{(n-r+1)(n-r)!(r-1)!} \times \left(x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)}\right). \quad (19)$$

Determination of the corresponding joint hazard function for (X, Y) can be achieved in connection with the join density function $f(x, y)$ and obtained as:

$$h(x, y) = \frac{a_1 x^{-a_1-1} ln(x) y^{-ln(x)-1}}{x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)}}. \quad (20)$$

Then, the conditional hazard function for $(X_{r;n}, Y_{[r;n]})$ can be produced:

$$h(y\,|x) = \frac{h(x, y)}{h(x)} = \frac{a_1 x^{-a_1-1} ln(x) y^{-ln(x)-1}}{a_1 x^{-a_1-1} \left(x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)}\right)} = \frac{x^{-a_1} ln(x) y^{-ln(x)-1}}{\left(x^{-a_1} - a_1 \frac{\left(x^{-a_1 - ln(y)} - 1\right)}{a_1 + ln(y)}\right)} \quad (21)$$

where $h(x) = a_1 x^{-1}$is the marginal hazard function of X. Putting all these results together, the consequent joint hazard function for $(X_{r;n}, Y_{[r;n]})$ is obtained:

$$
\begin{aligned}
h_{X_{r:n},Y_{[r:n]}}(x,y) &= h(y|x)h_{X_{r:n}}(x) \\
&= \frac{x^{-a_1} ln(x) y^{-ln(x)-1}}{\left(x^{-a_1} - a_1 \frac{\left(x^{-a_1-ln(y)}-1\right)}{a_1+ln(y)}\right)} \\
&\quad \times \frac{\frac{a_1 n!}{(r-1)!(n-r)!} \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} x^{-a_1(n-r+h+1)-1}}{\frac{x^{-a_1(n-r+1)} n!\ {}_2F_1\left(n-r+1,1-r,n-r+2,x^{-a_1}\right)}{(n-r+1)(n-r)!(r-1)!}} \\
&= \frac{ln(x) y^{-ln(x)-1}}{\left(x^{-a_1} - a_1 \frac{\left(x^{-a_1-ln(y)}-1\right)}{a_1+ln(y)}\right)} \\
&\quad \times \frac{a_1 \sum_{h=0}^{r-1} (-1)^h \binom{r-1}{h} x^{-a_1(n-r+h+1)-1}}{\frac{x^{-a_1(n-r)} {}_2F_1\left(n-r+1,1-r,n-r+2,x^{-a_1}\right)}{(n-r+1)}}
\end{aligned}
\tag{22}
$$

where ${}_2\mathrm{F}_1(\cdot)$ is what we have already defined for the equation in (16).

Having all the required tools and functions ready for implementations, we devote the next part of our paper to an example on the risky bonds of financial markets.

5 An Implementation Example on Defaultable Bonds

This section presents an application on defaultable bonds and their default generating events. A brief preminary explanation on the bonds and bond markets will be given first, then the implementation example that make use of our results will be provided.

Bonds are fixed income securities for investors and debt capital instruments for corporations and organizations. A bond is identified and specified by its issuer name, par or face value, coupons, coupon rate and maturity. Maturity of a bond is the length of time from its issue date to the preset date for its par value redemption. An optional call provision may also exist on a bond that allows a recall from its issuer prior to maturity. In case of failures in coupon payments and par value redemption for a bond, bond holders can exercise their right to put claims on the assets and incomes of its issuer. These claims are generally honored with a priority over the claims that may come from such securities like common and preferred stocks. This means that implicit value of a defaultable bond to an investor is also a function of the assets of its issuer. In this connection, bonds in debt capital markets are rated by the rating institutions with some standard risk qualifications. A full recovery or even a partial recovery of coupon payout and par redemption defaults of a high risk rated bond may not be fulfilled. Therefore, there is always a need for some reliable credit exposure estimates for the holders and potential investors of risk bearing bonds. In this respect, bond rating institutions publish some quick indicators for risky bonds. These indicators usually come out as a ratio of annual interest payout of a bond to its current market price, known also as the current return on a bond. A sudden and sharp decline in such a ratio is perceived as a strong signal for uprising default risks. In a more general perspective, a decreasing firm value of a bond issuer is a signal of perils for its bond holders. Such a situation is reflected out by an upward movement of a bond issuer company's debt-to-equity ratio, or simply debt ratio, for bond holders' attention.

It is known that value of a coupon bearing bond at its time of issue, say t_0, is expressed by

$$V_{t_0} = \sum_{t=1}^{T} \frac{c_t}{(1+k)^t} + \frac{m}{(1+k)^T} \tag{23}$$

with t:coupon payout period, T: maturity period, $\mathrm{c_t}$: coupon payable at the end of period t, m: par value of the bond that is redeemable at the end of its maturity period, and k: required rate of return (RRR) for a bond investor. If k in (23) secures that the value of a bond equals to its current market price at t_0, then $k$is named yield to maturity (YTM) for the bond. Coupon payout periods of bonds are given in terms of years, generally. When a bond's value at its issue time is equal to its par value, required rate of return and coupon interest rate values for the bond become the same. In the case that required rate of return of a bond exceeds its coupon interest rate, value of the bond falls below of its par value and then it becomes a discounted bond. In the opposite situation, a bonds gains a premium bond status. Credit rating institutions, like Moody's and S&P, rate a large number of bond issuers annually from the highest rating class of "AAA" to the junk bond rating class "D". Hence, default proneness of the bonds in a bond market is periodically informed to bond holders and investors in the way of credit ratings. As a result, a bond with a low rating can find buyers only at relatively cheaper prices as compared to higher rated ones.

There is a vast literature on defaultable bonds and other defaultable financial securities. In relation to our implementation example here, we refer to the works of Altman (1989), Altman and Heather (2000), Kijima (2003, 127-139) and Tapiero (2004). An interesting and highly informative account of the defaultable US corporate bonds was given lately by Giesecke et al. (2010) in a 150-year perspective.

Historical and current information on many bonds in debt capital markets come from the data bases of debt capital market authorities and bond rating institutions. A collection of bonds in a bond data base is called a bond cohort as long as they are contemporary, of the same maturity and marketability, and comparable with respect to par values and coupons they bear. Bond data bases provide information on the bonds usually in terms of bond cohorts that include in issuer names, risk ratings for issuers, issue dates, maturity dates, coupon interest rates, par values, primary market prices, realized loss amounts on coupons and par value redemptions, issuer companys' debt ratios, and secondary market bond prices. A bond cohorts data with this content of information in a current and past time perspective make it possible to conduct reliable bond default studies on the bonds of interest in a bond cohort given that the data at hand are valid and viable for future time evaluations in a current year, say year τ.

We assume for our example that investors in one year τ have an interest on "minimum investment grade-BBB" and "lower grade-BB" rating class bonds with no call provisions, among the other financial securities that they may have in their portfolios. Investors may associate a BB rated bond with a BBB rated one in their portfolios upon the rationale that by doing so the can abide by their budgetary constraints. In this way, investors may help themselves to optimize their portfolios on the higher priced BBB and lower priced BB rated bonds while staying within the limits of their investment budgets.

Bond defaults and loss amounts for BBB and BB rated bonds in a historical bond cohorts data base with a near past and present time spanning at a given time $\breve{\tau}$ can be described by the following two-tuples:

$$\{({}_{\ddot{\iota},\iota}\breve{c}_{t,BBB}, {}_{\ddot{\iota},\iota}\breve{m}_{BBB});\ \iota = 1, ..., n_{BBB-\ddot{\iota}-\breve{\tau}}, t = t_1, ..., T)\},$$
$$\{({}_{\ddot{\nu},\acute{\iota}}\breve{c}_{BB}, {}_{\ddot{\nu},\acute{\iota}}\breve{m}_{BB});\ \acute{\iota} = 1, ..., n_{BB-\ddot{\nu}-\breve{\tau}}, t = t_1, ..., T)\}$$

where the cohorts of BBB and BB rated bonds are indicated with indices $\ddot{\iota}$ =1,... ,N_{BBB} and $\ddot{\nu}$ =1,..., N_{BB}, respectively. Here, $\breve{\tau}$ denotes bond issue year, ι and $\acute{\iota}$ denote unique bonds in the cohorts, and ${}_{.,.}\breve{c}_{.,.}$ and ${}_{.,.}\breve{m}_{.}$ denote respective realized loss amount values on coupon payouts, c_t, and redeemable par values, m. These loss amount observations can be within the upfront known limits of $0\leq {}_{\ddot{\iota},\iota}\breve{c}_{t,BBB} \leq {}_{\ddot{\iota},\iota}c_{t,BBB}$, $0 \leq {}_{\ddot{\iota},\acute{\iota}}\breve{c}_{t,BBB} \leq, {}_{\ddot{\iota},\acute{\iota}}c_{t,BBB}$, $0 \leq {}_{\ddot{\iota},\iota}\breve{m}_{BBB} \leq {}_{\ddot{\iota},\iota}m_{BBB}$ and $0\leq {}_{\ddot{\nu},\acute{\iota}}\breve{m}_{BBB} \leq {}_{\ddot{\nu},\acute{\iota}}c_{BBB}$. The entirity of such a data can be qualified as a populaton data with total size $N_{BBB}+N_{BB}$ for the sample studies on BBB and BB rated defaultable bonds.

Random variables X and Y in the context of our paper are now let to stand for respective total default loss amounts for BBB and BB rated bond cohorts over the time duration $[t_0, T]$. So, we let X denote a total loss amount $X = \Sigma_t(\breve{c}_{t,BBB} + \breve{m}_{BBB})$ for any BBB rated bond, and Y a total loss amount $Y = \Sigma_t(\breve{c}_{t,BB} + \breve{m}_{BB})$for any$BB$ rated bond, where$t_1 \leq t \leq T$. Probability distributions of $X$and Y can be estimated from such a default loss amounts data that we have described above. Several parametric or non-parametric estimation techniques can be used up to this aim. We refer to London (1997), Altman (1989), Altman and Heather (2000)) for a good coverage of these techniques. We presume for our implementation example that the marginal Pseudo-Pareto distributions, given in equations (3) and (4), come out from such an estimation endeavor as the most plausible loss amount distribution models for the BBB and BB rated bonds that investors may consider for investment. In this regard, a sample $\{(X_i, Y_i)$, i =1,.,n$\}$ of size n on the BBB and BB rated bonds under concern, all with issue year $\breve{\tau}$, is nothing but a sample from the bivariate Pseudo-Pareto distribution that we wrote in expression (2). It is clear that the values of $X$and Y in this sample can be the observations from the value intervals $0 \leq X \leq \Sigma_t(\breve{c}_{t,BBB} + \breve{m}_{BBB})$ and $0 \leq Y \leq \Sigma_t(\breve{c}_{t,BB} + \breve{m}_{BB})$ depicting the possible loss totalities on coupon payments and par values.

An r-th order statistics from the sample $\{(X_i, Y_i)$, i=1,...,$n\}$ is a very important decision making variable for risk analysis on bond defaults since, as we said in Section 4, it is indicative of an r/n-th percentile for any distribution as well as the Pseudo-Pareto distribution of X under our concern. Risk positions for Y, being associated to X as a concomitant, are also indicated by the values of r and r-th order statistics. Hence, inverstors' choice on r becomes a sign of their risk taking behavior for the potential loss amounts X and Y because a large value of r points a large loss amount anticipation for the concerned bonds, especially for the more risky BB rated ones. Therefore, the pair of the r-th order statistic and its concomitant, $(X_{r;n}, Y_{[r;n]})$, as well as the main pair (X, Y) itself, are to be perceived in the risk of loss analysis as very crucial decision factors for defaultable bonds.This need can be satisfied in the best way by using the marginal distributions, complementing CDFs and joint complementing CDF of $(X_{r;n}, Y_{[r;n]})$ given in our expressions (11), (13), (15), (16) and (19), in respective order.

A further need emerges for the following conditional probabilities about $(X_{r;n}, Y_{[r;n]})$ in order to make the risk generating event probability statements that investors have to consider for their probable loss amounts:

$$\Pr\left(X_{r:n} > x + u \,|X_{r:n} > x\,\right) = \frac{S_{X_{r:n}}\,(x+u)}{S_{X_{r:n}}\,(x)} ={}_u\ p_x, \tag{24}$$

$$\Pr\left(Y_{[r:n]} > y + v \,\middle|\, Y_{[r:n]} > y\right) = \frac{S_{Y_{[r:n]}}(y + v)}{S_{Y_{[r:n]}}(y)} =_v p_y, \tag{25}$$

$$\Pr\left(X_{r:n} > x + u, Y_{[r:n]} > y + v \,\middle|\, X_{r:n} > x, Y_{[r:n]} > y\right) = \frac{S_{X_{r:n},Y_{[r:n]}}(x + u, y + v)}{S_{X_{r:n},Y_{[r:n]}}(x, y)} =_{u,v} p_{x,y}. \tag{26}$$

The first of these conditional probabilities state the probability that after realizing a loss amount x on $X_{r;n}$ for a BBB rated bond, an additional default loss amount may be realized leading to a loss that can be as much as x_{+}u. Similar interpretations can be made for the other two conditional probability expressions above, the last one being a conditional joint default loss event probability concerning $X_{r;n}$ and $Y_{[r;n]}$ in the loss anticipation concerns of investors. Computation of these conditional probabilites can be performed using the complementing CDF expressions given in (13), (16) and (19).

Along with the above given complementing CDFs, we need the hazard rate functions for $X_{r;n}$, $Y_{[r;n]}$ and $(X_{r;n}, Y_{[r;n]})$ that we gave in (14), (17) and (22), respectively. Hazard functions help us to express the severities of anticipated loss occurrences on BBB and BB rated bonds. Computation of such severities, as of a given year τ, can be performed by several present value functions that we will denote by PVτ (.), hereafter. Actually, these present values are expected risk expressions for anticipated future loss amounts. If loss amount variables $X$and Y are continuous random variables, computation of the expected risks can be achieved by:

$$PV_\tau(\overline{\overline{X}}) = \int_0^{C_{BBB}+m_{BBB}} (g_1(u))({}_u p_x)(h_{X_{r.n}}(x + u))du \tag{27}$$

for $X_{r:n}$,

$$PV_\tau(\overline{\overline{Y}}) = \int_0^{C_{BB}+m_{BBB}} (g_2(v))({}_u p_y)(h_{Y_{r.n}}(y + v))dv \tag{28}$$

for $Y_{[r:n]}$, and

$$PV_\tau(\overline{\overline{X}}, \overline{\overline{Y}}) = \int_0^{C_{BBB}+m_{BBB}} \int_0^{C_{BB}+m_{BB}} g_{12}(u, v)({}_{u,v} p_{x,y})(h_{X_{r,n},Y_{[r,n]}}(x+u, y+v))dvdu \tag{29}$$

jointly for $(X_{r;n}, Y_{[r;n)})$, where x and y can be be chosen by investors as their already incurred loss amounts that they stand by in view of future loss amounts. $C_{BBB} + m_{BBB}$ and $C_{BB} + m_{BB}$, upper limits of the integrations above, denote the respective sums of coupon payouts and par values of BBB and BB rated bonds. And, $g_1(u)$, $g_1(v)$, $g_{12}(u, v)$ denote year τ bound discounted loss amount values for the expected risk computations.

Random loss amounts X and Y are usually apprehended as discrete values in practice. This is due to the fact that bond value expressions, as shown in (23), contain annual coupon payout and redeemable par quantities usually in discrete values. Actually, investors may consider the largest possible integer valued loss amounts in these computations for more risk aversive decisions. Accordingly, it is a common necessity for investors to apply the following discrete loss amount counterparts of the continuous value expressions in (27) to (29):

$$PV_\tau\,(X) = \sum_{u=0}^{C_{BBB}+m_{BBB}-1} (g_1(u + 1))({}_u p_x)(q_{X_{r.n}}(x + u)) \quad , \tag{30}$$

$$PV_{\tau}(Y) = \sum_{v=0}^{C_{BB}+m_{BB}-1} (g_2(v+1))({}_{u}p_y)(q_{Y_{[r.n]}}(y+v)) \quad , \tag{31}$$

and

$$PV_{\tau}(X,Y) = \sum_{u=0}^{C_{BBB}+m_{BBB}-1} \sum_{v=0}^{C_{BB}+m_{BBB}-1} (g_{12}(u+1,v+1))({}_{u,v}p_{x,y})(q_{X_{r.n},Y_{[r;n]}}(x+u,y+v)) \tag{32}$$

where u and v are discrete loss amount values, and $q_{.}(.)$ and $q_{.,.}(.,.)$ are the discrete value case hazard functions:

$$q_{X_{r.n}}(x+u) = \frac{S_{X_{r;n}}(x+u) - S_{X_{r;n}}(x+u+1)}{S_{X_{r;n}}(x+u)}, \tag{33}$$

$$q_{Y_{[r.n]}}(y+v) = \frac{S_{Y_{[r;n]}}(y+v) - S_{Y_{[r;n]}}(y+v+1)}{S_{Y_{[r;n]}}(y+v)}, \tag{34}$$

$$q_{X_{r;n}Y_{[r.n]}}(x+u,y+v) = \frac{S_{X_{r;n},Y_{[r;n]}}(x+u,y+v) - S_{X_{r;n},Y_{[r;n]}}(x+u+1,y+v+1)}{S_{X_{r;n},Y_{[r;n]}}(x+u,y+v)}. \tag{35}$$

The first function $q_{X_{r.n}}(x+u)$ in (33) expresses a loss amount severity $(x+u+1)$ that a BBB rated bond investor may experince immediately after having realized a loss amount of $(x+u)$. In this sense, this function implies that loss occurrences will be ever going as u grows larger and as investors continue with a BBB rated bond in hand until its maturity. Similar interpretations can be made for the other hazard functions in (34) and (35), the latter being the discrete bivariate case hazard function.

It is better for investors to do these expected risk computations in a year τ that coincides with the first coupon payment period of the bonds. In this way, investors can do timely default loss amount evaluations and make effective decisions to mitigate expectable risks of loss. Outcoming values from these computations can be utilized by investors to make decisions on whether or not keep the less risky one or both of BBB and BB rated bonds in hand until the end of their maturity periods. As a result of such decisions, investors may choose to sell a comparatively more risky bond at once at its prevailing price in secondary bond markets. It may also be a preference for investors to sell both of the risky bonds immediately.

A numerical illustration of the example can be presented now using decision variables $(X_{r;n}, Y_{[r;n]})$ and utilizing the functions expressed in (27) to (35). Let us consider $n = 10$ counts of BBB and BB rated bond pairs that investors possess in year τ. Assume the following quantities for these bonds: Price(BBB)=7.75 with $k = 0.13$, Price(BB)=6.96 with $k = 0.15$, $T = 8$ years, $m = 10$ and the coupon rate is 10% so $c_t = 1$ for each bond. Then, probable total loss amounts for each bond may extend from 0 to 18 money units over their lifetime duration $[t_{.}, T]$. Assume further that coupon and par value payments are made at the end of each payment year. So, present values of the probable loss amounts can be calculated by using a discount factor $(1+\vartheta)^{t-1}$, $1 \le t \le T$, where the discount rate ϑ can be chosen by investors depending upon their judgement on the year τ cost of future losses. A sensible choice for ϑ can be the best possible interest rate on a concurrent government bond that is flat over $[t_0, T]$. With such a discount rate choice, investors can measure their missed gains regarding an alternative investment that they could have made on a government bond rather than BBB and BB rated bonds. Letting ϑ =0.05 per year, the year τ present values of the anticipated total losses out of 18 money units, that they at the start hope to receive on each bond, can be within the discounted values interval of [0, 13) money units since the present value of the possible total loss amount on a bond can be as much as $18/(1.05)^7 \approx 13$ as of the end of year τ. Parameters of the parent Pseudo-Pareto distributions of X and Y, given in (2), (3) and (4), can be determined then with respect to the value range [0, 13) assuming that investors trust these probability distributions as the true probability models for the probable losses on their current investments. It is implicit in this set up that investors retain a loss of 1 money unit on each bond since by assumption they are already in the first coupon payment period and have lost the first coupon payouts. Let the estimated parameter values of the concerned Pseudo-Pareto distributions be given as a_1 =2, $b_1 > 1$, $a_2 = ln(x) > 0$, $b_2 > 1$, and $X > b1$, $Y > 1$. And, suppose that investors choose r =5 as the order rank of their decision variable $X_{r;n}$ for risk evaluations. Under these circumstances, we can make risk assessments for investors by the use of the probability and hazard rate values that we tabulated in Appendix 2.

Present value expressions for the BBB and BB rated bonds are computed below as of the end of year τ =1 by applying the equations given in (30), (31) and (32) for the confrontable discrete and discounted loss amounts $\{1^+, 2,3,...,13\}$, the first value in the set being close to one, $1.1/(1.5) \approx 1.05$, as a retained loss amount on each bond. So, the loss amount evaluations here can made from the stand point of already incurred loss amounts of 1.05 for each bond, and our explicitly shown calculations produce the following results:

$$
\begin{aligned}
PV_1(X) &= 2\left(\frac{S_{X_{r:n}}(1.05) - S_{X_{r:n}}(2)}{S_{X_{r:n}}(1.05)}\right) \\
&\quad + \sum_{u=1}^{11}(2+u)\left(\frac{S_{X_{r:n}}(1+u) - S_{X_{r:n}}(2+u)}{S_{X_{r:n}}(1.05)}\right) = 1.96078, \\
PV_1(Y) &= 2\left(\frac{S_{Y_{[r:n]}}(1.05) - S_{Y_{[r:n]}}(2)}{S_{Y_{[r:n]}}(1.05)}\right) \\
&\quad + \sum_{v=1}^{11}(2+v)\left(\frac{S_{Y_{[r:n]}}(1+v) - S_{Y_{[r:n]}}(2+v)}{S_{Y_{[r:n]}}(1.05)}\right) = 2.37589, \\
PV_1(X,Y) &= 4\left(\frac{S_{X_{r:n},Y_{[r:n]}}(1.05, 1.05) - S_{X_{r:n},Y_{[r:n]}}(2,2)}{S_{X_{r:n},Y_{[r:n]}}(1.05, 1.05)}\right) \\
&\quad + \sum_{u=1}^{11}\sum_{v=1}^{11}(4+u+v)\frac{S_{X_{r:n},Y_{[r:n]}}(1+u, 1+v) - S_{X_{r:n},Y_{[r:n]}}(2+u, 2+v)}{S_{X_{r:n},Y_{[r:n]}}(1.05, 1.05)} \\
&= 10.0661
\end{aligned}
$$

where the following computational specifications are used for the discounted loss amount values; $g_1(u) = u$, $g_2(v) = v$ and $g_{12}(u+1,v+1) = (u+1)+(v+1)$.

Net expected payoffs for the final evaluations of investors on the bonds, as of time τ =1, can be assessed now. Net expected payoff quantities can be computed for each bond in line of the "present value of written coupons and par value total" net of "expected loss amount" and "paid bond price" calculation. Performing this calculation, the net expected payoffs for investor evaluations turn out as "13-2.96078-7.75 ≈2.30" money units on a BBB rated bond, and "13-3.37589-6.96≈2.66" money units on a BB rated one. These payoffs are much less than the price independent expected payoff values that follow from the "present value of written coupons and par value total" net of "probable loss amount" calculation. This calculation yields "13-7.75≈5.25" and "13-6.96≈6.04" values for the BBB and BB rated bonds, respectively. So, investors of the bonds have to decide promptly if they should sell one or both, or keep both of the bonds for further coupon payment years. Selected values of k have an obvious effect on the bond prices and investor evaluations. So are the effects of the values of r, the rank of decision variable $X_{r;n}$, and ϑ, the discount rate, on the computational results and pending investor decisions. Investors may try other r and ϑ values for their evaluations and see what they may come accross with under such simulations. At this point, we must mention the effect of another criterion that investor have to consider. This criterion is the strength of bond issuer companies which is measured in general in terms of a company's existing assets and equity amounts. Hence, if it is thought that the strength of a bond issuing company will not be able to cushion the default amounts that investors may incur, then selling the bonds of that company may become a prevailing decision for investors before they lose more.

The use of equation $PV_1(X, Y)$ for integer valued loss amounts is not exemplified here after all the antecedent results, although it is a useful but a more risk aversive expression for the expected loss amount calculations on BBB and BB rated bond pairs. Note that, similar computations like above can be pursued by using the equations in (27), (28) and (29) for loss amount expectations in continuous value terms. The essence of investor evaluations and decisions in the continous case stays as what is given for the discrete case and all aforesaid conclusions remain completely unchanged.

6 Conclusion

Bivariate Pseudo-Pareto distributions and distributions of their order statistics and concomitants can be used in many real life applications that require flexible and well established probability models. We have shown such a model with our bivariate Pseudo-Pareto distribution regarding a variable-concomitant variable pair (X, Y), as main variables, and an order statistic and its concomitant pair $(X_{r;n}, Y_{[r;n]})$, as decison variables. It is worth to go beyond this bivariate case and create some multivariate extensions of it. Inclusion of multifarious variable-concomitant variable multiplets in the model can increase its presently available modeling capacities for the decision making problems in multitude dimensions.

Different types of Pseudo-Pareto distributions can be developed on the grounds of what we have presented in this paper. Following the Pareto distribution families that already exist in the literature, several shape and scale parameter functions can be simultaneously introduced into this distribution in order to create more versatile versions of it. It is certain that there will be a need for more sophisticated computational tools in such endeavors.

Characterization of distributions by complementing CDFs, hazard functions and order statistics is another area of current research interest in probability. In this respect, characterization of the bivariate and multivariate Pseudo-Pareto distributions, given in this paper or elsewhere, is an appealing future research problem for us.

Appendix 1

This appendix presents instantaneous rate of change (IRC) expressions for the complementing CDF and hazard function of r-th order concomitant. Note that Polygamma and Riemann Zeta functions may take place in these expressions. The general definition for a polygamma function is $\psi^{(m)}(z) \equiv d^m/dz^m(\psi(z)) = d^{m+1}/dz^{m+1}ln(\psi(z))$ that we use in Section 3 in the digamma function form $\psi(\cdot)$. Riemann Zeta function is defined as $\zeta(h) = \Sigma_\upsilon(1/\upsilon^h)$, υ >1. This function is convergent as long as the real part of its complex variable h is greater than 1. We refer to Gradshteyn and Ryzhik (2007) for detailed information on these functions.

Instantaneous rate of change expressions are listed below with brief explanations and some numerical value tables:

$$\frac{\partial}{\partial a_1} S_{Y_{[r:n]}}(y) = \frac{\Gamma(n+1)\Gamma\left(n-r+(ln(y)/a_1)+1\right)ln(y)\left[\psi\left(n+(ln(y)/a_1)+1\right)-\psi\left(n-r+(ln(y)/a_1)+1\right)\right]}{a_1^2\Gamma(n-r+1)\Gamma\left(n+(ln(y)/a_1)+1\right)}$$

obtains decreasing values at each increasing a_1 value (n=10, r =5, y =5),

$$\frac{\partial}{\partial a_1} h_{Y_{[r:n]}}(y) = -\frac{\left(H_{\left(\frac{ln(y)}{a_1}+n\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r\right)}\right)}{a_1^2 y} - \frac{ln(y)\left(H_{\left(\frac{ln(y)}{a_1}+n,2\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)}\right)}{a_1^3 y}$$

obtains decreasing values at each increasing a_1 value (n=10, r =5, y =5),

$$\frac{\partial}{\partial r} S_{Y_{[r:n]}}(y) = \frac{\Gamma(n+1)\Gamma\left(n-r+(ln(y)/a_1)+1\right)\left[\psi\left(n-r+1\right)-\psi\left(n-r+(ln(y)/a_1)+1\right)\right]}{\Gamma(n-r+1)\Gamma\left(n+(ln(y)/a_1)+1\right)}$$

obtains increasing values at each increasing r value (n=10, a_1 =0.5, y =5),

$$\frac{\partial}{\partial r} h_{Y_{[r:n]}}(y) = \frac{\frac{\pi^2}{6} - H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)}}{a_1 y},$$

obtains increasing values at each increasing r value (n=10, a_1 =0.5, y =5),

$$\begin{aligned}\frac{\partial^2}{\partial a_1 \partial y} S_{Y_{[r:n]}}(y) = {} & \frac{\Gamma(n+1)\Gamma(n-r+(ln(y)/a_1)+1)}{a_1^2 y\Gamma(n-r+1)\Gamma(n+(ln(y)/a_1)+1)}\left(\psi\left(n+(ln(y)/a_1)+1\right)\right. \\ & - \frac{In(y)\psi(n+(ln(y)/a_1)+1)^2}{a_1} - \psi\left(n-r+(ln(y)/a_1)+1\right) + \frac{2In(y)\psi(n+(ln(y)/a_1)+1)\psi(n-r+(ln(y)/a_1)+1)}{a_1} \\ & \left. - \frac{In(y)\psi(n-r+(ln(y)/a_1)+1)^2}{a_1} + \frac{In(y)\psi^{(1)}(n+(ln(y)/a_1)+1)}{a_1} - \frac{In(y)\psi^{(1)}(n-r+(ln(y)/a_1)+1)}{a_1}\right)\end{aligned}$$

obtains decreasing values at each increasing (a_1, y) value pair (n=10, r =5),

$$\begin{aligned}\frac{\partial^2}{\partial a_1 \partial y} h_{Y_{[r:n]}}(y) = {} & \frac{H_{\left(\frac{ln(y)}{a_1}+n\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r\right)}}{(a_1 y)^2} - 2\left(\frac{H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)} - H_{\left(\frac{ln(y)}{a_1}+n,2\right)}}{a_1^3 y^2}\right) \\ & - ln(y)\left(\frac{H_{\left(\frac{ln(y)}{a_1}+n,2\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)}}{a_1^3 y^2}\right) - 2ln(y)\left(\frac{H_{\left(\frac{ln(y)}{a_1}+n,3\right)} - H_{\left(\frac{ln(y)}{a_1}+n-r,3\right)}}{a_1^4 y^2}\right)\end{aligned}$$

Table 1 Some IRC values for the complementing CDF of r-th order concomitant

y	$n = 10, r = 5$			$n = 10, a_1 = 1$		
	a_1			r		
	1	1.5	2	1	5	8
2	−0.4169	−0.2221	−0.1370	−0.1091	−0.4169	−0.6308
5	−0.0827	−0.0632	−0.0459	−0.0343	−0.0827	−0.2361
8	−0.0119	−0.0248	−0.0222	−0.0177	−0.0119	−0.1518

obtains abrubtly decreasing values at each increasing (a_1, y) value pair and turns to increase gradually as y gets larger (n=10, r =5),

$$\frac{\partial^2}{\partial r \partial y} S_{Y_{[r:n]}}(y) = \frac{\Gamma(n+1)\Gamma(n-r+(In(y)/a_1)+1)}{a_1 y \Gamma(n-r+1)\Gamma(n+(In(y)/a_1)+1)} \left(-\psi(n-r+1)\,\psi(n+(In(y)/a_1)+1)\right.$$
$$+\psi(n-r+1)\,\psi(n-r+(In(y)/a_1)+1) + \psi(n+(In(y)/a_1)+1)\,\psi(n-r+(In(y)/a_1)+1)$$
$$\left.-\psi(n-r+(In(y)/a_1)+1)^2 - \psi^{(1)}(n-r+(In(y)/a_1)+1)\right)$$

obtains increasing values at each increasing (r, y) value pair (n=10, a_1 =1.5),

$$\frac{\partial^2}{\partial r \partial y} h_{Y_{[r:n]}}(y) = -\frac{\frac{\pi^2}{6} - H_{\left(\frac{In(y)}{a_1}+n-r,2\right)}}{a_1 y^2} - \frac{2\left(-H_{\left(\frac{In(y)}{a_1}+n-r,3\right)} + \zeta(3)\right)}{a_1^3 y^3}$$

obtains decreasing values at each increasing (r, y) value pair (n=10, a_1 =1.5),

$$\frac{\partial^3}{\partial a_1 \partial r \partial y} S_{Y_{[r:n]}}(y) = -\frac{\Gamma(n+1)\Gamma(n-r+(ln(y)/a_1)+1)}{a_1 y \Gamma(n-r+1)\Gamma(n+(ln(y)/a_1)+1)} \left(\frac{\psi(n-r+1)}{a_1}\left(\psi(n-r+(ln(y)/a_1)+1) - \psi(n+(ln(y)/a_1)+1)\right.\right.$$
$$+\frac{ln(y)\psi(n+(ln(y)/a_1)+1)}{a_1}\left(\psi(n-r+(ln(y)/a_1)+1) - \psi(n+(ln(y)/a_1)+1)\right)$$
$$-\frac{ln(y)\psi(n-r+(ln(y)/a_1)+1)}{a_1}\left(\psi(n-r+(ln(y)/a_1)+1) - \psi(n+(ln(y)/a_1)+1)\right)$$
$$\left.\left.+\frac{ln(y)\left(\psi^{(1)}(n+(ln(y)/a_1)+1) - \psi^{(1)}(n-r+(ln(y)/a_1)+1)\right)}{a_1}\right)\right)$$
$$-\frac{ln(y)}{a_1^2}\left(\psi(n-r+(ln(y)/a_1)+1)\,\psi^{(1)}(n+(ln(y)/a_1)+1)\right.$$
$$+\psi(n+(ln(y)/a_1)+1)\,\psi^{(1)}(n-r+(ln(y)/a_1)+1) - \psi^{(2)}(n-r+(ln(y)/a_1)+1)$$
$$\left.-2\psi(n-r+(ln(y)/a_1)+1)\,\psi^{(1)}(n-r+(ln(y)/a_1)+1)\right)$$

captures the exemplary values in Table 1 for some (a_1, r, y) value triples,

Table 2 Some IRC values for the hazard function of r-th order concomitant

y	$n = 10, r = 5$			$n = 10, a_1 = 1$		
	a_1			r		
	1	1.5	2	1	5	8
2	−49.635	−15.652	−0.7002	−76.917	−49.635	−10.468
5	0.2762	0.0377	0.0067	0.4167	0.2762	0.1041
8	0.1569	0.0259	0.0068	0.2218	0.1569	0.0804

and

$$\frac{\partial^3}{\partial a_1 \partial r \partial y} h_{Y_{[r:n]}}(y) = \frac{\frac{\pi^2}{6} - H_{\left(\frac{ln(y)}{a_1}+n-r,2\right)}}{(a_1 y)^2} - \frac{6ln(y)\left(\frac{\pi^4}{60} - H_{\left(\frac{ln(y)}{a_1}+n-r,4\right)}\right)}{a_1^4 y^2} + \frac{\left(\zeta(3) - H_{\left(\frac{ln(y)}{a_1}+n-r,3\right)}\right)}{a_1^3 y^2}(4 - 2ln(y))$$

captures the exemplary values shown in Table 2 at the given (a_1, r, y) value triples.

Appendix 2

Table 3 Distribution and hazard function value points for the implementation example

Bivariate Pseudo-Pareto Distribution n =10, a_1 =2, a_2 =$ln(x)$, b_1 =1, b_2 =1, r =5

x	y	$F_X(x)$	$F_Y(y)$	$S_{X_{r:n}}(x)$	$h_{X_{r:n}}(x)$	$S_{Y_{[r:n]}}(y)$	$h_{Y_{[r:n]}}(y)$	$S_{X_{r:n},Y_{[r:n]}}(x,y)$	$h_{X_{r:n},Y_{[r:n]}}(x,y)$
2	2	0.7500	0.2574	0.0197	73.9051	0.8035	0.0685	0.0693	4.5119
3	3	0.8889	0.3545	2.6516e-004	29.0324	0.7102	0.0452	0.0018	0.4807
4	4	0.9375	0.4094	1.0047e-004	18.4074	0.6518	0.0336	1.0412e-004	0.0901
5	5	0.9600	0.4459	7.4824e-007	13.6604	0.6105	0.0267	1.1082e-005	0.0223
6	6	0.9722	0.4725	8.7614e-008	10.9360	0.5790	0.0222	1.7494e-006	0.0066
7	7	0.9796	0.4931	1.4138e-008	9.1517	0.5539	0.0189	3.6512e-007	0.0022
8	8	0.9844	0.5097	2.8955e-009	7.8848	0.5333	0.0165	9.3729e-008	8.3865e-004
9	9	0.9877	0.5235	7.1258e-010	6.9351	0.5158	0.0146	2.8213e-008	3.4228e-004
10	10	0.9900	0.5352	2.0289e-010	6.1947	0.5008	0.0131	9.6337e-009	1.4967e-004
11	11	0.9917	0.5452	6.5037e-011	5.6003	0.4876	0.0119	3.6437e-009	6.9361e-005
12	12	0.9931	0.5541	2.2997e-011	5.1120	0.4760	0.0109	1.4998e-009	3.3784e-005
13	13	0.9941	0.5619	8.8322e-012	4.7033	0.4656	0.0100	6.6278e-010	1.7180e-005

Note: “e-w” means 10 raised to power “-w” (e.g.2.6516e-004=2.6516*10^{-4} =0.00026516

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