A trip around octonions

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Abstract. In these expository notes, after a contemplation on the dawn of octonions, we give proofs for the Frobenius theorem and the Hurwitz theorem, we review the basics of Clifford algebras and spin groups, and exemplify the startling role played by the octonions in $7-$ and $8-$dimensional phenomena such as the special $3-$ and $4-$forms, the Bonan form, $Spin(7)$ and $Spin(8)$ groups and the mysterious triality.

1. The First Steps

The natural numbers were long around. Then came the rationals by the need to share. The Greeks took the lengths of segments as the reals. Everything positive, it goes without saying. And they made the terrifying discovery that there were "incommensurable" segments! Such as an edge and a diagonal of a regular pentagon. It might be this traumatic event that they stuck in the number mud and became masters of geometry.

After about a millennium the negative numbers (and zero!) made their appearance in the enigmatic India and made their way very slowly to Europe. In the Italy of the 16th Century cubic equations were solved in breathtaking contests among mathematicians and square-roots of negative numbers imposed themselves in hermetic formulas such as $\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$ for an equation $x^3 = 15x + 4$ with an obvious root. This nightmare of Cardano was only partly relieved some decades later by Bombelli, who noticed that $(2 + \sqrt{-1})^3 = 2 + \sqrt{-121}$ and $(2 - \sqrt{-1})^3 = 2 - \sqrt{-121}$, giving the obvious root 4. Even for the minds of Leibniz or even Euler these new "imaginary" ghosts were incomprehensible. Atiyah notes bitterly that it took centuries to understand the square root of -1.

Then came the fateful 19th Century. Everything was being rethought and rebuilt from scratch. We cannot tell here the dramatic story of this heroic epoch. But let us only remark that early on in the century the glorious Euclidean geometry was shattered, the futility to solve (in radicals) the quintic (and higher degree) polynomials were grasped, foundations of a trustworthy calculus were laid, and as a no more postponable issue, a solid basis for the numbers (the naturals, rationals, reals, negatives and the imaginaries) was devised, the reals being more troublesome than the imaginaries! (At the end of the century the Euclidean geometry was however rehabilitated and assigned to a more modest corner of mathematics.)

A "complex number" (as they were unluckily dubbed) was nothing more than a pair of real numbers, say (a, b). Addition and multiplication were defined by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. The number $(1, 0)$ was the unit of multiplication and the spooky square-root of -1 was nothing else than $(0, 1)$ since $(0, 1) \cdot (0, 1) = (-1, 0) = -(1, 0)$.

Since the objects $(a, 0)$ behave as the reals under addition and multiplication we can identify them with the reals (and write simply a again for them) and since by the above definitions $(a, b) = (a, 0) + (b, 0) \cdot (0, 1)$ holds, we can write $(a, b) = a + b \cdot i$, if we denote $(0, 1)$ by i, thus getting the more familiar form.

If you want to be more pragmatic, you can say that you introduce a new object i with $i^2 = -1$ and call the expressions $a + bi$ your new numbers which you add and multiply as $(a+bi)+(c+di) = (a+c)+(b+d)i$ and $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$. Nobody can query you about what your new object is, because at the end of the 19th Century (and now), not the objects, but only the relations are important, i.e. how you add and multiply them (or whatever the relations are).

Surely the complex numbers were interpreted as points of the plane and it was an irresistible desire to extend the complex number system to the 3-dimensional space. No less geniuses than Gauss and Hamilton were unable to multiply the triplets! Gauss himself writes [11]:

"The writer has reserved for himself ... the question why the relations between things that make up a manifold of more than two dimensions cannot provide quantities admissible in universal arithmetic."

(The case of Hamilton is well documented and we heartily recommend [2].)

What is a number system? Today we might give different answers, but in those days, arithmetic operations and their properties were the main guide. You ought to be able to add, subtract, multiply and divide (with non-zero numbers) and the accustomed properties should hold. Gauss had made a great leap in declaring a non-zero natural number to be zero in modular arithmetic, but the commutativity and associativity of the addition and multiplication were subliminal and not yet on the discussion table. (But seemingly not for Gauss as we will note shortly). In a survey of early attempts on multiplying the triplets and the impossibility proofs thereof, Kenneth May [11] writes, that *"the student can easily discover for himself if he experiments without getting bogged down (as did Hamilton!) in trying particular definitions of multiplication"*. We find this assessment somewhat merciless since it is always easier to be smart afterwards; one should take the *Zeitgeist* into account. Today it is a one-line exercise that on an odd-dimensional $\mathbb{R}^n (n > 1)$ there can be no real division algebra by considering an element as a linear transformation via multiplication and noticing that there is a real eigenvalue producing a zero-divisor. Nevertheless, one is compelled to admit that the following argument recalled by May might possibly have been accessible even in those days:

If we introduce a second object j to extend the complex numbers to the space and consider the triplets $a + bi + cj$ (with a, b, c real), we should first of all declare what ij is!

Let $ij = \alpha + \beta i + \gamma j$. Then, $i(ij) = i(\alpha + \beta i + \gamma j) = \alpha i - \beta + \gamma ij = \alpha i - \beta + \gamma(\alpha + \beta i + \gamma j) = (\alpha\gamma - \beta) + (\alpha + \beta\gamma)i + \gamma^2 j$. On the other hand, $i(ij) = (ii)j = -j$ (by the subliminally assumed associativity). By comparing the two expressions, we get $c^2 = -1$ for a real c, which is impossible! (Thus the complex numbers cannot be extended to a three-dimensional real associative algebra at all; irrespective of the division property.)

It seems somehow, that if we want to multiply the triplets, ij requires a separate room for itself! Let us allocate it and call it k. Then we must admit quartets in form $a + bi + cj + dk$. To multiply triplets, we need quartets! But how to multiply them, and to which price? Their multiplication rules were carved by Hamilton on a stone of a bridge in Dublin in 1843: $i^2 = j^2 = k^2 = ijk = -1$.

Now ij is k as planned: $(ijk)k = -k$, so that $ij = k$ since $k^2 = -1$. (Hamilton's sole writing "ijk" shows that "associativity" is innate!)

But what is ji? If we multiply $ijk = -1$ with ji, we get $ji(ijk) = -ji$ and hence $ji = -k$! Similarly, $jk = -kj = i$ and $ki = -ik = j$. The price is the loss of commutativity! It must have cost Hamilton some pain of overcoming. You can now multiply any two quartets using the (never disputed) distributivity and real-linearity. In the resulting system, called *Quaternions* by Hamilton, you can do also division by non-zero quaternions. But be careful: How do you "divide", for example k by i? Due to the loss of commutativity, the solutions of the equations $ix = k$ and $xi = k$ are different! (The discovery of the quaternions is generally credited solely to Hamilton, but as became known half a century after his death, Gauss had dared to dispense with the commutativity of multiplication and defined the quaternions as early as 1819! [1])

During the extensions from the reals to the complex numbers, and from them to quaternions, there is an unexpected accompanying phenomenon behind the scene: We first wanted to take the square-roots of negative numbers, then we wanted innocently to extend the arithmetic operations to the 3-space and landed at the 4-dimensional space, but we are rewarded with more structure: The simple absolute value $|a|$ of a real number (with the obvious property $|ab| = |a||b|$) has rather natural counterparts for the complex numbers and quaternions. We can surely assign an "absolute value" (or magnitude, modulus, norm, or whatever you call) for a complex number $a+bi$, namely the usual Euclidean norm $\|(a,b)\| = (a^2+b^2)^{1/2}$ of $(a,b) \in \mathbb{R}^2$; and for a quaternion $a+bi+cj+dk$ the norm of $(a,b,c,d) \in \mathbb{R}^4$, but who can hope that the multiplicative property $\|ab\| = \|a\|\ \|b\|$ holds for them? Somewhat miraculously, they come to be true as reflected in the following identities:

We defined the multiplication of two complex numbers by $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ and it holds $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$; we defined the multiplication of two quaternions by

$$\begin{aligned}(a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3)\\ + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)i\\ + (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1)j\\ + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)k\end{aligned}$$

and it holds

$$\begin{aligned}(a_0^2 + a_1^2 + a_2^2 + a_3^2)(b_0^2 + b_1^2 + b_2^2 + b_3^2) = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3)^2\\ + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)^2\\ + (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1)^2\\ + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)^2 \ !\end{aligned}$$

These identities were surely known before complex numbers and quaternions; the first since ancient times, the second since Euler. We will see a last (and more formidable) instance of this miracle shortly.

We would like to note as a historical side-remark that a Turkish general and solitary mathematician Vidinli Hüseyin Tevfik Paşa was also possessed by the quest to multiply the triplets and in a book titled "Linear Algebra" (published in 1882 in Istanbul, in English [14]) he defined $ij = -ji = 1$ (and $i^2 = j^2 = -1$), thus creating a non-commutative and non-associative algebra extending the complex numbers to $\mathbb{R}^3$, where the non-zero elements have a two-sided inverse. A norm multiplicativity holds e.g. for elements of the special type $a + bi + bj$, reflected in the partial 3-squares identity

$$(a^2 + b^2 + b^2)(c^2 + d^2 + d^2) = (ac - 2bd)^2 + (ad + bc)^2 + (ad + bc)^2,$$

(which is in fact a 2-squares identity for the considered elements which constitute a subalgebra isomorphic to the complex numbers since $a + bi + bj = a + \sqrt{2}b(\frac{1}{\sqrt{2}}(i + j))$ with $(\frac{i+j}{\sqrt{2}})^2 = -1$). Norm multiplicativity holds more generally for elements of type $a + bri + bsj$ for fixed $r, s \in \mathbb{R}$

(which also constitute a subalgebra isomorphic to $\mathbb{C}$ via $(\frac{ri+sj}{\sqrt{r^2+s^2}})^2 = -1$), resulting in the identity

$$(a^2+b^2r^2+b^2s^2)(c^2+d^2r^2+d^2s^2) = (ac-(r^2+s^2)bd)^2 + ((ad+bc)r)^2 + ((ad+bc)s)^2,$$

a narrow miss of the impossible 3-squares identity, might one think! He gives numerous geometric applications of this curious algebra. We hope to report on our forgotten fellow and his book elsewhere.

2. Where To Go From Here?

Having meanwhile three real associative algebras with division property on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^4$, what might happen in further search? One is tempted to imitate the pattern of quaternions (i.e. two new objects i and j with $i^2 = j^2 = -1$ and a new room k for ij with $ji = -ij$) with more objects, say, for example, with three new objects i, j, m with $i^2 = j^2 = m^2 = -1, \quad ij = -ji, \quad im = -mi, \quad jm = -mj$ and with new rooms for ij, im, jm and ijm. We have then a total of 8 rooms for $1, i, j, m, ij, im, jm, ijm$ (so that an arbitrary element of the candidate algebra will be a linear sum of these elements with real coefficients) and further multiplications are executed by associative (and distributive) simplification. For example, $(ij)(im) = i(ji)m = -i(ij)m = -(ii)jm = jm$, or, $(ijm)(ijm) = -jmjm = -mm = 1$.

We get on the whole an associative algebra on $\mathbb{R}^8$. Would this algebra allow divisions (with non-zero elements)? Unfortunately not! For example, $(1+ijm)(1-ijm) = 0$, so that there are zero-divisors! By the same reason, a generalization of this construction with n new objects yielding a real associative algebra on $\mathbb{R}^{2^n}$ will not be a division algebra either. However, this idea is not that bad and it will lead us to the Clifford algebras with their own merits.

Any further search is in vain, in fact. If we insist on associativity, there are no other real division algebras beyond the reals ($\mathbb{R}$), the complex numbers ($\mathbb{C}$) and the quaternions ($\mathbb{H}$).

Theorem (Frobenius 1878 [6]). *A finite-dimensional associative real division algebra is isomorphic either to $\mathbb{R}$, or to $\mathbb{C}$ or to $\mathbb{H}$.*

Before giving a proof of this theorem we want to define some notions we have been using more precisely:

Some Definitions: An algebra A over a field F is an F-vector space A with a bilinear product $A \times A \to A, \quad (a,b) \mapsto ab$. (We then have $(a_1+a_2)b = a_1b + a_2b, \quad a(b_1+b_2) = ab_1 + ab_2, \quad (\lambda a)b = a(\lambda b) = \lambda(ab)$ for $\lambda \in F$)

The algebra is called associative if $a(bc) = (ab)c$ holds. A is called an algebra with unit if there exists $1 \in A$ with $1a = a1 = a$ (When we talk of an algebra with unit we assume that $A \neq \{0\}$ so that the unit is necessarily different from zero; we can anyway discard that trivial case from start). F is naturally an algebra over itself and the map $F \to A, \lambda \mapsto \lambda 1$ is an algebra isomorphism and thus F can be thought lying embedded in A.

An algebra $A \neq \{0\}$ is called a division algebra if for any $a, b,\ a \neq 0$, the equations $ax = b$ and $ya = b$ have unique solutions; in other words, the left and right multiplication maps $L_a : A \to A, \quad x \mapsto ax$ and $R_a : A \to A, \quad x \mapsto xa$ are linear-isomorphisms.

A division algebra has no zero-divisors: If $ab = 0$, then either a or b must be zero: Say $a \neq 0$, then the unique-solution property of $ay = 0$ would give $b = 0$ since $a0 = 0$. If the algebra A is finite-dimensional as a vector space over F, then the converse is also true: If there are no zero-divisors, then $ax = b$ and $ya = b$ have unique solutions (for proof consider the maps L_a and R_a: Injectivity implies surjectivity).

We will consider in these notes only real (i.e. $F = \mathbb{R}$), finite-dimensional division algebras with unit.

The algebra is full of little gems: An associative division algebra has automatically a unit! To see this, consider $ax = a$ for an $a \neq 0$. This has a unique solution. Call it e. Thus, $ae = a$ (and

$e \neq 0$). We have $(ae)e = ae$. By associativity, $a(ee) = ae$. Hence $a(ee - e) = 0$. As there are no zero-divisors, $ee - e = 0$. Now, for any x, $\quad x(ee - e) = 0, \quad x(ee) - xe = 0, \quad (xe)e - xe = 0, \quad (xe - x)e = 0, \quad xe = x$, thus e is a right-unit. Similarly, $(ee - e)x = 0$ and and $ex = x$, and e is a unit. We used above a weaker version of associativity, called alternativity: $a(bc) = (ab)c$ holding when two of a, b, c are equal. Thus we have seen:

Remark: An alternative division algebra has necessarily a unit.

Now, we give a proof of the Frobenius theorem (which is a streamlined version of the scattered proof in the beautiful book *Numbers* [5]; for another elementary proof see [13]).

Proof of the Frobenius Theorem. Given a finite-dimensional, associative, real division algebra A, any element $a \in A$ can be decomposed as $a = r + x$, where r is real (since A has necessarily a unit, $\mathbb{R}$ is embedded in A) and x^2 is a non-positive real. To see this we can consider the powers of a (up to the dimension of A) and obtain a polynomial which splits into first and second degree factors over the reals. Since there are no zero-divisors, one of the factors must vanish. If a is not real, it must satisfy a quadratic polynomial, say, $a^2 - 2ra + s = 0$ (with $r^2 - s < 0$). Now we get, $(a - r)^2 = a^2 - 2ra + r^2 = r^2 - s < 0$.

The surprising thing is that elements of A, which, when squared, become non-positive reals, constitute a linear subspace of A! Calling this subset A', it is obvious that a multiple of an element of A' belongs to A', so let $u, v \in A'$ be two linearly independent elements. Then, u, v and 1 are also linearly independent (if $\lambda u + \mu v + \nu 1 = 0$, then $\lambda^2 u^2 = \mu^2 v^2 + 2\mu\nu v + \nu^2$ yielding $\lambda = \mu = \nu = 0$ since u^2 and v^2 are reals) and consequently, $u + v$ and $u - v$ are not reals. Let the quadratic polynomials they satisfy be $(u+v)^2 - 2p(u+v) + q = 0$ and $(u-v)^2 - 2p'(u-v) + q' = 0$. Opening the squares and adding up the equations gives, by linear-independence of u, v and 1, $p = p' = 0$, hence $u + v \in A'$, showing that A' is indeed a linear subspace of A. We thus get a fine decomposition of the algebra as $A = \mathbb{R} \oplus A'$ with $x^2 \leq 0$ for $x \in A'$.

We note that for $u, v \in A'$, $uv + vu$ is real: $uv + vu = (u + v)^2 - u^2 - v^2$.

Now, if $A' = \{0\}$, then $A = \mathbb{R}$; if $\dim(A') = 1$, choose $u \in A'$ with $u^2 = -1$ and we get $A = \mathbb{C}$. If $\dim(A') \geq 2$, then we can construct a so-called Hamiltonian triple, that is a triple of elements with the same multiplicative relationships as the i, j and k of the quaternions: First choose $u \in A'$ with $u^2 = -1$ and a $v' \in A'$ such that u and v' are linearly independent. Then $uv' + v'u = r$ with $r \in \mathbb{R}$. Now take $v = v' + \frac{r}{2}u$. We get, $uv + vu = uv' + \frac{r}{2}u^2 + v'u + \frac{r}{2}u^2 = 0$. Now it can be easily seen that u, v and $W = uv$ constitute a Hamiltonian triple. For example, $w^2 = (uv)(uv) = ((uv)u)v = (u(vu))v = -(u(uv))v = -(u^2v)v = v^2 = -1$; or, $vw = v(uv) = -v(vu) = -v^2u = u$, etc. This shows that there is an embedded $\mathbb{H}$ in A. Now let x be any element of A'. We have $xu + ux = r$, $xv + vx = s$ and $xw + wx = t$ for some $r, s, t \in \mathbb{R}$. From the first equation we get $(xu)v + (ux)v = rv$, from the second, $u(xv) + u(vx) = su$, and subtracting them (vital use of associativity! For Hamiltonian triples we could dispense with associativity and get by alternativity) we get $(xu)v - u(vx) = x(uv) - (uv)x = xw - wx = rv - su$. Adding up this equation with $xw + wx = t$ we find $2xw = rv - su + t$ and by multiplying from the right with w we obtain $x = \frac{1}{2}(ru + sv + tw)$! That means, there is nothing more in A' than the span of u, v and w, in other words, $A = \mathbb{H}$. $\square$

3. Where Did The Octonions Come From?

A short (and rather naive) narrative like this can hardly follow the inscrutable ways of the actual discovery process. As we try to understand the division algebras and to see what could be done after Hamilton, we come to admit that the octonions were discovered long before Frobenius listed the associative division algebras. A few months after learning the discovery of the quaternions from a letter of Hamilton, J.T. Graves, a friend of Hamilton, had promptly extended the quaternions to the 8-dimensional space, and called them octaves, as the unhappily delayed attestment of Hamilton of the year 1847 shows [7]:

My present object is to mention that J. T. Graves, to whom I had previously communicated my theory of *quaternions*, was early led, by his extension of Euler's theorem, to conceive an analogous theory of *octaves*, involving *seven* distinct imaginaries, or square roots of negative unity, namely, *four* new roots, which he denoted by the letters l, m, n, o, to be combined with my *three* letters, i, j, k, into one common imaginary or symbolic system. Thus, as I had already (in October and November, 1843) communicated to him and to the Academy the fundamental equations of quaternions, namely,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = -1, \\ ij = k,\ jk = i,\ ki = j, \\ ji = -k,\ kj = -i,\ ik = -j, \end{array}\right\} \qquad \text{(a)}$$

which may be concisely summed up in the formula

$$i^2 = j^2 = k^2 = ijk = -1; \qquad \text{(b)}$$

so he proposed to found a theory of octaves on the following equations,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1, \\ i = jk = lm = on = -kj = -ml = -no, \\ j = ki = ln = mo = -ik = -nl = -om, \\ k = ij = lo = nm = -ji = -ol = -mn, \\ l = mi = nj = ok = -im = -jn = -ko, \\ m = il = oj = kn = -li = -jo = -nk, \\ n = jl = io = mk = -lj = -oi = -km, \\ o = ni = jm = kl = -in = -mj = -lk; \end{array}\right\} \qquad \text{(c)}$$

which he communicated to me, in a letter dated January 4, 1844, and which may be concisely expressed by the single but continued equation,

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1 \\ = ijk = ilm = ion = jln = jmo = klo = knm. \end{array}\right\} \qquad \text{(d)}$$

Meanwhile, also apparently intrigued by the quaternions, Cayley has already published a cryptic note in 1845:

Cambridge, January 16, 1845. **A. CAYLEY.**

P.S. ***On Quaternions.***

It is possible to form an analogous theory with seven imaginary roots of (-1) (? with $\nu = 2^n - 1$ roots when ν is a prime number). Thus if these be $\iota_1, \iota_2, \iota_3, \iota_4, \iota_5, \iota_6, \iota_7$, which group together according to the types

$$123,\ 145,\ 624,\ 653,\ 725,\ 734,\ 176,$$

***i. e.* the type 123 denotes the system of equations**

$$\iota_1\iota_2 = \iota_3,\quad \iota_2\iota_3 = \iota_1,\quad \iota_3\iota_1 = \iota_2,$$
$$\iota_2\iota_1 = -\iota_3,\quad \iota_3\iota_2 = -\iota_1,\quad \iota_1\iota_3 = -\iota_2,$$

&c. We have the following expression for the product of two factors:

$$\begin{aligned}
&(X_0 + X_1\iota_1 + \dots X_7\iota_7)(X'_0 + X'_1\iota_1 + \dots X'_7\iota_7)\\
&= X_0X'_0 - X_1X'_1 - X_2X'_2 \dots - X_7X'_7.\\
&+ [\overline{23} + \overline{45} + \overline{76} + (01)]\iota_1 \qquad \text{where } (01) = X_0X'_1 + X_1X'_0\\
&+ [\overline{31} + \overline{46} + \overline{57} + (02)]\iota_2 \qquad \vdots\\
&+ [\overline{12} + \overline{65} + \overline{47} + (03)]\iota_3 \qquad \overline{12} = X_1X'_2 - X_2X'_1\\
&+ [\overline{51} + \overline{62} + \overline{47} + (04)]\iota_4 \qquad \text{\&c.}\\
&+ [\overline{14} + \overline{36} + \overline{72} + (05)]\iota_5\\
&+ [\overline{24} + \overline{53} + \overline{17} + (06)]\iota_6\\
&+ [\overline{25} + \overline{34} + \overline{61} + (07)]\iota_7
\end{aligned}$$

Both Graves and Cayley, took seven imaginaries, gave a multiplication table and derived as a bonus an eight-square identity! (or was it maybe the other way around?) Is it that easy? Why seven imaginaries? Why those multiplication tables? How does the mysterious 8-squares identity pop up? All in all, it seems that it was a clever cast. The ensuing intense search in higher (especially 2^n) dimensions for new number systems is an indication that there was no conceptual frame yet. Nevertheless, you can take the multiplication table of Graves or Cayley (but beware: there is a misprint in Cayley's table!), express a new number as, say,

$$a = (a_0 + a_1i + a_2j + a_3k + a_4l + a_5m + a_6n + a_7o) \in \mathbb{R}^8,$$

take another number

$$b = (b_0 + b_1i + b_2j + b_3k + b_4l + b_5m + b_6n + b_7o),$$

multiply them according to the table and get

$$ab = c = (c_0 + c_1i + c_2j + c_3k + c_4l + c_5m + c_6n + c_7o),$$

and then you will discover the incredible identity

$$\begin{aligned}
(a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)&(b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2)\\
&= (c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2)!
\end{aligned}$$

That is, $\|ab\| = \|a\|\,\|b\|$, where $\|a\|$ is the standard norm of a in $\mathbb{R}^8$. Moreover, you can do the divisions $ax = b$ and $ya = b$ within the system. You will at some point notice that the element $\overline{a} = (a_0 - a_1i - a_2j - a_3k - a_4l - a_5m - a_6n - a_7o)$ is the analogue of the complex conjugate, as useful as that ($a\overline{a} = \overline{a}a = \|a\|^2$), making the division such easy:

$$ax = b, \quad \overline{a}(ax) = \overline{a}b, \quad (\overline{a}a)x = \overline{a}b, \quad \|a\|^2x = \overline{a}b, \quad x = \frac{1}{\|a\|^2}(\overline{a}b).$$

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You were however lucky in this computation (using the associativity). You will soon sadly realize that the multiplication of the new number system is generally not associative. But if division is more important for you, you will accept unwillingly the loss of associativity.

We can elucidate at this point some of the mystery of n-square identities. If we have an identity of the form

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) = (c_1^2 + c_2^2 + \cdots + c_n^2),$$

where the tuple $(c_1, c_2, \ldots, c_n)$ is bilinear in $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, then we can define an algebra on $\mathbb{R}^n$ by

$$(a_1e_1 + a_2e_2 + \cdots + a_ne_n)(b_1e_1 + b_2e_2 + \cdots + b_ne_n) = (c_1e_1 + c_2e_2 + \cdots + c_ne_n),$$

where the e_i are the standard basis vectors of $\mathbb{R}^n$. The Euclidean norm is then multiplicative for this algebra. If, conversely, we can find an algebra on $\mathbb{R}^n$, for which the Euclidean norm is multiplicative, then we find an n-square identity via this multiplicativity.

This observation brings a bunch of problems which we cannot address here. If the algebra thus defined has a unit, life becomes easy. In that case, Hurwitz showed that the only possible cases are the already known $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. We will dwell on his theorem in more detail below. Otherwise, i.e. if there does not exist a unit, there are an infinitude of algebras with multiplicative Euclidean norm (but again in these somehow privileged four dimensions 1, 2, 4 and 8.) For example, the multiplication

$$(a_1e_1 + a_2e_2)(b_1e_1 + b_2e_2) = (a_1b_1 + a_2b_2)e_1 + (a_2b_1 - a_1b_2)e_2$$

defines an algebra on $\mathbb{R}^2$, without a unit, but still with multiplicative norm:

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_1 + a_2b_2)^2 + (a_2b_1 - a_1b_2)^2.$$

We would like to remark that an algebra on $\mathbb{R}^n$ with a multiplicative norm is necessarily a division algebra since it can not have zero-divisors. If we forget about the norms and ask for division algebras on $\mathbb{R}^n$, then the possible dimensions are still only the 1, 2, 4 and 8! This was shown by Milnor and Kervaire (independently) in 1958 [12] by rather sophisticated topological means. The classification of division algebras in these dimensions is however subject of ongoing research and not yet completed.

Before going to the discussion of normed algebras, we want to recall another look on octonions. Dickson noticed in 1919 [4] that the octonions can be understood as pairs of quaternions, as the complex numbers were conceived as pairs of real numbers by Hamilton (likewise, the quaternions can be defined as pairs of complex numbers). Dickson defined the product of quaternion pairs as follows:

$$(a, b)(c, d) = (ac - \overline{d}b, da + b\overline{c}),$$

where a, b, c and d are quaternions, $\overline{d}, \overline{c}$ are the conjugates of d and c. (This product rule is a refinement of Hamilton's complex multiplication rule and can be applied verbatim to define the quaternions as complex number pairs.) If we denote the elements $(1, 0), (i, 0), (j, 0)$ and $(k, 0)$ by $e_0 = 1, e_1, e_2$ and e_3; and the elements $(0, 1), (0, i), (0, j)$ and $(0, k)$ by e_4, e_5, e_6 and e_7, we obtain the multiplication table on the standard basis of $\mathbb{R}^8$.

$e_0 = 1$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Now, at the point we arrived, after sacrificing first commutativity and then associativity, we have four real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, with a unit and with a multiplicative norm, which comes from the standard inner products on the respective Euclidean spaces. It turns out that this is the end of the road in the following sense:

Theorem (Hurwitz 1898 [10]). *A finite-dimensional, real algebra with unit and with a multiplicative norm coming from an (bilinear, symmetric and positive definite) inner product is isomorphic either to $\mathbb{R}$, or to $\mathbb{C}$ or to $\mathbb{H}$ or to $\mathbb{O}$.*

A normed algebra is free of zero-divisors and, by finite-dimensionality, a division algebra. Normed algebras have a simple and useful theory. The inner product simplifies some of the foregoing considerations and enables new nice relationships. We will now consider them and give a proof of the Hurwitz theorem following the beautiful references [9] and [8].

4. An Excellent Set-up: Normed Algebras Over the Reals

Definition. A normed algebra (over $\mathbb{R}$) is a finite dimensional algebra A ($\neq \{0\}$) over $\mathbb{R}$ with identity and with an inner product $\langle x, y\rangle$ such that the associated norm ($\|x\| = \langle x, x\rangle^{1/2}$) satisfies $\|xy\| = \|x\|\|y\|$.

The embedded $\mathbb{R} \subset A$, which consists of the real multiples of the identity of A, is called the real part of A and denoted by $Re(A)$ (sometimes by ReA). The norm of the identity $\mathbf{1}$ of A is 1 since $\|\mathbf{1}\|\|\mathbf{1}\| = \|\mathbf{1}\|$ (note that $\mathbf{1} \neq 0$ and thus $\|\mathbf{1}\| > 0$). We identify $Re(A)$ with $\mathbb{R}$ and denote the identity of A henceforth also with 1. The orthogonal complement of $Re(A)$ is called the imaginary part of A, denoted by $Im(A)$ (sometimes by ImA) and thus A is decomposed as $Re(A) \oplus Im(A) = \mathbb{R} \oplus Im(A)$. Any element $x \in A$ can be written uniquely as $x = x_1 + x'$ with $x_1 \in \mathbb{R}$ and $x' \in Im(A)$. We will write $x_1 = Re(x)$ and $x' = Im(x)$.

We define a conjugation by $\overline{x} = x_1 - x'$. Thus,

$$x_1 = Re(x) = \frac{1}{2}(x + \overline{x}), \qquad x' = Im(x) = \frac{1}{2}(x - \overline{x}).$$

The left and right multiplication maps L_a and R_a on A with $a \in A$ satisfy the following equations:

Lemma. *We have $\langle L_a x, L_a y\rangle = \langle ax, ay\rangle = \langle x, y\rangle\|a\|^2$ and $\langle R_a x, R_a y\rangle = \langle xa, ya\rangle = \langle x, y\rangle\|a\|^2$.*

Proof. Let us show the latter (the first being the same). We use repeatedly the multiplicative property of the norm.

$$\begin{aligned}
\|(x+y)a\|^2 &= \|x+y\|^2\|a\|^2 \\
\langle (x+y)a, (x+y)a\rangle &= \langle x+y, x+y\rangle\|a\|^2 \\
\langle xa, xa\rangle + 2\langle xa, ya\rangle + \langle ya, ya\rangle &= (\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle)\|a\|^2 \\
\|xa\|^2 + 2\langle xa, ya\rangle + \|ya\|^2 &= (\|x\|^2 + 2\langle x, y\rangle + \|y\|^2)\|a\|^2 \\
\|x\|^2\|a\|^2 + 2\langle xa, ya\rangle + \|y\|^2\|a\|^2 &= (\|x\|^2 + 2\langle x, y\rangle + \|y\|^2)\|a\|^2 \\
\langle xa, ya\rangle &= \langle x, y\rangle\|a\|^2.
\end{aligned}$$

□

Lemma. *We have $\langle ax, y\rangle = \langle x, \overline{a}y\rangle$ and $\langle xa, y\rangle = \langle x, y\overline{a}\rangle$, in other words, the adjoint of L_a is $L_{\overline{a}}$ and the adjoint of R_a is $R_{\overline{a}}$.*

Proof. Consider the first. Inserting $a = a_1 + a'$, this reduces to $\langle a'x, y\rangle = \langle x, -a'y\rangle$. Now, working out the equality $\langle (1+a')x, (1+a')y\rangle = \langle x, y\rangle\|1 + a'\|^2$ (of the previous lemma), yields the desired result. □

The following lemma is very handy (and of omnipotent use!):

Lemma. *For $x, y \in A$, the followings hold:*

(i) $\overline{\overline{x}} = x$,
(ii) $\langle \overline{x}, \overline{y} \rangle = \langle x, y \rangle$,
(iii) $\langle x, y \rangle = Re(x\overline{y}) = Re(\overline{x}y)$,
(iv) $\overline{xy} = \overline{y}\ \overline{x}$,
(v) $Re(x\overline{y}) = \frac{1}{2}(x\overline{y} + y\overline{x})$,
(vi) $x\overline{x} = \overline{x}x = \|x\|^2 = x_1^2 - x'^2$.

Proof. *(i)* is obvious. For *(ii)*: insert $x = x_1 + x'$ and $y = y_1 + y'$. For *(iii)*: $\langle x, y \rangle = \langle x, 1y \rangle = \langle x\overline{y}, 1 \rangle = \langle Re(x\overline{y}) + Im(x\overline{y}), 1 \rangle = Re(x\overline{y})$; likewise, $\langle x, y \rangle = Re(\overline{x}y)$. For the surprising and important *(iv)*:

$$\langle \overline{xy}, z \rangle = \langle \overline{\overline{xy}}, \overline{z} \rangle = \langle xy, \overline{z} \rangle = \langle x, \overline{z}\,\overline{y} \rangle = \langle zx, \overline{y} \rangle = \langle z, \overline{y}\,\overline{x} \rangle = \langle \overline{y}\,\overline{x}, z \rangle.$$

For *(v)*: $Re(x\overline{y}) = \frac{1}{2}(x\overline{y} + \overline{x\overline{y}}) = \frac{1}{2}(x\overline{y} + y\overline{x})$. As a consequence we have $x\overline{y} = -y\overline{x}$ and $\overline{x}y = -\overline{y}x$ for $\langle x, y \rangle = 0$.
For *(vi)*: $\|x\|^2 = \langle x, x \rangle = Re(x\overline{x}) = \frac{1}{2}(x\overline{x} + x\overline{x}) = x\overline{x} = (x_1 + x')(x_1 - x') = x_1^2 - x'^2$. □

We will need also the following:

Lemma. *For $x, y, z, w \in A$ the followings hold:*

(i) $\langle xw, yz \rangle + \langle xz, yw \rangle = 2\langle x, y \rangle \langle w, z \rangle$,
(ii) $x(\overline{y}w) + y(\overline{x}w) = 2\langle x, y \rangle w$,
(iii) $(w\overline{y})x + (w\overline{x})y = 2\langle x, y \rangle w$.

Proof. *(i)*: Let us take the equality $\langle xw, yw \rangle = \langle x, y \rangle \|w\|^2$, and put $w + z$ for w; then,

$$\begin{aligned}
\langle x(w+z), y(w+z) \rangle &= \langle x, y \rangle \langle w+z, w+z \rangle \\
\langle xw + xz, yw + yz \rangle &= \langle x, y \rangle (\|w\|^2 + \|z\|^2 + 2\langle w, z \rangle) \\
\langle xw, yw \rangle + \langle xw, yz \rangle + \langle xz, yw \rangle + \langle xz, yz \rangle &= \langle x, y \rangle (\|w\|^2 + \|z\|^2 + 2\langle w, z \rangle) \\
\langle x, y \rangle \|w\|^2 + \langle xw, yz \rangle + \langle xz, yw \rangle + \langle x, y \rangle \|z\|^2 &= \langle x, y \rangle (\|w\|^2 + \|z\|^2 + 2\langle w, z \rangle) \\
\langle xw, yz \rangle + \langle xz, yw \rangle &= 2\langle x, y \rangle \langle w, z \rangle.
\end{aligned}$$

(ii): Using the equality in *(i)*,

$$\begin{aligned}
\langle xw, yz \rangle + \langle xz, yw \rangle &= 2\langle x, y \rangle \langle w, z \rangle \\
\langle \overline{y}(xw), z \rangle + \langle z, \overline{x}(yw) \rangle &= \langle 2\langle x, y \rangle w, z \rangle \\
\overline{y}(xw) + \overline{x}(yw) &= 2\langle x, y \rangle w.
\end{aligned}$$

Putting $\overline{x}$ for x and $\overline{y}$ for y we find *(ii)* and *(iii)* is similar. □

As a consequence we get $x(\overline{y}w) = -y(\overline{x}w)$ and $(w\overline{y})x = -(w\overline{x})y$ for $\langle x, y \rangle = 0$.

A normed algebra need not be associative as the example of octonions shows. But a weaker form associativity, the alternativity necessarily holds:

Lemma. *A normed real algebra A is alternative, i.e. the associativity $(xy)z = x(yz)$ holds if any two of x, y and z coincide.*

Proof. We first remark that if any one of x, y or z is real, then $(xy)z = x(yz)$ obviously holds. Now let us assume $x = y$. Inserting $x = x_1 + x'$ we see that it is enough to show $(x'x')z = x'(x'z)$, or what the same is, $(\overline{x'}x')z = \overline{x'}(x'z)$. It suffices to show that $\langle(\overline{x'}x')z - \overline{x'}(x'z), w\rangle = \langle\|x'\|^2 z - \overline{x'}(x'z), w\rangle = 0$ for any $w \in A$:

$$\begin{aligned}\langle\|x'\|^2 z - \overline{x'}(x'z), w\rangle &= \langle\|x'\|^2 z, w\rangle - \langle\overline{x'}(x'z), w\rangle \\ &= \|x'\|^2\langle z, w\rangle - \langle x'z, x'w\rangle \\ &= \|x'\|^2\langle z, w\rangle - \|x'\|^2\langle z, w\rangle = 0.\end{aligned}$$

The other cases ($x = z$ or $y = z$) are similar. Note that associativity holds even if any two of x, y and z coincide only up to conjugation. □

We already know that a normed algebra is a division algebra. But with the meanwhile developed apparatus we can do the divisions easily:

Lemma. *Let A be a normed real algebra (with unit). Then the equations $ax = b$ and $ya = b$ (with $a \neq 0$) have the following unique solutions:*

$$x = \frac{1}{\|a\|^2}\overline{a}b \quad \text{and} \quad y = \frac{1}{\|a\|^2}b\overline{a}.$$

Furthermore, any non-zero $a \in A$ has a unique right and left inverse $a^{-1} = \frac{1}{\|a\|^2}\overline{a}$.

Proof. If $ax = b$, then $\overline{a}(ax) = \overline{a}b$ and by the above lemma $(\overline{a}a)x = \overline{a}b$, hence $\|a\|^2 x = \overline{a}b$ and $x = \frac{1}{\|a\|^2}\overline{a}b$ (and this is indeed a solution, thus the unique one). Similarly for $ya = b$. To find the right and left inverses of $a \neq 0$ take $b = 1$. □

The following lemma shows that the (Cayley-)Dickson product we met in the foregoing section is immanent in the structure of a normed algebra:

Lemma. *Suppose that B is a subalgebra (with $1 \in B$) of the normed real algebra A and $\varepsilon \in B^\perp$ with $\|\varepsilon\| = 1$. Then $B\varepsilon$ is orthogonal to B and*

$$(a + b\varepsilon)(c + d\varepsilon) = (ac - \overline{d}b) + (da + b\overline{c})\varepsilon$$

for all $a, b, c, d \in B$.

Proof. Since $1 \in B$, $x \in B$ if and only if $\overline{x} \in B$. $\langle a, b\varepsilon\rangle = \langle\overline{b}a, \varepsilon\rangle = 0$ since $\overline{b}a \in B$ for $a, b \in B$. This shows that $B \perp B\varepsilon$.

Since $\langle\varepsilon, 1\rangle = 0$, $\varepsilon \in Im(A)$, so that $\|\varepsilon\|^2 = \varepsilon\overline{\varepsilon} = -\varepsilon^2$ and thus $\varepsilon^2 = -1$. We examine the right-hand side of the equation

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (b\varepsilon)(d\varepsilon) + a(d\varepsilon) + (b\varepsilon)c,$$

and show that $(b\varepsilon)(d\varepsilon) = -\overline{d}b$, $a(d\varepsilon) = (da)\varepsilon$ and $(b\varepsilon)c = (b\overline{c})\varepsilon$. We use in the following computations the properties $x(\overline{y}w) + y(\overline{x}w) = 0$ and $(w\overline{y})x + (w\overline{x})y = 0$ for $\langle x, y\rangle = 0$ proven above:

$$\begin{aligned}(b\varepsilon)(d\varepsilon) &= -\overline{d}((\overline{b\varepsilon})\varepsilon) = -\overline{d}((\overline{\varepsilon}\ \overline{b})\varepsilon) = \overline{d}((\varepsilon\overline{b})\varepsilon) = -\overline{d}((\varepsilon\overline{\varepsilon})b) = -\overline{d}b, \\ a(d\varepsilon) &= a(-\overline{\varepsilon}\ \overline{d}) = a(\varepsilon\overline{d}) = -\overline{\varepsilon}(\overline{a}\ \overline{d}) = \varepsilon(\overline{a}\ \overline{d}) = -(\overline{(\overline{a}\ \overline{d})})\overline{\varepsilon} = (da)\varepsilon, \\ (b\varepsilon)c &= -(b\overline{c})\overline{\varepsilon} = (b\overline{c})\varepsilon.\end{aligned}$$

□

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We are now ready to prove the Hurwitz Theorem:

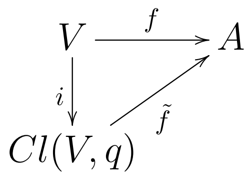
Proof of the Hurwitz Theorem. Let A be a normed algebra. Let $B_1 = Re(A) = \mathbb{R}$. If $B_1 = A$, we are done. If not, choose $\varepsilon_1 \in B_1^\perp$ with $\|\varepsilon_1\| = 1$ and let $B_2 = B_1 + B_1\varepsilon_1$, which is isomorphic to $\mathbb{C}$. If $B_2 = A$, we are done. If not, choose $\varepsilon_2 \in B_2^\perp$ with $\|\varepsilon_2\| = 1$ and $B_3 = B_2 + B_2\varepsilon_2$, which is isomorphic to $\mathbb{H}$. If $B_3 = A$, we are done. If not, choose $\varepsilon_3 \in B_3^\perp$ with $\|\varepsilon_3\| = 1$ and $B_4 = B_3 + B_3\varepsilon_3$, which is isomorphic to $\mathbb{O}$. We claim $B_4 = A$. If not, choose $\varepsilon_4 \in B_4^\perp$ with $\|\varepsilon_4\| = 1$ and let $B_5 = B_4 + B_4\varepsilon_4$. $B_5 \cong \mathbb{O} \oplus \mathbb{O}$, which is not alternative (a small exercise; you will hit probably by the first search upon a non-alternative pair!) and thus it could not be normed. □

5. A Short Look at the Clifford Algebras

You will recall that we were close to discover the Clifford algebras at the stage of looking around after the discovery of the quaternions as happened also historically. It seems that Clifford's work remained fragmentary after his untimely death in 1879 and these algebras were rediscovered independently by Lipschitz shortly thereafter.
The octonions are not a Clifford algebra since the latter are associative but the octonions are not. Nevertheless, there are intimate relationships between them and we give a short look at them.

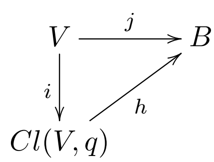
Let V be a (real) vector space with a non-degenerate quadratic form $q : V \to \mathbb{R}$. (i.e. $q(v) = b(v, v)$ for a non-degenerate symmetric, bilinear map $b : V \times V \to \mathbb{R}$.) Clifford algebras can be characterized by a universal property:

The Clifford algebra on the vector space V (with the quadratic form q), denoted by $Cl(V, q)$, is an associative algebra with unit, together with a linear injection $i : V \to Cl(V, q)$ satisfying the property $i(v)^2 = -q(v) \cdot 1$, such that for any associative algebra A with unit and any linear map $f : V \to A$ satisfying $f(v)^2 = -q(v) \cdot 1$, there exists a *unique* algebra homomorphism $\tilde{f} : Cl(V, q) \to A$ such that the following diagram commutes:



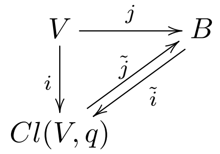
$$(\tilde{f}i = f, \quad \tilde{f} \ \ extends \ \ f)$$

$Cl(V, q)$ is uniquely determined by this property, i.e. if B is another associative algebra with unit, together with an injection $j : V \to B$ with $j(v)^2 = -q(v).1$ and the above universal property is satisfied also for B, then there exists an algebra isomorphism $h : Cl(V, q) \to B$ making the following diagram commutative:

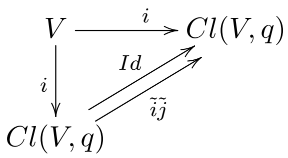


(The algebras are not only isomorphic, but there exists an isomorphism respecting the embeddings.)

To see this, use i and j as test maps. By universality of both $Cl(V,q)$ and B we get the algebra homomorphisms $\tilde{i}$ and $\tilde{j}$:



Note that $\tilde{i}$ and $\tilde{j}$ are inverse to each other: Use i as test map ($f = i$):



Both Id and $\tilde{i}\tilde{j}$ extend i:

$$(\tilde{i}\tilde{j})i = \tilde{i}(\tilde{j}i) = \tilde{i}j = i.$$

By the uniqueness of the extension we get $\tilde{i}\tilde{j} = Id_{Cl(V,q)}$. Similarly we get $\tilde{j}\tilde{i} = Id$.

Universal properties show that if a certain object exists, then it is unique up to isomorphism. But does the object exist? There are various ways to show this. We will give an intuitive, explicit construction.

Let $V = \mathbb{R}^{r,s}$ and $q(x) = x_1^2 + x_2^2 + \cdots + x_r^2 - x_{r+1}^2 - x_{r+2}^2 - \cdots - x_{r+s}^2$, with $r + s = n$. (The general case is equivalent to this. Choosing an appropriate base of $\mathbb{R}^{r,s}$, a non-degenerate quadratic form can be expressed as such.)

We will denote $Cl(\mathbb{R}^n, q)$ by $Cl_{r,s}$, $Cl_{n,0}$ by Cl_n and $Cl_{0,n}$ by Cl'_n. To construct an algebra, it is enough to give a vector space, choose a basis and define the products of the basis elements in terms of these basis elements. One can then extend this product distributively. The emerging algebra is associative if and only if the product is associative on basis elements. (The same is true for commutativity and for the existence of a unit.)

In this spirit, to define $Cl_{r,s}$ we first consider the real vector space generated by the (abstract) basis elements

$$\begin{aligned}
&1, e_1, e_2, \ldots, e_n,\\
&e_1e_2, e_1e_3, \ldots, e_1e_n, e_2e_3, \ldots, e_2e_n, \ldots, e_{n-1}e_n,\\
&e_1e_2e_3, e_1e_2e_4, \ldots, e_{n-2}e_{n-1}e_n,\\
&\vdots\\
&e_1e_2e_3\cdots e_n.
\end{aligned}$$

In short, we take the vector space

$$\langle e_{i_1}e_{i_2}\cdots e_{i_k} \mid i_1 < i_2 < \cdots < i_k\rangle_{\mathbb{R}},$$

where the element for the empty index set is taken as 1. We now define the product:

$$\begin{aligned}
&1 \text{ acts as unit},\\
&e_i^2 = -1 \text{ for } 1 \le i \le r,\ e_i^2 = 1 \text{ for } r+1 \le i \le r+s = n,\\
&e_i \cdot e_j = e_ie_j \text{ for } i < j \text{ and } e_i \cdot e_j = -e_je_i \text{ for } i > j.
\end{aligned}$$

For any two basis elements $e_{i_1}e_{i_2}\cdots e_{i_k}$ and $e_{j_1}e_{j_2}\cdots e_{j_l}$, we write

$$e_{i_1}e_{i_2}\cdots e_{i_k}e_{j_1}e_{j_2}\cdots e_{j_l}$$

and cancel and rearrange according to the above rules, which results in an associative multiplication.

The injection $\mathbb{R}^n \to Cl_{r,s}$ is given by $e_i \mapsto e_i$. Image of $x = \sum x_i e_i$ satisfies $x^2 = -q(x).1$. We must verify the universal property. Let $f : \mathbb{R}^n \to A$ be a test map with $f(x)^2 = -q(x).1$. We must extend f to $\tilde{f} : Cl_{r,s} \to A$. First note that

$$\begin{aligned} f(x+y)^2 &= -q(x+y) \\ (f(x)+f(y))^2 &= -b(x+y, x+y) \\ f(x)^2 + f(y)^2 + f(x)f(y) + f(y)f(x) &= -q(x) - q(y) - 2b(x,y) \\ f(x)f(y) + f(y)f(x) &= -2b(x,y). \end{aligned}$$

Thus for $e_i, e_j \in \mathbb{R}^n$ $(i \neq j)$ it holds $f(e_i)f(e_j) = -f(e_j)f(e_i)$. Now we define $\tilde{f}$ on the linear generators and extend it linearly:

$$\tilde{f}(1) = 1, \quad \tilde{f}(e_i) = f(e_i), \quad \tilde{f}(e_{i_1}e_{i_2}\cdots e_{i_k}) = f(e_{i_1})f(e_{i_2})\cdots f(e_{i_k}).$$

It is easy to see that this map is multiplicatively homomorphic:

$$\begin{aligned} \tilde{f}(e_ie_j) =& f(e_i)f(e_j) = \tilde{f}(e_i)\tilde{f}(e_j) \text{ for } i<j, \\ \tilde{f}(e_je_i) =& \tilde{f}(-e_ie_j) = -\tilde{f}(e_ie_j) \\ =& -f(e_i)f(e_j) = f(e_j)f(e_i) = \tilde{f}(e_j)\tilde{f}(e_i) \text{ for } i<j, \\ \tilde{f}(e_{i_1}e_{i_2}\cdots e_{i_k}\cdot e_{j_1}e_{j_2}\cdots e_{j_l}) =& \tilde{f}(\pm e_{h_1}e_{h_2}\cdots e_{h_m}) \\ =& \pm f(e_{h_1})f(e_{h_2})\cdots f(e_{h_m}) \\ =& f(e_{i_1})\cdots f(e_{i_k})\cdot f(e_{j_1})\cdots f(e_{i_l}) \\ =& \tilde{f}(e_{i_1}e_{i_2}\cdots e_{i_k})\cdot \tilde{f}(e_{j_1}e_{j_2}\cdots e_{j_l}). \end{aligned}$$

We have now the Clifford algebra $Cl_{r,s}$ at our disposal.

It is fun to see that we have $Cl_1 = \langle 1, e_1\rangle_{\mathbb{R}}$ with $e_1^2 = -1$ so that $Cl_1 \cong \mathbb{C}$; and $Cl_2 = \langle 1, e_1, e_2, e_1e_2\rangle_{\mathbb{R}}$ with $e_1^2 = e_2^2 = -1, e_1e_2 = -e_2e_1$, so that $Cl_2 \cong \mathbb{H}$:

$$\begin{aligned} (e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1, \\ e_1(e_1e_2) = -(e_1e_2)e_1, \quad e_2(e_1e_2) = -(e_1e_2)e_2. \end{aligned}$$

so that the map

$$e_1 \mapsto i, \quad e_2 \mapsto j, \quad e_1e_2 \mapsto k$$

gives an algebra isomorphism.

What about Cl_3, or $Cl_1^{'}$, $Cl_2^{'}$ or $Cl_{1,1}$? These are good exercises and we will possibly need some of them below.

For example, $Cl_1{'} = \langle 1, e_1\rangle_{\mathbb{R}}$ with $e_1^2 = 1$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ via $e_1 \mapsto (1,-1)$ and $Cl_2{'} = \langle 1, e_1, e_2, e_1e_2\rangle_{\mathbb{R}}$ with $e_1^2 = e_2^2 = 1$ is isomorphic to $\mathbb{R}(2)$ (the algebra of 2×2 matrices) via

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. The universality property enables us to construct an algebra homomorphism $Cl_{r,s} \to A$ by way of giving the images of $e_1, e_2, \ldots, e_n \in \mathbb{R}^n \subset Cl_{r,s}$ satisfying

$$\begin{aligned} f(e_i)^2 &= -1 && for\ \ 1 \le i \le r, \\ f(e_i)^2 &= 1 && for\ \ r+1 \le i \le r+s = n, \\ f(e_i)f(e_j) + f(e_j)f(e_i) &= 0 && for\ \ i \neq j, \end{aligned}$$

because, in such a case we can first extend f linearly to $\mathbb{R}^n$ and since

$$f(x)^2 = f\left(\sum x_i e_i\right)^2 = \left(\sum x_i f(e_i)\right)^2 = -q(x) \ ,$$

we can then extend f to $Cl_{r,s}$ by the universal property. This is a very useful device in Clifford algebra theory.

Mapping e_i to $-e_i$ gives rise to an involution $\alpha : Cl_{r,s} \longrightarrow Cl_{r,s}$ yielding a decomposition into *even* and *odd* parts, $Cl_{r,s} = Cl^0_{r,s} \oplus Cl^1_{r,s}$ with $\alpha(u) = u$ for $u \in Cl^0_{r,s}$ and $\alpha(u) = -u$ for $u \in Cl^1_{r,s}$. Note that $Cl^0_{r,s}$ is a subalgebra of $Cl_{r,s}$.

Proposition. *There exists an algebra isomorphism* $Cl_{r,s} \cong Cl^0_{r+1,s}$. *In particular,* $Cl_n \cong Cl^0_{n+1}$.

(For a proof of the case we need, $Cl_n \cong Cl^0_{n+1}$, consider $e_i \mapsto e_0 e_i$ for $\mathbb{R}^n = \langle e_1, e_2, \ldots, e_n\rangle$ and $\mathbb{R}^{n+1} = \langle e_0, e_1, e_2, \ldots, e_n\rangle$.)

The following reduction properties will enable us to compute all Clifford algebras.

Theorem (Reduction Theorem)**.** *There exist isomorphisms*

(i) $Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2}$,

(ii) $Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}$,

(iii) $Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}$.

As an example, we give a proof for the first case in detail:

We will define an isomorphism $Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}$ which will come from a map $\mathbb{R}^{n+2} \to Cl_{n,0} \otimes Cl_{0,2}$. Choose the basis $e_1, e_2, \ldots, e_{n+2}$ for $\mathbb{R}^{n+2}$ with $q(e_i) = -1$ and denote the standard generators of $Cl_{n,0}$ by $e'_1, e'_2, \ldots, e'_n$, those of $Cl_{0,2}$ by e''_1, e''_2. Let f be given as

$$\begin{aligned} f: \quad \mathbb{R}^{n+2} &\longrightarrow Cl_{n,0} \otimes Cl_{0,2} \\ e_i &\mapsto e'_i \otimes e''_1 e''_2 \ for\ 1 \le i \le n \\ e_{n+1} &\mapsto 1 \otimes e''_1 \\ e_{n+2} &\mapsto 1 \otimes e''_2. \end{aligned}$$

Let us check the necessary relations:

$$\begin{aligned} f(e_i)^2 &= (e'_i \otimes e''_1 e''_2) \cdot (e'_i \otimes e''_1 e''_2) = {e'_i}^2 \otimes e''_1 e''_2 e''_1 e''_2 = (-1) \otimes (-1) = 1 \otimes 1, \\ f(e^2_{n+1}) &= (1 \otimes e''_1)(1 \otimes e''_1) = 1 \otimes {e''_1}^2 = 1 \otimes 1, \\ f(e^2_{n+2}) &= (1 \otimes e''_2)(1 \otimes e''_2) = 1 \otimes {e''_2}^2 = 1 \otimes 1. \end{aligned}$$

$$\begin{aligned} f(e_i)f(e_j) + f(e_j)f(e_i) &= (e'_i \otimes e''_1 e''_2)(e'_j \otimes e''_1 e''_2) + (e'_j \otimes e''_1 e''_2)(e'_i \otimes e''_1 e''_2) \\ &= e'_i e'_j \otimes (-1) + e'_j e'_i \otimes (-1) \\ &= 0 \quad for \quad 1 \le i, j \le n, \quad i \ne j. \end{aligned}$$

$$\begin{aligned} f(e_i)f(e_{n+1}) + f(e_{n+1})f(e_i) &= (e'_i \otimes e''_1 e''_2)(1 \otimes e''_1) + (1 \otimes e''_1)(e'_i \otimes e''_1 e''_2) \\ &= e'_i \otimes e''_1 e''_2 e''_1 + e'_i \otimes e''_1 e''_1 e''_2 \\ &= e'_i \otimes (-e''_2) + e'_i \otimes e''_2 = 0. \end{aligned}$$

$$\begin{aligned} f(e_i)f(e_{n+2}) + f(e_{n+2})f(e_i) &= (e'_i \otimes e''_1 e''_2)(1 \otimes e''_2) + (1 \otimes e''_2)(e'_i \otimes e''_1 e''_2) \\ &= e'_i \otimes e''_1 e''_2 e''_2 + e'_i \otimes e''_2 e''_1 e''_2 \\ &= e'_i \otimes e''_1 + e'_i \otimes (-e''_1) = 0. \end{aligned}$$

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$$\begin{aligned} f(e_{n+1})f(e_{n+2}) + f(e_{n+2})f(e_{n+1}) &= (1\otimes e_1'')(1\otimes e_2'') + (1\otimes e_2'')(1\otimes e_1'') \\ &= 1\otimes e_1''e_2'' + 1\otimes e_2''e_1'' = 0. \end{aligned}$$

Thus we get an algebra homomorphism $Cl_{0,n+2} \longrightarrow Cl_{n,0}\otimes Cl_{0,2}$.

The terms $1\otimes e_1'',\ 1\otimes e_2''$ and $(e_i'\otimes e_1''e_2'')(1\otimes e_1'')(1\otimes e_2'') = e_i'\otimes(-1)$ appear in the image and they generate $Cl_{n,0}\otimes Cl_{0,2}$. The map is then surjective and by dimensional reasons an isomorphism.

With the help of the reduction theorem we can now prove the startling periodicity theorem:

Theorem (Periodicity Theorem)**.** *We have*

(i) $Cl_{n+8} \cong Cl_n \otimes Cl_8 \cong Cl_n \otimes \mathbb{R}(16)$.

(ii) $Cl'_{n+8} \cong Cl'_n \otimes Cl'_8 \cong Cl'_n \otimes \mathbb{R}(16)$.

Proof. Let us show the first case (the second being similar). Using repeatedly the reduction theorem we get

$$\begin{aligned} Cl_{n+8} &\cong Cl'_{n+6}\otimes Cl_2 \\ &\cong Cl_{n+4}\otimes Cl'_2\otimes Cl_2 \\ &\cong Cl'_{n+2}\otimes Cl_2\otimes Cl'_2\otimes Cl_2 \\ &\cong Cl_n\otimes \overbrace{Cl'_2\otimes Cl_2\otimes Cl'_2\otimes Cl_2}^{Cl_8} \\ &\cong Cl_n\otimes \mathbb{R}(2)\otimes\mathbb{H}\otimes\mathbb{R}(2)\otimes\mathbb{H} \\ &\cong Cl_n\otimes(\mathbb{H}\otimes\mathbb{H})\otimes\mathbb{R}(2)\otimes\mathbb{R}(2) \\ &\cong Cl_n\otimes\mathbb{R}(4)\otimes\mathbb{R}(4) \\ &\cong Cl_n\otimes\mathbb{R}(16). \end{aligned}$$

We owe to the reader a justification of the final steps: $\mathbb{R}(n)\otimes\mathbb{R}(m)\cong\ \mathbb{R}(nm)$ (where $\mathbb{R}(n)$ is the algebra of real $n\times n$ matrices) and $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}\cong\ \mathbb{R}(4)$. For the first take

$$(a_{ij})\otimes B \mapsto \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

and for the second consider the $\mathbb{R}$-bilinear map

$$\begin{aligned} \psi:\quad \mathbb{H}\times\mathbb{H} &\longrightarrow Hom_{\mathbb{R}}(\mathbb{H},\mathbb{H})\cong\mathbb{R}(4) \\ (q_1,q_2) &\longmapsto \quad \psi(q_1,q_2)(x) = q_1x\bar{q_2}, \end{aligned}$$

which gives

$$\begin{aligned} \tilde{\psi}:\quad \mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} &\longrightarrow Hom_{\mathbb{R}}(\mathbb{H},\mathbb{H}) \\ (q_1,q_2) &\longmapsto \quad \tilde{\psi}(q_1\otimes q_2)(x) = q_1x\bar{q_2}, \end{aligned}$$

which can be checked to be an algebra isomorphism. □

The Cl_7, which we will talk about more below, can be computed to be $Cl_7\cong {Cl_1}'\otimes Cl_2\otimes {Cl_2}'\otimes Cl_2 = (\mathbb{R}\oplus\mathbb{R})\otimes\mathbb{H}\otimes\mathbb{R}(2)\otimes\mathbb{H} = (\mathbb{R}\oplus\mathbb{R})\otimes\mathbb{R}(8) = \mathbb{R}(8)\oplus\mathbb{R}(8)$.

6. Spin Groups

We now want to give the definition of spin groups and a minimum of basic facts about them. We consider only the positive-definite case. Spin groups live in Clifford algebras and can be defined as

$$Spin(n) = \{v_1 v_2 \cdots v_{2k} \mid v_i \in \mathbb{R}^n, \|v_i\| = 1\} \subset Cl_n.$$

There is however another more structured and illuminating definition which we want to consider and reduce to the former.

Let us define

$$\Gamma = \{u \in Cl_n^* \mid \alpha(u)xu^{-1} \in \mathbb{R}^n \ \text{ for all } \ x \in \mathbb{R}^n\}$$

and $\Gamma^0 = \Gamma \cap Cl_n^0$, where Cl_n^* denotes the group of invertible elements of Cl_n. α is the grading automorphism defined in the previous section.

Lemma. *Γ is a subgroup of Cl_n^*.*

Proof. For $u_1, u_2 \in \Gamma$, $u_1 \cdot u_2 \in \Gamma$:

$$\alpha(u_1 u_2)x(u_1 u_2)^{-1} = \alpha(u_1)\alpha(u_2)xu_2^{-1}u_1^{-1} = \alpha(u_1)(\alpha(u_2)xu_2^{-1})u_1^{-1}$$

For $u \in \Gamma$, $u^{-1} \in \Gamma$: Define

$$\begin{aligned} \widetilde{Ad}(u):\ \ \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \alpha(u)xu^{-1}. \end{aligned}$$

This map is one-to-one: $\alpha(u)xu^{-1} = 0$ implies $x = 0$. So, it is onto. Given $y \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ with $\alpha(u)xu^{-1} = y$, i.e. $x = \alpha(u^{-1})yu = \alpha(u^{-1})y(u^{-1})^{-1}$. It means that $u^{-1} \in \Gamma$. □

Now we define the *twisted adjoint representation* (which is a group homomorphism):

$$\begin{aligned} \widetilde{Ad}:\ \ \Gamma &\longrightarrow GL(\mathbb{R}^n) \\ u &\longmapsto \widetilde{Ad}(u). \end{aligned}$$

Lemma. *$\widetilde{Ad}(u) \in O(n)$.*

Proof.

$$\begin{aligned}
\langle \widetilde{Ad}(u)(x), \widetilde{Ad}(u)(x) \rangle &= \langle \alpha(u)xu^{-1}, \alpha(u)xu^{-1} \rangle \\
&= -(\alpha(u)xu^{-1})^2 \quad \textit{since} \ \ \alpha(u)xu^{-1} \in \mathbb{R}^n \\
&= -(\alpha(u)xu^{-1})(\alpha(u)xu^{-1}) \\
&= \alpha(u)xu^{-1}\alpha(\alpha(u)xu^{-1}) \quad \textit{since} \ \ \alpha(v) = -v \ for \ v \in \mathbb{R}^n \\
&= \alpha(u)xu^{-1}u\alpha(x)\alpha(u^{-1}) \\
&= \alpha(u)x(-x)\alpha(u^{-1}) \\
&= \alpha(u)\langle x, x \rangle \alpha(u^{-1}) \\
&= \langle x, x \rangle.
\end{aligned}$$

□

So we have a map $\widetilde{Ad} : \Gamma \longrightarrow O(n)$. We note two facts:

- $v \in \mathbb{R}^n, v \neq 0$, belongs to Γ:

$$\begin{aligned}\alpha(v)xv^{-1} &= (-v)xv^{-1} = -(vx)v^{-1} = -(-xv - 2\langle v, x\rangle)v^{-1} \\ &= x + 2\langle v, x\rangle v^{-1} = x - 2\frac{\langle v, x\rangle}{\langle v, v\rangle}v \in \mathbb{R}^n.\end{aligned}$$

- $\widetilde{Ad}(v)$ is a reflection across $v^{\perp}$ in $\mathbb{R}^n$:

$$\widetilde{Ad}(v)(v) = -v \text{ and } \widetilde{Ad}(v)(x) = x \text{ for } \langle v, x\rangle = 0.$$

  We denote this reflection along $v^{\perp}$ by $Ref(v)$.

We now want to compute the kernel of $\widetilde{Ad}$.

Lemma. *The kernel of $\widetilde{Ad} : \Gamma \longrightarrow O(n)$ consists of non-zero scalars.*

Proof. Let $\widetilde{Ad}(u) = Id$, i.e. $\alpha(u)xu^{-1} = x$ for all $x \in V$. Decompose u into even and odd parts: $u = u_0 + u_1$, so that we have $\alpha(u) = u_0 - u_1$. Then, inserting this into $\alpha(u)x = xu$ we get $u_0x = xu_0$ (odd degrees) and $-u_1x = xu_1$ (even degrees). Especially, $u_0e_i = e_iu_0$ and $-u_1e_i = e_iu_1$ for the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

Fixing one e_i temporarily, u_0 can be expressed as $u_0 = a_i + e_ib_i$, where the terms a_i and b_i do not contain e_i. Inserting this into $u_0e_i = e_iu_0$ and inspecting the degrees of the terms, one obtains $b_i = 0$ and thus $u_0 = a_i$. This means that u_0 does not contain e_i! As this is true for all i, u_0 must be a scalar.

Similarly, let $u_1 = c_i + e_id_i$, where e_i is fixed and c_i and d_i do not contain e_i. Inserting this into $-u_1e_i = e_iu_1$, one obtains $d_i = 0$. This means that u_1 does not contain any e_i and thus must be zero as an odd-degree element. Consequently, u is a scalar. As $u \in \Gamma$, it must be a non-zero scalar. □

Lemma. *$\widetilde{Ad} : \Gamma \longrightarrow O(n)$ is onto.*

Proof. Any element $f \in O(n)$ is, by Cartan-Dieudonné theorem, a product of reflections, say $f = Ref(v_1)Ref(v_2)\cdots Ref(v_k)$. As $Ref(v) = \widetilde{Ad}(v)$, we get

$$f = \widetilde{Ad}(v_1)\widetilde{Ad}(v_2)\cdots\widetilde{Ad}(v_k) = \widetilde{Ad}(v_1v_2\cdots v_k).$$ □

We are now in a position to clarify the structure of Γ. Let $u \in \Gamma$. We have $\widetilde{Ad}(u) = \widetilde{Ad}(v_1v_2\cdots v_k)$ for suitable $v_i \in \mathbb{R}^n$. Then

$$\begin{aligned}\widetilde{Ad}(u^{-1}v_1v_2\cdots v_k) &= Id \\ u^{-1}v_1v_2\cdots v_k &= \text{non-zero scalar!} \\ u &= \lambda v_1v_2\cdots v_k, \quad v_i \in \mathbb{R}^n, \quad v_i \neq 0, \ \lambda \in \mathbb{R}, \ \lambda \neq 0 \\ u &= \underbrace{\lambda\|v_1\|\|v_2\|\cdots\|v_k\|}_{\mu}\frac{v_1}{\|v_1\|}\frac{v_2}{\|v_2\|}\cdots\frac{v_k}{\|v_k\|}, \quad v_i \neq 0, \lambda \neq 0 \\ u &= \mu w_1w_2\cdots w_k, \quad w_i \in \mathbb{R}^n, \ \|w_i\| = 1, \ \mu \neq 0.\end{aligned}$$

So we can write

$$\Gamma = \{\lambda v_1v_2\cdots v_k \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, \lambda \in \mathbb{R}^*\}$$

and

$$\Gamma^0 = \Gamma \cap Cl_n^0 = \{\lambda v_1v_2\cdots v_{2k} \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, \lambda \in \mathbb{R}^*\}.$$

As a consequence we see that Γ^0 is mapped by $\widetilde{Ad}$ onto $SO(n)$ with kernel consisting of the non-zero scalars. We want to get rid of the factor λ in an invariant way.

We define the *norm* function

$$\begin{aligned} N:\ \ Cl_n &\longrightarrow Cl_n \\ u &\longmapsto u.\alpha(u^t) \end{aligned}$$

where t is the transpose-antiinvolution of Cl_n, i.e. $(e_{i_1}e_{i_2}\cdots e_{i_k})^t = e_{i_k}e_{i_{k-1}}\cdots e_{i_1}$.

Lemma. *$N(u) \in \mathbb{R}^*$ for $u \in \Gamma$.*

Proof.

$$\begin{aligned} u &= \lambda v_1 v_2 \cdots v_k \quad with\ v_i \in \mathbb{R}^n, \|v_i\| = 1, \lambda \neq 0 \\ u^t &= \lambda v_k v_{k-1} \cdots v_1 \\ \alpha(u^t) &= \lambda(-v_k)(-v_{k-1})\cdots(-v_1) \\ u \cdot \alpha(u^t) &= \lambda v_1 v_2 \cdots v_k.\lambda(-v_k)(-v_{k-1})\cdots(-v_1) = \lambda^2. \end{aligned}$$

□

Now consider the set

$$\{u \in Cl_n^* \mid \alpha(u)xu^{-1} \in \mathbb{R}^n \ for\ x \in \mathbb{R}^n,\ N(u) = 1\} \ \cap\ Cl_n^0.$$

This set coincides with the $Spin(n) = \{v_1 v_2 \cdots v_{2k} \mid v_i \in \mathbb{R}^n, \|v_i\| = 1\} \subset Cl_n$, given as an ad hoc definition in the beginning of this section. Because, in the first place, u must have the form $u = \lambda v_1 v_2 \cdots v_k$. The condition $N(u) = 1$ necessitates $\lambda = \pm 1$ (and $\lambda = -1$ can be put in v_1). As $u \in Cl_n^0$, k must be even. We have thus arrived at a better understanding of $Spin(n)$. Since $\alpha(u) = u$ and $N(u) = uu^t$ for $u \in Cl^0$, we can express $Spin(n)$ also as the set

$$\{u \in Cl_n^0 \mid uxu^{-1} \in \mathbb{R}^n \ \ for\ \ x \in \mathbb{R}^n,\ uu^t = 1\}$$

(the condition $uu^t = 1$ implies that u is invertible). The restriction of $\widetilde{Ad}$ thus gives a surjective homomorphism $Spin(n) \longrightarrow SO(n)$ with kernel $\{\pm 1\}$.

We remark that, as a topological group, $Spin(n)$ is a 2:1 covering space of $SO(n)$ and for $n \geq 3$ it is simply connected and thus the universal covering space of $SO(n)$.

7. A Closer Look at Seven and Eight Dimensions

In this section we want to give an octonionic model for Cl_7 and Cl_8.

To apply the universal property of Clifford algebras, we consider the 8-dimensional real vector space $V = \mathbb{O} \cong \mathbb{R}^8$, the algebra $A = End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$, and the map

$$f : \mathbb{O} \longrightarrow End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O}),\ f(u) = \begin{pmatrix} 0 & R_u \\ -R_{\overline{u}} & 0 \end{pmatrix},$$

R_u being the right octonionic multiplication $R_u : \mathbb{O} \to \mathbb{O},\ R_u(x) = xu$. We have

$$f(u)^2 = \begin{pmatrix} -R_u R_{\overline{u}} & 0 \\ 0 & -R_{\overline{u}} R_u \end{pmatrix}$$

and since $R_u R_{\overline{u}}(x) = (x\overline{u})u = x(\overline{u}u) = \|u\|^2 x$ for $x \in \mathbb{O}$ (and $R_{\overline{u}} R_u(x) = \|u\|^2 x$) we get

$$f(u)^2 = \begin{pmatrix} -\|u\|^2 I_8 & 0 \\ 0 & -\|u\|^2 I_8 \end{pmatrix} = -\|u\|^2 I_{16}$$

(where $I_8 = Id_{\mathbb{O}}$ and $I_{16} = Id_{\mathbb{O}\oplus\mathbb{O}}$). Thus we obtain by the universal property an algebra homomorphism

$$\tilde{f} : Cl_8 = Cl(\mathbb{O}) \longrightarrow End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O}).$$

We already know that Cl_8 is isomorphic to the matrix algebra $\mathbb{R}(16)$ and as such, a simple algebra (i.e. it has no non-trivial two-sided ideals. This is a good exercise.) The kernel of $\tilde{f}$, being a two-sided ideal, must be zero, as it cannot be the whole of Cl_8 because $\tilde{f}$ is non-zero. Hence $\tilde{f}$ is one-to-one and by dimensional reasons, it is an isomorphism.

Let $e_0, e_1, \ldots, e_7$ be the standard basis of $\mathbb{O}$. We should make careful distinction between octonionic multiplication and Clifford multiplication, e.g. e_0 is the unit of octonionic multiplication, but the unit of Clifford multiplication is the external $1 \in \mathbb{R}$. Note that $\mathbb{O} = \langle e_0, e_1, \ldots, e_7 \rangle_{\mathbb{R}}$ and

$$Cl_8 = \langle 1, e_0, e_1, \ldots, e_7, e_0e_1, \ldots, e_6e_7, \ldots, e_0e_1 \ldots e_7 \rangle_{\mathbb{R}}.$$

We have

$$f(e_0) = \tilde{f}(e_0) = \begin{pmatrix} 0 & I_8 \\ -I_8 & 0 \end{pmatrix} \text{ and } \tilde{f}(1) = \begin{pmatrix} I_8 & 0 \\ 0 & I_8 \end{pmatrix} = I_{16}.$$

We know that Cl_7 is isomorphic to the even part Cl_8^0 of Cl_8. What is it under the isomorphism $\tilde{f} : Cl_8 \longrightarrow End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$?

We first remark that, in any even-dimensional Clifford algebra Cl_n, the even part Cl_n^0 consists exactly of elements commuting with the *volume element* $e_1e_2 \cdots e_n$, in our case, with $e_0e_1 \cdots e_7$, since

$$e_i \cdot (e_0e_1 \cdots e_7) = -(e_0e_1 \cdots e_7) \cdot e_i \text{ and } (e_ie_j) \cdot (e_0e_1 \cdots e_7) = (e_0e_1 \cdots e_7) \cdot (e_ie_j).$$

Now, what is the image of $e_0e_1 \cdots e_7$ under $\tilde{f}$ in $End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$?

$$\begin{aligned} \tilde{f}(e_0e_1 \cdots e_7) &= \tilde{f}(e_0)\tilde{f}(e_1) \cdots \tilde{f}(e_7) \\ &= \begin{pmatrix} 0 & I_8 \\ -I_8 & 0 \end{pmatrix} \begin{pmatrix} 0 & R_{e_1} \\ -R_{\overline{e}_1} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & R_{e_7} \\ -R_{\overline{e}_7} & 0 \end{pmatrix} \\ &= \begin{pmatrix} R_{e_1}R_{e_2} \cdots R_{e_7} & 0 \\ 0 & -R_{e_1}R_{e_2} \cdots R_{e_7} \end{pmatrix} \end{aligned}$$

A straightforward check shows that $R_{e_1}R_{e_2} \cdots R_{e_7} = I_8$! We thus get

$$\tilde{f}(e_0e_1 \cdots e_7) = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix}.$$

Now we can identify Cl_8^0 inside $End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$: A matrix commuting with this one must be of type $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. We can express this fact also as $Cl_8^0 \cong End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$.

Let us recall the identification of Cl_n with Cl_{n+1}^0 and apply it to the present case:

$$\begin{aligned} Cl_7 \longrightarrow Cl_8^0 &\longrightarrow End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O}) \subset End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O}), \\ e_i \longmapsto e_0e_i &\longmapsto \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & R_{e_i} \\ R_{e_i} & 0 \end{pmatrix} = \begin{pmatrix} R_{e_i} & 0 \\ 0 & -R_{e_i} \end{pmatrix}, \\ v = \sum \lambda_i e_i \longmapsto \sum \lambda_i e_0 e_i &\longmapsto \begin{pmatrix} R_v & 0 \\ 0 & -R_v \end{pmatrix} \end{aligned}$$

The resulting isomorphism $Cl_7 \cong End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$ can thus be viewed as the extension of the map

$$\begin{aligned} Im(\mathbb{O}) \cong \mathbb{R}^7 &\longrightarrow End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O}) \\ v &\longmapsto (R_v, -R_v). \end{aligned}$$

(We could define this map from the start, but we wanted to relate the octonionic models of Cl_7 and Cl_8 and wanted to understand how Cl_7 sits in $End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$.)

We now want to look at $Spin(7) \subset Cl_7^0$ and see how it sits in $End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O})$. Recall that $Spin(7) = \{v_1 v_2 \cdots v_{2k} \mid v_i \in Im(\mathbb{O}), \|v_i\| = 1\}$ so that we get

$$\begin{aligned} v_1 \cdot v_2 \cdots v_{2k} \longmapsto & (R_{v_1}, -R_{v_1})(R_{v_2}, -R_{v_2}) \cdots (R_{v_{2k}}, -R_{v_{2k}}) \\ &= (R_{v_1} R_{v_2} \cdots R_{v_{2k}}, R_{v_1} R_{v_2} \cdots R_{v_{2k}}). \end{aligned}$$

Since R_{v_i} are orthogonal (by $\|v_i\| = 1$), we get a diagonal embedding

$$Spin(7) \subset O(8) \oplus O(8) \subset End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O}).$$

Interestingly, $Spin(7)$ is generated by (R_v, R_v) (for $v \in Im(\mathbb{O}) \cong \mathbb{R}^7$ and $\|v\| = 1$) also, because R_v itself is a product of an even number of right multiplications, $R_v = R_{e_1} R_{e_2} \cdots R_{e_7} R_v$, since, as we remarked above, $R_{e_1} R_{e_2} \cdots R_{e_7} = Id_{\mathbb{O}}$.

Projecting onto the first factor we obtain a very useful result:

Proposition. *$Spin(7)$ is isomorphic to the subgroup of $O(8)$ generated by octonionic right multiplications R_v for $v \in Im(\mathbb{O})$, $\|v\| = 1$.*

8. Special Forms in Seven and Eight Dimensions

A calibrated n-manifold is a Riemannian manifold equipped with a closed,non-degenerate p-form for some $0 \leq p \leq n$.Theory of calibrations in general is due to Harvey and Lawson. However, much earlier in 1966 Edmond Bonan introduced G_2 and $Spin(7)$ holonomy manifolds that are seven and eight dimensional, respectively. He constructed all the parallel forms on these exceptional holonomy manifolds and showed that they are Ricci-flat. The "associative" 3- and "co-associative" 4-forms on $\mathbb{R}^7$ $(= Im(\mathbb{O}))$ and the Bonan form on $\mathbb{R}^8$ $(= \mathbb{O})$, which play an eminent role in exceptional G_2 geometry (G_2 being the automorphism group of the octonions) can most succinctly be defined in terms of octonions, indicating that the octonions are their natural home:

$$\begin{aligned} \varphi \in \Lambda^3(Im\mathbb{O})^*, &\ \varphi(x, y, z) = \langle x, yz \rangle, \\ \psi \in \Lambda^4(Im\mathbb{O})^*, &\ \psi(x, y, z, w) = \langle x, y \times z \times w \rangle, \\ \Phi \in \Lambda^4\mathbb{O}^*, &\ \Phi(x, y, z, w) = \langle x, y \times z \times w \rangle \end{aligned}$$

(so that $\psi = \Phi \mid_{Im\mathbb{O}}$), where $y \times z \times w = \frac{1}{2}[y(\overline{z}w) - w(\overline{z}y)]$ is the triple cross product for $y, z, w \in \mathbb{O}$.

This triple cross product might seem strange and indeed it is not equal to $(y \times z) \times w$, where $y \times z$ is defined by $Im(\overline{z}y) = \frac{1}{2}(\overline{z}y - \overline{y}z)$.

For $y, z \in Im(\mathbb{O})$, we have $y \times z = \frac{1}{2}((-z)y - (-y)z) = \frac{1}{2}(yz - zy)$. We know for $y, z \in \mathbb{O}$, $\langle y, z \rangle = Re(y\overline{z}) = \frac{1}{2}(y\overline{z} + z\overline{y})$, so that for $y, z \in Im(\mathbb{O})$ it holds $\langle y, z \rangle = -\frac{1}{2}(yz + zy)$, whence we get $y \times z = yz + \langle y, z \rangle$.

For $y, z \in \mathbb{O}$ and $\langle y, z \rangle = 0$ we have $y \times z = \frac{1}{2}(\overline{z}y - \overline{y}z) = \overline{z}y$ so that it holds $\|y \times z\| = \|y\|\|z\|$. Now consider three pair-wise orthogonal elements $y, z, w \in \mathbb{O}$. Then, $y(\overline{z}w) = -z(\overline{y}w)$ since y

and z orthogonal; $z(\overline{y}w) = -z(\overline{w}y)$ since y and w are orthogonal; $z(\overline{w}y) = -w(\overline{z}y)$ since z and w orthogonal, so that we have $y(\overline{z}w) = -w(\overline{z}y)$, whence we get $y \times z \times w = y(\overline{z}w)$ and hence $\|y \times z \times w\| = \|y\|\|z\|\|w\|$. This might be seen as a rationale for the definition of the triple cross product. (This property can not be expected from the iterated double cross product: e.g. e_1, e_2 and $e_3 \in \mathbb{O}$ are pair-wise orthogonal, but $e_1 \times e_2 = Im(\overline{e_2}e_1) = Im(-e_2e_1) = Im(e_3) = e_3$ and $(e_1 \times e_2) \times e_3 = e_3 \times e_3 = Im(\overline{e}_3e_3) = Im(-e_3^2) = Im(1) = 0$. On the other hand, $e_1 \times e_2 \times e_3 = e_1(\overline{e}_2e_3) = -e_1(e_2e_3) = -e_1e_1 = 1$.)

The double and triple cross products can easily be seen to be alternating and likewise, the multi-linear functions φ, ψ and Φ can be seen to be alternating by showing that they vanish if two of the variables coincide. For many amazing details about these forms we refer to the seminal work of Harvey-Lawson [9] and to the book Harvey [8]; we will give below only a few indications of proofs.

To show, for example, that φ is alternating, consider the cases $x = y,\ x = z$ and $y = z$:

For $x = y$ we have $\langle x, xz\rangle = \|x\|^2\langle 1, z\rangle = 0$ since $z \in Im(\mathbb{O})$. Similarly for $x = z$. For $y = z$ we have $\langle x, y^2\rangle = \langle x, -\overline{y}y\rangle = -\langle x, \|y\|^2\rangle = -\|y\|^2\langle x, 1\rangle = 0$.

We recommend as an exercise to see whether $\psi = *\varphi$ with respect to the standard inner product on $Im(\mathbb{O}) = \mathbb{R}^7$.

$G_2 = Aut(\mathbb{O})$ is a compact, connected, simple Lie group of dimension 14. We can guess the dimension easily, but let us first note that G_2 can be viewed as a subgroup of the orthogonal group $O(Im(\mathbb{O})) = O(7)$: Let $g \in G_2$. $1 \in \mathbb{O}$ can be seen to go to 1 under g, and since $x^2 \in \mathbb{R}(= Re(\mathbb{O}))$ iff $x \in \mathbb{R}$ or $x \in Im(\mathbb{O})$, $g(x)$ must belong to $Im(\mathbb{O})$ for $x \in Im(\mathbb{O})$, so that g respects the conjugation, $\overline{g(x)} = g(\overline{x})$; and consequently, $\|g(x)\|^2 = g(x)\overline{g(x)} = g(x)g(\overline{x}) = g(x\overline{x}) = g(\|x\|^2) = \|x\|^2 g(1) = \|x\|^2$, showing $g \in O(7)$.

Now, to construct an automorphism of $\mathbb{O} = \langle 1, e_1, e_2, \ldots, e_7\rangle_{\mathbb{R}}$, where 1 goes necessarily to 1, we can map e_1 to any element of the unit sphere in $Im(\mathbb{O}) = \langle e_1, e_2, \ldots, e_7\rangle_{\mathbb{R}}$ (giving a 6-dimensional choice) and then map e_2 to any element in this sphere orthogonal to e_1 (an additional five dimensional choice). Now $e_3 = e_1e_2$ has to go to $g(e_1)g(e_2)$. We have a last 3-dimensional choice for e_4 (orthogonal to e_1, e_2 and e_3). The images of e_5, e_6 and e_7 are then determined by the multiplication table and we get 6+5+3=14.

The forms φ and ψ are invariant under G_2:

Let $g \in G_2$. Then, $g\,|_{Im(\mathbb{O})}: Im(\mathbb{O}) \longrightarrow Im(\mathbb{O})$ and we have

$$g^*(\varphi)(x, y, z) = \varphi(g(x), g(y), g(z)) = \langle g(x), g(y)g(z)\rangle = \langle g(x), g(yz)\rangle = \langle x, yz\rangle,$$

so that $g^*(\varphi) = \varphi$ and

$$\begin{aligned}
g^*(\psi)(x, y, z, w) &= \psi(g(x), g(y), g(z), g(w)) = \langle g(x), g(y) \times g(z) \times g(w)\rangle \\
&= \langle g(x), \frac{1}{2}[g(y)(\overline{g(z)}g(w)) - g(w)(\overline{g(z)}g(y))]\rangle \\
&= \langle g(x), \frac{1}{2}[g(y(\overline{z}w)) - g(w(\overline{z}y))]\rangle \\
&= \langle g(x), g[\frac{1}{2}(y(\overline{z}w) - w(\overline{z}y))]\rangle \\
&= \langle x, \frac{1}{2}(y(\overline{z}w) - w(\overline{z}y))\rangle \\
&= \langle x, y \times z \times w\rangle,
\end{aligned}$$

so that $g^*(\psi) = \psi$. Similarly, one sees $g^*(\Phi) = \Phi$ since g respects the octonionic multiplication and conjugation and it is obviously orthogonal also on $\mathbb{O} = Re(\mathbb{O}) \oplus Im(\mathbb{O})$ ($\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$), fixing the first component.

Surprisingly, the Bonan form Φ is invariant even under $Spin(7)$! We want to give only the proof-idea for this property. There is a four-fold cross product for octonions:

$$x \times y \times z \times w = \frac{1}{4}[\overline{x}(y \times z \times w) + \overline{y}(z \times x \times w) + \overline{z}(x \times y \times w) + \overline{w}(y \times x \times z)],$$

where $x, y, z, w \in \mathbb{O}$. This cross product is alternating and for pair-wise orthogonal vectors it can be expressed as $x \times y \times z \times w = \overline{x}(y(\overline{z}w))$. Moreover, it can be shown that $\langle x, y \times z \times w\rangle = Re(x \times y \times z \times w)$. Furthermore, it can be shown with the help of the so-called Moufang identities that, for $v \in Im(\mathbb{O})$, $\|v\| = 1$ and $x, y, z, w \in \mathbb{O}$ the following equality holds: $(xv) \times (yv) \times (zv) \times (wv) = v(x \times y \times z \times w)\overline{v}$.

Now we can see the invariance of the Bonan form (sometimes called the Cayley calibration) Φ under $Spin(7)$ as follows: Recall that $Spin(7) \subset O(\mathbb{O}) = O(8)$ is generated by octonionic right multiplications R_v for $v \in Im(\mathbb{O})$, $\|v\| = 1$. So, it will be enough to see $(R_v^*\Phi)(x, y, z, w) = \Phi(x, y, z, w)$ for $x, y, z, w \in \mathbb{O}$:

$$\begin{aligned}(R_v^*\Phi)(x, y, z, w) &= \Phi(xv, yv, zv, wv)\\ &= \langle xv, yv \times zv \times wv\rangle\\ &= Re(xv \times yv \times zv \times wv)\\ &= Re(v(x \times y \times z \times w)\overline{v})\\ &= Re(x \times y \times z \times w)\\ &= \langle x, y \times z \times w\rangle\\ &= \Phi(x, y, z, w),\end{aligned}$$

whereby we used the simple fact

$$Re(vt\overline{v}) = \frac{1}{2}[vt\overline{v} + \overline{vt\overline{v}}] = \frac{1}{2}[vt\overline{v} + v\overline{t}\overline{v}] = \frac{1}{2}v(t + \overline{t})\overline{v} = \frac{t + \overline{t}}{2}v\overline{v} = Re(t).$$

for any $t \in \mathbb{O}$.

For the interesting roles the forms φ, ψ and Φ play in the context of calibrations we refer again to [9].

9. A Glimpse into Triality

As a last topic we want to give a look at the interesting phenomenon of triality. Recall that in Section 7 we constructed an algebra isomorphism

$$\tilde{f} : Cl_8 = Cl(\mathbb{O}) \longrightarrow End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O}) \text{ with } f(v) = \begin{pmatrix} 0 & R_v \\ -R_{\overline{v}} & 0 \end{pmatrix} \text{ for } v \in \mathbb{O}.$$

We have also seen that Cl_8^0 is identified with $End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O}) \subset End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$ under this isomorphism. Let us now determine $Spin(8) \subset Cl_8^0$. As we have seen,

$$Spin(8) = \{u \in Cl_8^0 \mid uxu^{-1} \in \mathbb{O} \;\; for \; x \in \mathbb{O} \;\; and \;\; uu^t = 1\}.$$

One can check that the transpose-antiinvolution in Cl_8^0 is translated by $\tilde{f}$ to taking the adjoint in the endomorphism-algebra: $\tilde{f}(u^t) = (\tilde{f}(u))^t$ for $u \in Cl_8^0$. This means that under the identification by $\tilde{f}$ we can write:

$$Spin(8) = \Bigg\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in End_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O}) \mid \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^t = I_{16} \text{ and for } x \in \mathbb{O}$$
$$\text{there exists } v \in \mathbb{O} \text{ such that } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & R_x \\ -R_{\overline{x}} & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & R_v \\ -R_{\overline{v}} & 0 \end{pmatrix} \Bigg\}.$$

Note that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^t = I_{16}$$

gives $AA^t = BB^t = I_8$ and we can write

$$Spin(8) = \{(A,B) \in O(8) \times O(8) \mid \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & R_x \\ -R_{\overline{x}} & 0 \end{pmatrix} \begin{pmatrix} A^t & 0 \\ 0 & B^t \end{pmatrix} = \begin{pmatrix} 0 & R_v \\ -R_{\overline{v}} & 0 \end{pmatrix}\},$$

or,

$$Spin(8) = \{(A,B) \in O(8) \times O(8) \mid \text{for } x \in \mathbb{O} \text{ there exists } v \in \mathbb{O} \text{ such that } AR_xB^t = R_v \text{ and } BR_{\overline{x}}A^t = R_{\overline{v}}\}.$$

The second condition is the transpose of the first and by the connectedness of $Spin(8)$ we get

$$Spin(8) = \{(A,B) \in O(8) \times O(8) \mid \text{for } x \in \mathbb{O} \text{ there exists } v \in \mathbb{O} \text{ such that } AR_xB^t = R_v\}.$$

We note that the assignment $x \in \mathbb{O} \longmapsto v \in \mathbb{O}$ is nothing else than $\widetilde{Ad}(u)$, where $u \in Cl_8^0$ is now in the guise of $(A,B) \in End_{\mathbb{R}}(\mathbb{O}) \oplus End_{\mathbb{R}}(\mathbb{O}) \cong Cl_8^0$.

We can express v in terms of x as follows:

$$AR_xB^t(e_0) = R_v(e_0) = e_0 \cdot v = v$$

and thus $v = A(B^t(e_0) \cdot x)$, where the dots denote the octonionic multiplication. (A and B are called the positive and negative spinor representations and C the associated vector representation.)

If we define $C = A\ L_{B^t(e_0)}$, then $C \in SO(8)$ and $v = C(x)$.

Applying the condition $AR_xB^t = R_v$ to an arbitrary element $z \in \mathbb{O}$, we get $(AR_xB^t)(z) = z{\cdot}v$, or, $A(B^t(z){\cdot}x) = z{\cdot}C(x)$. Denoting $B^t(z)$ by $y \in \mathbb{O}$ we obtain the following beautiful relationship between A, B and C:

For any $x, y \in \mathbb{O}$ it holds $A(y \cdot x) = B(y) \cdot C(x)$, where the dots denote the octonionic multiplication.

The same computation shows, that given A, B and $C \in SO(8)$ satisfying the relationship in this lemma, then $(A,B) \in Spin(8)$, with C the associated element to (A,B). We thus get the following result:

Proposition. *Let $A, B \in SO(8)$. Then $(A,B) \in Spin(8)$ if and only if there exists $C \in SO(8)$ such that $A(y \cdot x) = B(y) \cdot C(x)$ for all $x, y \in \mathbb{O}$, where the dots denote the octonionic multiplication.*

The actions of $Spin(8)$ on the first and second components of $\mathbb{O} \oplus \mathbb{O}$ via projections $(A,B) \mapsto A : \mathbb{O} \to \mathbb{O}$ and $(A,B) \mapsto B : \mathbb{O} \to \mathbb{O}$ are called the positive and negative spinor representations and the associated action on $\mathbb{O}$ via C is called the associated vector representation.

We now proceed to construct the triality automorphism of $Spin(8)$.

Let $E'(x) = \overline{E(\overline{x})}$ for $E \in End(\mathbb{O})$ and $x \in \mathbb{O}$. Now, let us be given $(A,B) \in Spin(8)$ with the associated C. Then,

$$A'(y \cdot x) = \overline{A(\overline{y \cdot x})} = \overline{A(\overline{x} \cdot \overline{y})} = \overline{B(\overline{x}) \cdot C(\overline{y})} = \overline{C(\overline{y})} \cdot \overline{B(\overline{x})} = C'(y) \cdot B'(x).$$

This means that (A', C') belongs to $Spin(8)$ and is associated with B'.

Lemma. *The map* $\alpha : Spin(8) \longrightarrow Spin(8), \quad (A, B) \longmapsto (A', C')$ *is an automorphism and* $\alpha^2 = Id$.

The proof is straightforward.

It might be practical to carry the associated vector representation as the third component of a triple and write $\alpha : (A, B, C) \mapsto (A', C', B')$.

Now start again with $(A, B) \in Spin(8)$ with the associated C satisfying $A(y \cdot x) = B(y) \cdot C(x)$ for $x, y \in \mathbb{O}$ and insert $y \cdot x$ for y and $\overline{x}$ for x:

$$\begin{aligned} A((y \cdot x) \cdot \overline{x}) &= B(y \cdot x) \cdot C(\overline{x}), \\ A(y \cdot \|x\|^2) &= B(y \cdot x) \cdot C(\overline{x}), \\ \|x\|^2 A(y) &= B(y \cdot x) \cdot C(\overline{x}), \\ B(y \cdot x) &= \|x\|^2 A(y) \cdot \overline{C(\overline{x})} / \|C(\overline{x})\|^2 = A(y) \cdot C'(x). \end{aligned}$$

This means that (B, A) also belongs to $Spin(8)$ with the associated representation C'. We get:

Lemma. *The map* $\beta : Spin(8) \longrightarrow Spin(8), \quad (A, B) \longmapsto (B, A)$ *is an automorphism and* $\beta^2 = Id$.

In this case we could write in the triple notation, $\beta : (A, B, C) \mapsto (B, A, C')$.

Combining these two automorphisms we find the triality automorphism:

Theorem. $\tau = \alpha\beta : Spin(8) \longrightarrow Spin(8)$ *is an automorphism with* $\tau^3 = Id$.

Indeed, $\tau(A, B, C) = (\alpha\beta)(A, B, C) = \alpha(B, A, C') = (B', C, A')$. Then we get $\tau(B', C, A') = (C', A', B)$ and finally, $\tau(C', A', B) = (A, B, C)$, so that we find $\tau^3 = Id$!

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References

[1] Altmann, S.L., 1989. *Hamilton, Rodrigues, and the quaternion scandal.* Mathematics Magazine, **62(5)**, pp.291-308.
[2] Baez, J., 2002. *The octonions.* Bulletin of the American Mathematical Society, **39(2)**, pp.145-205.
[3] Bonan,E., 1966. *Sur les varietés riemanniennes a groupe d'holonomie* G_2 *ou* $Spin(7)$. CR Acad.Scie.Paris. **262** pp.127-129
[4] Dickson, L.E., 1919. *On quaternions and their generalization and the history of the eight square theorem.* Annals of Mathematics, pp.155-171.
[5] Ebbinghaus H.D., Hermes H., Hirzebruch F., Koecher M., Mainzer K., Neukirch J., Prestel A., Remmert R., 1991. *Numbers*, Springer Verlag.
[6] Frobenius, H., 1878. *Über lineare Substitutionen und bilineare Formen.* Journal für die reine und angewandte Mathematik (Crelles Journal), **1878(84)**, pp.1-63.
[7] Hamilton W. R. 1847. *The Transactions of the Royal Irish Academy*, pp.338–341,
[8] Harvey, F.R., 1990. *Spinors and calibrations.* Elsevier.
[9] Harvey, R. and Lawson, H.B., 1982. *Calibrated geometries.* Acta Mathematica, **148**, pp.47-157.
[10] Hurwitz, A., 1898. *Über die Composition der quadratischen Formen von belibig vielen Variablen.* Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pp.309-316.
[11] May, K.O., 1966. *The impossibility of a division algebra of vectors in three dimensional space.* The American Mathematical Monthly, **73(3)**, pp.289-291.
[12] Milnor, J., 1958. *Some consequences of a theorem of Bott.* Annals of Mathematics, pp.444-449.
[13] Palais, R.S., 1968. *The classification of real division algebras.* The American Mathematical Monthly, **75(4)**, pp.366-368.
[14] Hüseyin Tevfik Paşa, 1882. *Linear Algebra*, Constantinople.